Conceptual Changes in Mathematics Majors' Understanding of Completed Infinite Iterative Processes

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1. Introduction

The teaching and learning of topics related to the concept of mathematical infinity has been researched extensively. Numerous researchers have indicated that infinity in its many forms is problematic for students of all ages, and noted that the Cantorian treatment of infinity has proven especially difficult because students find it counterintuitive (e.g. Fischbein, Tirosh & Hess, 1979; Tall, 1980; Wheeler & Martin, 1988; Falk & Ben-Lavy, 1989; Monaghan, 2001). In the area of infinite iteration, existing research involving students up to and including college level suggests that the vast majority of students provide non-normative answers to tasks that require them to define a state at infinity for an infinite iterative process (Dubinsky et al., 2008; Brown et al., 2008; Dubinsky et al., 2005; Stenger et al, 2005; Ely, 2007; Mamolo & Zazkis, 2008). Given that historically, infinite iteration played a crucial role in the development of how infinity is treated in mathematics, reasoning normatively about infinite iterative processes can help students in developing normative conceptions of all aspects of infinity and the many mathematical concepts related to it (e.g., infinite sequences and their limits, series, Cantorian set theory) (Dubinsky et al., 2005; Lakoff & Nunez, 2000). However, there is very little research exploring *how* students may come to reason about tasks involving infinite iteration in a normative manner. Our study aims to start addressing this gap.

More specifically, the two research questions addressed by this study are:

- What are the main types of initial arguments provided by mathematics majors in reasoning about infinite iterative processes and their states at infinity?
- Through what learning paths can mathematics majors come to reason about infinite iteration in normative ways?

2. Formalizing the definition of *state at infinity*

While several researchers have investigated students' solutions to tasks about infinite iterative processes (e.g., Dubinsky et al., 2008; Brown et al., 2008; Stenger et al, 2005; Ely, 2007; Mamolo & Zazkis, 2008), none of these studies has defined clearly (from a mathematical point of view) what characterizes a normative state at infinity for infinite iteration. In order to avoid this ambiguity, in our study we used only infinite iterative processes for which the intermediary states are described as sets of objects. More precisely, in this study an infinite iterative process consists of an initial set S_0, together with an infinitely countable ordered set of actions $\{A_n\}_{n\in N}$, where an action consists of one or finitely many operations on sets. It is assumed that A_1 is applied to the initial set S_0 producing a set S_1, then A_2 is applied to S_1 and the result is denoted by S_2, and so on. For the rest of the paper, we will use the term "intermediary state" to refer to an arbitrary S_n.

Following Allis & Koetsier (1995), we take the position that the information provided by the intermediary states of an infinite iterative process does not logically determine a state at infinity, unless we add an assumption that mandates that the objects on which the process acts have continuous space-time paths. In other words, we need to assume that: A1) any object that belongs to all intermediary states from a step on must belong to the state at infinity; A2) any object that *is not* part of any of the intermediary states from a step on is not part of the state at infinity; and A3) if there exists an object that is not part of either of the two previous categories, then by definition there is no state at infinity for the process in question. We believe that these three assumptions clearly define the notion of state at infinity for infinite iteration. From here on, the phrase "normative solutions" will be used to refer to arguments that lead to states at infinity that are in agreement with this definition, usage which is consistent with what other researchers have called normative solutions to tasks involving infinite iteration.

For an application of this definition to a task involving infinite iteration let us consider what we named The Original Tennis Ball Problem, which was borrowed from the existing literature on infinite iteration (Falk, 1994; Dubinsky et al, 2005):

> Suppose you are given an infinite set of numbered tennis balls (1, 2, 3,…) and two bins of unlimited capacity, labeled A and B.
>
> At step 1 you place balls 1 and 2 in bin A and then immediately move ball 1 to bin B.
> At step 2 you place balls 3 and 4 in bin A and immediately move ball 2 to bin B.
> At step 3 you place balls 5 and 6 in bin A and immediately move ball 3 to bin B.
> …………………………………………………………………………………….
>
> This process is continued in this manner ad infinitum. Now assume that ALL steps have been completed. What are the contents of the two bins at this point?

