Lesson Study as a Tool for Research: A Case of Undergraduate Calculus

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1. Introduction

The study of student learning of and difficulties with undergraduate calculus has been a major part of mathematics education research for over a decade (Ferrini-Mundy & Graham, 1994). This study used the Lesson Study experience of a group of mathematics Ph.D. students to investigate the potential of lesson study as a research tool for mathematics educators interested in research in calculus and undergraduate learning of mathematics. The idea behind the practice of lesson study is to improve education by having a group of teachers study the processes involved in teaching and learning (Fernandez, 2002).

Lesson Study consists of four major stages: study, planning, teaching and reflecting. In the study phase, a topic to improve is selected and studied by the lesson study group. During the planning phase, the results of the study process are used to devise a lesson to address the topic in such a way as to improve learning. The lesson is then taught by one of the lesson study team members during the teaching phase, while the other lesson study team members observe the lesson. In the reflection phase, the lesson study team debriefs in order to find ways in which to improve the lesson for when it is taught again (Fernandez, 2002). For a more detailed account of Lesson Study, see Kaplan, Corcoran and Cervello (2009). This paper will detail the study and reflection enacted by three mathematics education Ph.D. students in the course of one cycle of Lesson Study. The reflections contribute as study background for the next possible cycle of Lesson Study.

2. The Case

The Lesson Study reported here was carried out as part of a one semester graduate class called Teaching College Mathematics that was designed as professional development for mathematics education Ph.D. students who plan to teach in a mathematics department in their future academic careers. The Lesson Study team was comprised of five graduate students and a faculty member in mathematics education. The study phase of the Lesson Study was enacted in the month of September and included reading and discussions about lesson study and calculus learning as well as the writing of individual literature reviews by the Ph.D. students. The planning phase of the Lesson Study took place in October. There was a practice teaching of the study lesson and a short reflection and planning cycle prior to the actual teaching and reflection phases, which took place in November. For a more detailed description of the enactment of this Lesson Study see Kaplan, et al. (2009).

The study lesson, on applications of the derivative, was designed for a calculus I course at Michigan State University. The professor who allowed us to teach this study lesson during one of the class periods described the class as a typical undergraduate engineering calculus class that was generally interactive, with students who were open to thinking about the mathematics. The 20 students in attendance the day the study lesson was taught were mainly freshmen and mostly engineering majors, but with a variety of other majors as well. The overarching goal under which the particular lesson was designed was to provide opportunities for the students to move between and work toward an understanding of different representations of a mathematical concept. In addition, there was an understanding on the part of the Lesson Study group to ensure that the students were engaged in authentic tasks.

The topic of the study lesson was optimization. The study lesson had four student learning goals: students will be able to (a) estimate solutions to optimization problems using tables, graphs, and dynamic graphs, (b) solve optimization problems using algebra, (c) describe affordances of using tables, graphs, and algebra to solve optimization problems, and (d) describe relationships between solutions found using different methods. The selection of calculus I and the focus on representations of the derivative as a topic for the study lesson was based on the derivative's foundational status in undergraduate mathematics and substantial research related to student thinking about the derivative (Ferrini-Mundy & Graham, 1994; Monk, 1994; Monk & Nemirovsky, 1994; White & Mitchelmore, 1996; Williams, 1991; Zandieh, 2000). With increasing interest in learning and teaching collegiate mathematics, many studies about students' understanding of calculus concepts have been done. Among calculus concepts, the derivative seems to be one of the most complex for students to understand because it is related to many other concepts (e.g., ratio, limit, & function) and can be represented in various ways: the slope of the tangent line, an instantaneous rate of change in a physical context, and Leibniz notation. Existing literature shows students' preference of the algebraic representation over other representations of the derivative and, further, a lack of understanding between representations (e.g., Hahkioniemi, 2005).

The study lesson had three parts or examples; the first two problems were designed as interactive presentations and the third was designed for student group work. The first task, shown in figure 1, was designed to show the students the affordances and disadvantages of tabular, graphical, and algebraic representations in finding local minima and maxima. It was expected

What are the extreme values of $f(x) = \dfrac{2\sin(\frac{\pi x}{2})}{x^{\frac{1}{3}}}$?

Figure 1. The first problem

that students would take the derivative of *f(x),* and set the equation of *f'(x)*=0. The algebraic solution to this equation is not trivial to compute, which provided the motivation to use graphical and tabular methods to estimate the local extrema.

The second task, shown in figure 2, was based in the family of functions known as the normal density functions. Because the students had not yet learned the derivative formula for the exponential function, the problem was designed to see how students approach a problem when they cannot take the derivative algebraically. The presentation used a graphical and tabular approach to show the relationship between the parameters *a* and *b* and the location of the extrema.

How could we find the maximum of this function? $f(x)=\frac{1}{b\sqrt{2\pi}}e^{\frac{-(x-a)^2}{2b^2}}$

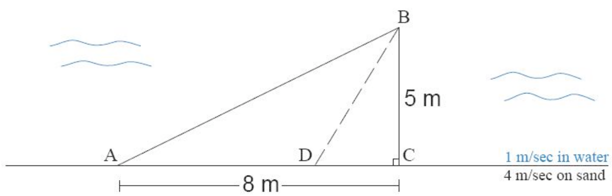
Figure 2. The second problem

The last problem, shown in figure 3, was designed to be done by the students in small groups. This small-group activity was built around a problem adapted from an article appearing in the *College Mathematics Journal* (Pennings, 2003). As depicted in figure 3, a dog begins at point *A* and wants to fetch a tennis ball at point *B*. The dog runs 4 meters per second on the sand and swims 1 meter per second in the water. The students worked individually to determine the time required for the dog to reach the ball swimming directly along *AB* and the time required if the dog ran from *A* to *C* and then swam from *C* to *B*, questions 1 and 2 of the task. Once it was clear that the students understood the calculations associated with the task, they worked in small groups to answer the question: What is the optimal path, with regard to time, for the dog to take to the ball? They were instructed to begin by discussing and deciding on an approach to the problem (algebraic, graphical, or tabular). The groups had been predetermined by the Lesson

Study group and were indicated using color coding on name tags that had been taped to the desks in the classroom.

Math on the Beach

Prof Kurn and her dog Molly enjoy playing fetch with a tennis ball on the beach. After several trips to the lake and several observations of Molly running along the shore for a certain distance before diving in and swimming toward the ball, Prof Kurn began to wonder—Is Molly minimizing the time that it takes to get to the ball? Can dogs do calculus?



To answer this question, let us make the following assumptions:

- Molly starts at point A and the tennis ball is at point B
- Molly's speeds and the situation's distances are as indicated in the figure above
- Molly's speeds do not vary over time or distance
- the shoreline is straight
- the wave action is negligible

1) How much time would it take for Molly to reach the ball if she swam straight to it?
2) How much time would it take for Molly to reach the ball if she ran from A to C and then swam from C to B?
3) What is the optimal path, with regard to time, for Molly to take? (In other words, where is the point D such that running from A to D and swimming from D to B minimizes the time it takes to reach the ball?) Be prepared to discuss how you determined your solution.

Figure 3. The third problem

On the day of the study lesson, the research team had five observers, four video cameras, and one audio recorder in the classroom. One video camera was focused on the instructor of the lesson; the other three cameras were focused on the students. During group work, the video cameras captured the work of two different groups and the audio recorder captured the work of a third group. In addition, the worksheets on which the students completed the third problem were collected, copied and returned to the students. The data on which the research projects below were based are the transcripts of the audio- and videotapes and the written work submitted by the students.

3. The Research Projects

3.1 Differing Enactments of the Small Group Task

During the small-group activity in the second half of our study lesson, the students formed six pre-assigned small groups. The goal of the task was to provide the students with an opportunity to engage in authentic mathematical ability and gain first-hand experience with at least one of the optimization approaches emphasized in the lesson: algebraic, graphical, or tabular. This section addresses the following question: How did different small groups interact with one another and engage with the mathematical task during the small-group activity?

The teacher circulated the room during this period of the lesson and thus was able to get snapshots of the groups' progress but not a full picture of any one of the groups. The rest of the lesson study group members were positioned as observers on the periphery of the classroom and were able to gather video data of two small groups and audio data of a third. We also collected their written work. It is to these three small groups that we now turn our attention, the analysis of which we hope demonstrates the dual function of lesson study with respect to practice and research; practice because the process and results of the analysis are significant opportunities for teacher learning on the part of those in the lesson study group, and research based on the rich data sourced directly in classrooms (Fernandez, 2002; Roback, Chance, Legler, & Moore, 2006).