The pattern implied by the "…" is that at step n (where n is an arbitrary natural number), the two balls with the "smallest labels" are taken from outside of the bins and placed in bin A, after which the ball with the "smallest label" in bin A is moved to bin B. Assuming this pattern, one can prove by induction that for an arbitrary natural number n, **at step n** ball n is taken out of bin A and placed in bin B, and that none of the subsequent steps affects its position. By A1, ball n belongs to the final state corresponding to the contents of bin B, and by A2 it does *not* belong to the final state corresponding to the contents of bin A. As n was chosen arbitrarily, we can infer that after all steps have been completed, bin A is empty and bin B contains all the balls we started with.

3. Related literature

3.1 Early research on infinite iteration

Piaget and Inhelder (1956) reported on children's (ages 5-12) understanding of infinite divisibility. In this study, the tasks used involved repeatedly splitting a given geometrical figure into smaller parts (e.g., a segment was split into smaller segments by halving; a square was split into 4 smaller squares). The children were asked to predict what would happen if the process of division were continued mentally, predict the form of the "final elements" of such a division process, if considered completed, and discuss the reconstruction of the original figure starting from the final elements. The researchers concluded that only in the abstract (formal) operational thought stage

(age 11-12) did children conceptualize the division process as infinite, and the “final elements” as points. Fischbein (1963) replicated some of the Piagetian techniques and his results somewhat confirmed those obtained by Piaget and Inhelder (1956), but in Fischbein’s study only half of the subjects aged 11-12 (which are considered in the abstract operational thought stage, in Piaget terms) saw the division process as infinite.

Fischbein, Tirosh & Hess’ (1979) own infinite iteration study had 470 students in grades 5-9 solve tasks involving repeated division of a segment as well as other infinite processes set in a geometrical context. This study focused more on whether the students viewed these processes as finite or infinite, rather than on what a state at infinity would be. Infinitist views were considered to be displayed by answers such as “the process never ends” and “the process ends but theoretically it is infinite”. At all grade levels, finitist views (“the process comes to an end after finitely many steps”) were displayed by the majority of the students. Over all grade levels, 55% of the students had finitist views.

3.2 Recent theoretical studies

The Basic Metaphor of Infinity. Lakoff and Nunez (2000) propose that an infinite iterative process can be seen as completed if a metaphorical final state is added to it. This addition can be done by drawing a parallel between finite processes (that have a well-defined final state) and infinite processes. As both types of processes have an initial state and a clear procedure for obtaining the next state from an existing state, one can extend the parallel by imagining that the infinite processes also have a final, unique state that follows all intermediary state. This extension is what Lakoff and Nunez (2000) call the Basic Metaphor of Infinity. The metaphorical process thus obtained has infinitely many intermediary states and a metaphorical final state that is *unique* (that is, there is no distinct previous state within the process that both follows the completion stage of the process yet precedes the final state, and there is no other state of the process that both results from the completion of the process and follows the final state).

The APOS approach. Researchers embracing the APOS (Action Process Object Schema) learning theory (e.g. Dubinsky et al, 2008; Brown et al., 2008; Steger et al., 2005) propose that in order for one to construct an infinite iterative process, one needs to first be able to construct a process of iterating completely through N, which can be encapsulated into an object (conventionally labeled ∞) as one attempts to apply an action of evaluation to the process in trying to determine "what comes next". Reaching a process view of an arbitrary infinite iterative process then involves coordinating this completed iteration through N with a transformation that assigns an object to each natural number; an object view of this process (once seen as a totality) is reached by applying an action of evaluation to it, the obtained object being "a state at ∞" and understood as beyond the objects that correspond to the natural numbers (a transcendent object)

Both of these theoretical approaches will be discussed in more detail in the context of the empirical studies presented in the next section, as well as in the Discussion section concluding this paper.