Observations

Group 1: Individual workers in close proximity

Dagney sits in front of Gwen and to their respective rights are Hank and John. Dagney turns her desk slightly toward the middle of the group, the other desks remain unmoved. Gwen talks about assigning the variable x to DC and Hank says that they might want to start with $a^2+b^2=c^2$ but, unfortunately, he "forgot how to do it." Without having discussed a general approach to the problem (e.g., algebraic, graphical, or tabular) and without having discussed what a solution would conceptually consist of, the group works individually for more than a minute. It appears that the de facto approach is algebraic, which is verified by looking at their written work.

Dagney turns toward Gwen and says, "I put my variables opposite to yours." Gwen looks at Dagney's paper and then erases her own in favor of AD being labeled x and DC being labeled $8 - x$. Note that there is no discussion about advantages of one variable assignment over the other and, as we will see shortly, it is plausible that their desire for matching variables stems from the fact that it will make future comparisons of progress more clear. Dagney and Gwen then talk about the time spent on land, t_1, and the time spent in the water, t_2. They agree on expressions for these quantities in terms of x and Dagney says that they should be added together. She continues, "You have to take the derivative and set it equal to zero." With this pronouncement of next steps, the group again retires to their individual workspace for the next three minutes.

After this second independent work session, Dagney turns toward Gwen to compare work. Both have correctly taken the derivative of their time function and, after checking cancellations, they agree that their derivatives match. Individual work then continues for 1 minutes and 20 seconds until Dagney and Gwen compare work again. They have each attempted to solve the equation that resulted from setting the derivative equal to 0 but both were unsuccessful. Dagney points with her pencil on Gwen's paper as they search through each step of the calculation for possible mistakes. At this point, Hank leans over to see what they are talking about but does not say anything himself. He has used the individual work time to attempt implicit differentiation of his $a^2+b^2=c^2$ equation and substitutions of x and $8-x$, to no avail. The small group session ends before any further progress is made.

In summary, Group 1 behaved remarkably like students working on an individual homework assignment. This was evidenced from the beginning by the fact that their desks remained essentially in individual positions and at the end by the fact that a majority of time was spent doing quiet independent work. Two of the group members, Dagney and Gwen, conferred several times, but not to discuss the conceptualization or meaning of the problem; they merely set up matching notation, communicated a procedure to follow, and then checked progress along the way. Another group member, Hank, was only peripherally involved. Furthermore, the group's fourth member, John, never interacted with the rest of the group. Working alone, he correctly solved the optimization problem. Ironically, his labeling was different from the rest of the group and he used a calculator to solve the equation that had caused the rest of the group trouble, bypassing difficult algebra to make progress on the calculus. This group's enactment of the task did not include the doing of rich, collaborative mathematics, nor did it include discussion of the relations between the optimization in different representations, which were the goals of the study lesson.

Group 2: Recreators of a worked example.

Daisy, Catherine, Jay, and Nick laugh together as they move their desks, creating a sort of small-group table. Daisy and Jay talk about a past example completed as a whole class that resembles the current optimization problem. The group discusses using it, but Daisy says, "Well, I feel like I'm cheating or something." They laugh. Daisy continues by suggesting that they "figure out the relationship between the times," but this is not the last we will see of the past worked example.

The teacher stops by and asks what approach they are taking. The group answers, "Algebraic." Jay says that it is the easiest, and Daisy adds that they've done it this way before. The teacher asks how they plan to start, and Daisy responds that they will "set up an equation." It appears that Daisy, and perhaps others, feel the task will be easy to complete.

After the teacher leaves, the group's attention does turn toward setting up an equation. Nick points to *DB* and says they're trying to find that distance (see figure 3). Jay adds that they also need *AD*. Nick notices that Jay has labeled *DC* as x and says, "So this distance [*AD*] must be eight minus x." At this point Daisy abruptly places her notebook in the center of the group table facing Jay and Nick. It is opened to the past worked example and she taps the page twice with her pencil. After a brief pause, Jay and Nick continue their joint reasoning by talking about the placement of x and $8 - x$. Nick declares that "it doesn't really matter" which is which, and Jay says it is a case of "tomato, tomahto." The group then decides to use x as a label for *AD* rather than *DC* and y as a label for *DB*. Daisy leans to look at her notebook and asks, "Then what does she [the course professor] do [in the worked example]?" Jay, without looking at the notebook or acknowledging Daisy's question, suggests that they use the equation $x^2 + 5^2 = y^2$, which he writes on his paper. Nick looks at this and asks the group, "If we knew what y was, could we solve for x?"

Here Jay and Nick seem to be resisting the temptation to simply follow the worked example and are trying to continue along steps that make sense to them. Unfortunately, the group is unsure of what to do with Jay's equation and how to answer Nick's question. Daisy uses this uncertainty to again push for reliance on the worked example. She points to her notebook and asks, "How about we just follow what she did?" Jay leans back and jokingly says, "But I'm *thinking* about it." The group laughs and Jay finally relents. "Yeah, we can." Group 2 shifts to a complete focus on recreating the worked example.

With Daisy momentarily in the lead, she posits that the length of *AD* will be changing at a rate of 4 meters per second and asks, "Is that right?" That is not right. Nick then explains how to use the given speeds together with the labeled distances to create a time function (see figure 4). Catherine verifies that distance divided by speed is correct. Daisy, however, raises a concern because this function does not look like the one in her notebook, which has square roots. Jay points to the square-root term and notes that "this is if you solve for *y*," which he has done using his equation from earlier. Nick confirms this computation and Daisy writes down the new function including the square-root term (see figure 4).

$$\frac{x}{4} + \frac{y}{1} = t \qquad T = \frac{x}{4} + \frac{\sqrt{5^2 + (8-x)^2}}{1}$$

Figure 4. The time functions of Nick and Daisy, respectively.

Group 2 agrees that they have an appropriate function for the time the dog would take to reach the tennis ball, so Daisy says, "Okay, well let's just find the derivative then." They work together to identify the various pieces that will go into the derivative function (e.g., the "inside" expression, the "derivative of the inside") but Daisy expresses frustration at the fact that it seems to be a "crappy derivative." She looks up from her work to the example in her notebook. The group then works individually for a minute to execute the derivative computation.

Pulling the group back together, Daisy says, "Now we are supposed to solve this equal to zero. We must not have done something right, or just in a difficult way." She seems to be concerned because the group's derivative looks more complicated than the one in her notebook. (It is worth noting that Daisy, Nick, and Catherine have all computed the derivative correctly.)

Daisy then looks up at the board in the front of the classroom where the three general approaches are recorded and says, “We could do a table.” Jay interjects, “No, let’s do it algebraically.” He mock-pounds his fist and continues, “Let’s think through this. We just need to *think* about it.” The group laughs. They then spend the remaining minutes talking about an upcoming exam.

In summary, Group 2 began with Jay and Nick making some progress on the problem in a manner that made sense to them. At the continual prompting of Daisy, however, the group eventually changed their strategy to one of following the example laid forth in Daisy’s notebook. Ironically, this mimicking approach, which Daisy initially thought would be easy to the point of cheating, resulted in the abandonment of a correct derivative. Additionally, Jay at two different points referred to the importance of “thinking” through the problem (perhaps for the benefit of the video camera) but did so in an obviously joking way, as if he knew that this was an officially appropriate position but an unofficially ridiculous one. This group differed from Group 1 in that they worked together for the vast majority of the activity, but they resembled Group 1 in that they never seriously considered what another approach could offer their understanding of the problem and they were in many ways not deeply engaged in the doing of mathematics; rather, they were engaged in an attempt to recreate mathematics from the past.

Group 3: Partners working based on contextual cues

Joe and Frank pull two desks together in a back corner of the room. Frank begins by suggesting that they “just make a table.” This is fine with Joe. Frank starts a sentence, “Okay, so we already have,” and Joe completes it, “zero and we already have eight.” (This refers to calculations that were completed as a class prior to the small-group activity using the direct path along *AB* and the path comprising *AC* and *CB*; see figure 3.) They write out a blank table and confer with each other regarding the meaning of each column. Joe suggests that x represent the distance traveled on land (*AD*) and y the distance traveled in water (*DB*). After labeling these columns, they decide to add a third column for the total time.