3.3 Recent empirical studies

Several different studies examined college students' (mathematics, mathematics education, computer science, or engineering majors) reasoning on versions of what we called the Original Tennis Ball Problem. Dubinsky et al. (2008) and Mamolo & Zazkis (2008) used timed versions of this task (i.e., the steps of the process were time-indexed such that each step took half the time necessary to complete the previous step, and so infinitely many steps could be completed in finite time). Using an APOS perspective, both studies reported that the vast majority of the participants had a process view of the infinite iteration involved in the tasks, and either could not conceptualize of the infinite process as completed or attempted to do so (and define a state at infinity) in non-normative manners; the students in the latter category attempted to describe the state at infinity by plugging $"\infty"$ either in the algebraic expression indicating the cardinality of the n^{th} intermediate state (obtaining, for example, "9∞" as the number of elements in the final state, in the case of the problem used by Mamolo & Zaskis), or in the expression that represented the range of balls in bin A after n steps (obtaining, for example, $\infty+1-2\infty$ as the range of balls in bin A "at the end", in Dubinsky et al.'s study).

Ely (2007) reported similar types of reasoning for the tennis ball problem (out of 225 Calculus undergraduates, only three solved the problem successfully, while 73% either claimed that the process did not end or gave an answer that was categorized as meaning "infinitely many balls in bin A"). However, the author suggested that in the case of the students in the "infinitely many" category, APOS does not accurately explain their non-normative solutions. Ely (2007) claimed that these students did encapsulate the process into an object, but a non-normative object by metaphorically extending the "how many" property from the intermediate states to obtain a final state, which he interpreted as evidence for that claim that BMI provides a more plausible account for the different types of student answers.

Brown and her colleagues (2008) explored college students' reasoning on infinite iteration by challenging them to prove or disprove $\bigcup_{n=1}^{\infty} P(\{1, 2, 3, ..., n\}) = P(N)$. All twelve students in the study attempted to make sense of the infinite union on the left of the equation by approaching it as an infinite iterative process and noticed that $\bigcup_{k=1}^{n} P(\{1, 2, 3, ..., k\}) = P(\{1, 2, 3, ..., n\})$, consequently initially deciding that the state at infinity of the infinite iterative process was indeed P(N). Interviews of the students in the 6 groups revealed that only one student had an object view of the infinite iterative process in this problem (infinite union), and was successful in disproving the given statement. The other students in the study gave explanations which were coded as showing an action or process view of the infinite union, and could not successfully complete the problem. The authors conclude that the APOS approach to constructing infinite iteration provides adequate terms for explaining the various stages of understanding reached by the students in making sense of

$$\bigcup_{n=1}^{\infty} P(\{1, 2, 3, \ldots, n\}).$$

Brown et al. (2008) and Dubinsky et al. (2008) noted that although some aspects of the observed student reasoning are consistent with the mode of thinking predicted by BMI, their data indicated that successfully solving problems that require defining a resultant final state for infinite iteration may require more than metaphorical thinking.

We will discuss our position with respect to the APOS and BMI approaches in the Discussion section, in light of the data obtained in this study.

4. Research paradigm

In order to be able to closely examine students' initial conceptions of infinite iteration as well as investigate possible learning paths that students may take toward developing normative understandings, we employed a design research paradigm (in the sense of Cobb et al., 2003 and Gravemeijer, 1998). This type of research involves a cyclic process in which the researcher formulates "the significant disciplinary ideas and forms of reasoning that constitute the prospective goals or endpoints for student learning" (Cobb et al., 2003, p.11), after which a hypothetical learning trajectory and an associated instructional sequence are designed; the instruction is implemented with one or more students and this implementation is carefully observed and analyzed, and the hypothetical learning trajectory and instructional activities are subsequently revised based on this analysis. These steps are then repeated in a new cycle, with a new student or group of students.