Joe and Frank quickly discuss and then fill in the first row of their tables using the previous whole-class work. Moving on to the second row, Frank says, "Let's say they're doing one meter on land. How much are they doing in water?" After a moment of computation, Joe responds that *DC* would have "seven left." He then uses the Pythagorean theorem to determine that the associated y-value is $\sqrt{74}$. As Frank works to verify this calculation, Joe comments on their tabular approach, pointing out that, "whatever answer we get is not going to be the most appropriate answer, it's just going to be an approximation." This echoes the preceding whole-class discussion in which the advantages and disadvantages of different approaches were identified. Frank agrees with the limitation. Joe asks, "Is that alright?" Frank responds that it is.

At this point the teacher visits Group 3 and notices that they are taking a tabular approach to the problem. The group members affirm this observation but do not respond further as they are working to determine the time associated with $x = 1$ and $y = \sqrt{74}$. These individual calculations are markedly different than those in the groups above because Frank and Joe consistently call out the operations they are performing on the calculator for the benefit of their partner. Joe declares that the "time is going to be 8.85, approximately." The pair then moves on to $x = 2$. Frank describes the calculation he is attempting to execute on his calculator and Joe begins to offer a way to double-check it when Frank stops him. "Hold on a second." Frank brings the attention back to the $x = 1$ case because he has noticed that there is more than one way to think about x. It seems as though Frank was not aware of this choice when they talked earlier about x being the distance run on land. Joe explains that he is thinking of x as the distance run on land, that is, *AD* rather than *DC*. Joe continues by verbalizing his understanding of Frank's concern, "What you were saying is that we could have made x the seven meters." Frank responds, "Right." Joe begins to try to further clarify with the diagram when Frank says, "Hold on, let me get my mind around this. Okay, so you're saying that this [*DC*] is seven and this [*CB*] is five so then *this* [*DB*] is eight-point-six, right?" Joe responds, "Yeah." Frank is now ready to fully move on to $x = 2$.

They fill the second row quickly and move systematically down the remainder of the table. After a minute, Joe begins looking through his notebook and, perhaps realizing that Frank should have no trouble with the rest of the table and remembering that the tabular approach is limited in its the solution, makes the following suggestion: "Do you mind if you do the table and I'll get the direct answer so we'll have both?" Frank answers, "Alright." Joe adds that he found the example with the "direct answer" in his notebook. (This is the same worked example used by Group 2.) Frank asks Joe how he will get the direct answer, and Joe explains that he will set up the function that goes with their table and then differentiate it. "So I'm doing it algebraically while you're doing it table-y." They laugh. Joe continues, "And then we're going to take the two and see how close we were." The pair uses the rest of the time to work on their separate tasks. Frank completes the table with $x = 7$ corresponding to the approximate minimum time value but the small-group activity ends before Joe finishes, though he has progressed without any errors.

In summary, Group 3 used cues from the context of the lesson (e.g., calculations just prior to the small-group activity, themes in the whole-class discussions) and an attention to each other's thinking to progress successfully, though not completely, through the problem. Their time working independently at the end was distinct from that of Group 1 because they laid out separate and complementary tasks and intended to compare their solutions meaningfully rather than simply checking them for agreement. Their use of the worked example was distinct from that of Group 2 because it came only after more than 6 minutes of an approach marked by sense-making and they were going to use the worked example to verify this previous work rather than to guide the entire solution process. The way in which Group 3 worked together also seemed to differ in quantity from Group 1 and in quality from Group 2 as Frank and Joe seemed to be more attuned to the alignment of his thinking with that of the other. There was much evidence of this, none stronger than the interaction in which they both verbalized what they heard the other saying (when they revisited the $x = 1$ case). This group was able to meet some of our lesson study goals for the activity in that they engaged in a non-trivial amount of collaborative mathematics and discussed the fact that one approach to a solution could address the shortcomings of another.

Reflections

The structure of lesson study allowed us to follow closely the work of three separate groups during a small-group activity, something that would have been difficult for an individual teacher or researcher. The significant differences in the ways that the groups enacted the task have implications for both practice and research. For teachers, it emphasizes the importance of reflecting on not only the tasks that will be assigned but also the manner in which the students may engage in the tasks. Furthermore, while circulating during a small-group activity teachers can look for not only how far the groups have progressed but also the way in which the group members are working with one another. For researchers, this analysis raises several interesting questions. What would be contained in a comprehensive account of the different ways in which small groups of undergraduates enact mathematical tasks? What are the factors that influence these different enactments? What is the relationship between the enactments and student outcomes such as achievement or attitudes toward mathematics? What is the relationship between students' mathematical histories and the way in which they engage in small-group activity? Specific conjectures in response to this final question would be that the interactions of

Group 1 relate to a history of mathematics learning as an individual endeavor with a bounty of homework and a scarcity of collaborative sense-making, or that the interactions of Group 2 relate to a history of executing procedures that have been recently demonstrated by the teacher.

3.2 Finding Extreme Values using Algebraic, Graphical and Tabular Representation

In this project, the instructor's and students' discourse in the classroom was analyzed to answer the question: What are the patterns that emerge when the teacher and students find extreme values of functions using three different representations: algebraic, graphical and tabular? Studies about students' understanding of various representations of the derivative involve hierarchical frameworks (Hahkioniemi, 2005; Santos & Thomas, 2003; Zandieh, 2000). Santos and Thomas (2003) and Zandieh (2000) use matrix-form frameworks with columns of several understanding layers and rows of different representations of the derivative. Using these frameworks, the authors conclude that students show different levels of understanding depending on representations of the derivative. For example, a student who can explain all three concept-layers of the derivative in one representation cannot explain the same level in another representation (Zandieh, 2000). Hahkioniemi (2005) investigated students' conceptual knowledge by analyzing how they explained relations among different representations of the derivative. While students who had weak understanding of the relations only could explain the meaning of each representation separately, students with strong understanding could explain the relations among different representations, adopting various interpretations.

The method of discourse analysis used in this study was inspired by Sfard's (2008) notion of *routine*. Sfard (2008) defined *routine* as "a set of meta-rules that describe a repetitive discursive action" (p. 208). In other words, routine delineates rules that describe patterns in mathematical discourse. A routine may be divided into two subsets of meta-rules. *Routine-how* are rules that determine, or constrain, the path of the discursive performance (the *course-of-action*). R*outine-when* are a collection of meta-rules that determine, or constrain, situations in which one would consider a discursive performance as proper to start (*applicability condition)* or to finish (*closure).* By analyzing routine-when and routine-how, I attempted to find patterns that repeatedly and consistently emerged across problems during the enactment of the study lesson. To find the repeated patterns, I analyzed the classroom and small group discussions.

Observations

Different routines were identified when the teacher and students were working with different representations. When working with algebraic representations, the pattern that emerged was to (a) take the derivative of a given equation of a function, (b) set the derivative equal to zero, (c) solve the equation, and (d) make the definite statement about the method such as, "with all that, we find what the exact maximum is" (T60c). The analysis of the patterns of discourse that emerged when the participants were working with problems graphically are shown as an example in Appendix A with excerpts from the transcript. Using the graphical representation, the routine that emerged was to (a) graph the given equation, (b) find the extremes by either using the function key in the calculator (S9b) or by looking at the graph (S20, T36, S185, T186, S187, T188, & S189), (c) check end points of a given interval (T21a, b, & T36c), (d) mention the slope of the tangent line on extreme values (T29a, b), and (e) make a general statement of the relationship between the graphs of a function and its derivative, such as, "whenever…the graph of the derivative…crosses [the] *x*-axis, we've got a max [or] min there" (T30). When using tabular representations, the routine that emerged was to (a) make a table of values of a function with respect to several *x* values, (b) look at patterns of values in the table, (c) find the extremes of the function in a table, (d) mention values of the derivative, (e) look for numbers close to zero in the derivative column, (f) look for values of the function around x-values almost zero, and (g) estimate extremes.

In summary, the patterns of discourse differed according to the representations used. For example, the teacher mentioned checking the end points of the given interval when he approached the problems with graphs and tables, but he did not mention it using algebraic representations. Also, when using an algebraic approach, the subjects always included in the discourse a notion of setting the derivative equal to zero. When using a graphical or tabular method, however, they did not even mention the derivative in some of the problems.

Reflections

In the classroom discourse, the *routine-when* prompts and closing conditions were mostly determined by the teacher, so the research did not provide opportunities to learn about situations in which students choose one representation over others or move to other representations while working on problems involving extreme values of functions. Similarly, *routine-how* courses of action were mostly led by the teacher, which made it difficult to obtain information about students' reasoning behind the choice of representations and changes they would make in problem solving processes. In order to learn more about student generated discourse in a future lesson, I would plan more specific prompts and opportunities for students to participate in classroom discourse, particularly about *why* students would choose one representation. In addition, I would include in the lesson plan questions a teacher might ask while students were working in groups that would help students consider changing representations and elicit reasons that they might decide to employ a different representation.

This study suggests that the choice of representation could make a difference in the observed patterns of discourse. Since optimization was introduced as a part of the application of the derivative, the students seemed to rely heavily on the algebraic method. By representing functions and derivatives in different ways, students might be encouraged to try other methods which do not necessarily include the procedure of "taking" the derivative when they estimate the maximum or minimum. On the other hand, they also realized a benefit of the algebraic method, namely that they can calculate exact extreme values compared to graphical and tabular methods, which lead to estimates of extremes.

3.3 Quantitative Research on Lesson Study

This research project aimed to design a quantitative study of the effects of Lesson Study on student learning. Just as there are four phases to lesson study, a quantitative program of research can be seen as a four-phase process. The first of these phases is *needs assessment*, where the needs of the quantitative program of research are developed. Some question to look at is specified, along with operationalized goals and a definition of how it is known that those goals are reached. The second phase is, as in lesson study, *planning*. During this phase, items are written and piloted and instruments are developed. The third phase, *execution*, is when the instruments are administered. During the final phase, *evaluation*, the program of research is analyzed. Questions such as, did the items function properly, what results can be stated based on performance on those items and how can we improve the program during the next Lesson Study cycle, are answered during the evaluation phase.

Observations

Needs Assessment

The needs assessment phase takes place at roughly the same time as the study phase of a lesson study. During the study phase, the lesson study team defines an overarching goal, selects a question, reads the literature and draws on personal teaching experiences to learn about what is already known about the question. During our lesson study, we did define an overarching goal and select a question. Goals of the lesson study were set and operationalized. Our lesson study topic was optimization in the context of different types of representations. This is a rather complicated topic, so both the definition of success and the assessment to measure success must address that complexity.

In addition, the needs assessment should address the current baseline. For example, if studying students, how much knowledge related to the topic in the lesson do they currently have or do they gain in typical classroom? If studying the instructor, what is the current behavior? If the goals of the lesson study are already satisfied in the population, then the goals of the quantitative program of research need to be revised, as well as the goals of the lesson study.

Planning

After the goals have been specified, the next step is to plan the actual design and administration of the instrument. The design should be sufficiently related to the plans of the Lesson Study team in order to make sure that the instrument and the lesson are aligned. The administration of the instrument needs careful consideration, because any task(s) that are administered during the lesson need to be considered during the lesson planning. It cannot be stated with enough emphasis that, if studying the effect of the lesson study on the students, the length of time for the planning of the instruments must be significantly longer than that of the Lesson Study. The piloting of instruments needs a large amount of time to be executed and it cannot be done until the full content of the lesson plan has been laid out, so as to make sure that the items actually test what is intended to be taught.

Besides suitable piloting or use of previously validated measures, it is also important to consider the complexity of the items and to investigate sources of “irrelevant variance” (Standard 3.17), which may confound the results. For example, while this study lesson had as a goal student learning about optimization, there were other factors that could interfere with the ability to perform the tasks, both analytic and algebraic. In addition, assessment items must be examined for “the degree to which scores included a speed component” (Standard 3.18). In this iteration of assessment, students complained that the assessment task was difficult to engage in, meaning that it had not be presented in “sufficient detail so that test takers can respond to a task in the manner that the test developer intended” (Standard 3.20).

Execution

The execution phase of the research program should take place before, during and after the teaching phase of the lesson study. Depending on the defined research questions, instruments should be administered to other graduate teaching assistants or to other similar classes of students who are not participating in the study in order to control for other relevant variables. Care should be taken during the administration of the items to make sure that each administration is similar in duration, who is answering each question and how. Without such administrations it is not possible to discern whether gains in performance associated with the lesson.

Assess

Finally, the program of research should be assessed. After all of the data have been collected, it should be determined whether and to what extent the goals of the lesson study were accomplished. Results from this phase can then inform the next cycle of research: Did the items perform well? Was our model sufficiently defined?) In addition, the next cycle of Lesson Study could be informed by considering which, if any, goals were not satisfied. The study lesson could be revised after considering how those goals could be addressed. Particular consideration might be given to the question of whether there could there be other reasons that explain poor performance on certain items or with regard to progress on certain student learning goals.

Model specification is crucial in the assessment phase of a quantitative research program. One possible model is a value-added model, which requires pre-lesson data on the participants. Another possible model that might be used to assess the outcomes of a Lesson Study is a hierarchical linear model with measurements nested within students nested within classrooms. Based on my past experience, I think that there will be some degree of variance between sections, especially if one is the control section.

Reflections

In summary, it would appear that lesson study is ripe for study through quantitative methodology in addition to qualitative methods. The phases of planning for a quantitative program of research run parallel to the lesson study process and, in most cases, there is interaction between the processes.

4. Overall Reflections

In Kaplan et al. (2009) the authors illuminate value of Lesson Study as a professional development activity for mathematics teaching assistants. In this companion paper, we have made the case that Lesson Study is also a valuable activity for developing the classroom research skills of future researchers in undergraduate mathematics education. Incorporating individual literature reviews on the part of the Ph.D. students involved in the Lesson Study group helped the student researchers to focus on certain aspects of the learning that would take place in the classroom, both during the development and the teaching of the study lesson. In addition, a study lesson provides a unique opportunity for classroom data collection because the lesson can be manipulated to meet the needs of the research and because of the opportunity for multiple researchers in the classroom.

The three research ideas presented in this paper are varied in scope. In the first project, the video data collected of the calculus students' group work was analyzed using an approach similar to grounded theory, in which the researcher allowed themes to emerge from the data. This project has generated several future research ideas for the Ph.D. student who enacted the study. In the second study, a more senior Ph.D. student was able to collect data that allowed her to apply a framework with which she is familiar and has been working for some significant amount of time to data acquired in the course of an actual class. In the final study, a Ph.D. student who is interested in incorporating quantitative methods into mathematics education research was able to learn about such an implementation through enacting a pilot study of such a project. While the research described represents a significant time commitment above the normal parameters for research and lesson planning, the participants in this project agree that the benefits, both for our teaching and research, were well worth the time invested.

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Appendix A

Table 1. *Patterns with graphical representations*

Case (equations)		$f(x) = \frac{(2\sin\frac{\pi x}{2})}{x^{\frac{1}{3}}}$		$f(x) = \frac{1}{b\sqrt{2\pi}} e^{\frac{-(x-a)^2}{2b^2}}$	$f(x) = \frac{8-x}{4} + \sqrt{x^2+5^2}$ $T = \frac{x}{4} + \frac{\sqrt{5^2+(8-x)^2}}{1}$
Prompt		T4.… anyone tried any other way?		You can find the maximum if you look to graph.	T177b. Is anybody trying making a graph?
Course of Action	Graph	S5. I graphed it			T178. Take the equation you wrote and try sticking in calculator see what comes up. See if you get that same minimum.
	How to graph	T6. What…you did when you graphed it? S9a. You plugged it in the calculator			
	Obtain graph		T17a. If you graphed it, this should be pretty closed to what you got./ T17b. Showing the graph of f(x)		
	Extremes	S9b. Second, trace, calculates max, min	T18. What kind of extremes do you see?		T182. Did you find the minimum time using your calculator?
	Obtain extremes		S20. You are gonna have two local maxima which look to be equal and the local minimum, so you got three critical points.	T36a. We can see that it looks like it has a maximum and that maximum is right there T36b. Pointing the maximum point of the curve.	S185. *x* value or *y*? / T186. The *y* value. S187. 2.486 / T188. 2.486… so what was the *x* value? S189. 8?

	Check end points		T21a. when you are checking, you also have to check the end points of your interval, here and here T21b. pointing to two end points of graph on screen	T36c. We don't know that this is...the maximum because we don't know what happens if we kept going that way(-∞) or…that way (+∞)	
	Discuss meaning				T190. What is that *x* represents? S191. Time might end up in?
	slopes		T29. want to know what the value of the slope at a point		
	0 slopes		T29b. We are getting close to what it looks like a minimum, and notice, the value of the slope is about 0 .		
Closure			T30. Wherever…the graph of the derivative…crosses x-axis, we've got a max [or] min there.		(Time is up).