For this particular study (which focuses only on the first cycle of a multi-cycle teaching experiment), our instructional goals were for the participants to be able to conceptualize an infinite iterative process as completed (that is, be able to imagine infinitely many steps as having been performed) and provide normative arguments regarding the state at infinity of an infinite iterative process across a variety of tasks. The sequence of tasks we used was semi-structured (i.e., some tasks were designed before the study to address common non-normative answers documented in the literature, while others were added during the study in response to the specific types of arguments displayed by the participating students), and was created by formulating variations of tasks such as the Original Tennis Ball Problem described above, as well as geometrical construction tasks inspired by the work of Fischbein, Tirosh & Hess (1979). In addition to the types of tasks already used by other researchers, we also used tasks in which the described process did not have a state at infinity (in the sense described in section 2). The rationale behind the creation of these variations was to obtain a collection of tasks that were similar enough to each other to inspire students to make connections between the types of reasoning used for each task, but also different enough to potentially trigger a variety of types of student arguments, which we as researchers could use to create cognitive conflict for the students. Our hope was that the students' attempts to resolve such conflict would lead to conceptual changes, as Piaget (1963) suggests.

The learning environment created for this study was typical of student-centered instruction, following Maher (2002). The participating students were encouraged to work together on tasks, to provide justifications for their answers, and to question the correctness of their answers as well as those of their partner. At the same time, the researcher did not validate or invalidate the answers given by the students, and did not force the students' investigations in predetermined directions. Instead, the researcher asked clarifying questions regarding the students' answers, and at times pointed out a conflict in the ideas presented and encouraged the students to discuss how the conflict could be resolved.

5. Methods

5.1 Participants

A written pre-test containing the Original Tennis Ball Problem was administered to fourteen mathematics majors[1] enrolled in a Problem Solving course at a large university in northeastern United States. Out of the thirteen students who provided non-normative answers to this problem, two[2] were randomly chosen. Both of these students, which we will call Max (a junior) and Tom[3] (a senior) for the purpose of this study, had already completed the Calculus 1-3 sequence, Linear Algebra, and a proof techniques course.

5.2 Procedure

Following the written pre-test, Max and Tom were interviewed separately by the first author and asked to explain in detail their reasoning on the pre-test problem. The two students then worked collaboratively for a total of six problem-solving sessions lasting approximately two hours each, during which they progressed through the task sequence at their own pace. These sessions, as well as the initial interviews, were videotaped. The students were then individually administered a written post-test containing tasks similar to those used in the main task sequence.

5.3 Data Analysis

The pre-test interviews and the six problem-solving sessions (totaling 14 hours of video) were transcribed. These transcripts, together with the worksheets used by the students, were used to identify chains of critical events (in the sense of Powell, Francisco & Maher, 2003) with respect to the nature of each student's initial arguments for each task, as well as the context in which changes in each student's reasoning occurred, if any. As the analysis progressed, we noted

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[1] We decided to work with mathematics majors in order to ensure that the participants had a certain maturity regarding proof techniques, and also because a mathematics major would have already been exposed to a wide variety of mathematical concepts, which gave us more mathematical contexts to choose from for our task variations.
[2] We worked only with two students at a time in order to be able to closely follow each student's reasoning, both during each session and later during data analysis.
[3] All student names are pseudonyms.

that students' reasoning was consistent with the behavior suggested by Wagner's (2006) "transfer in pieces" theory. Wagner proposes that transfer of knowledge is a complex process during which an initially topical set of principles is constantly refined to *account for* (and not ignore) the new contexts of the new problems encountered. Thus, the acquisition of abstract knowledge is a *consequence* of transfer and not a required initial component for it to happen. Furthermore, according to Wagner (2006), deciding what the mathematical structure of a problem is and whether it is structurally similar to a previously encountered problem is intimately connected with the problem-solving process itself; as a working set of principles is refined to account for new contexts, structural commonalities of the growing class of examples are gradually formulated, which in turn helps with the formulation of an abstract principle or set of principles applicable to the entire class. This view of knowledge transfer served as a guide as we revisited and refined our previous analysis.

6. Results

The first half of this section describes the main types of arguments displayed by the students when challenged to describe a "final state" for an infinite iterative process. In the second half we discuss the various ways in which the students employed references to other tasks (whether from this study or their previous mathematical experience), and how these references often triggered changes in the students' conceptions of completed infinite iteration.

6.1. Initial types of arguments

6.1.1 Generalizing global properties from the intermediary states to the final state

When confronted with a new task, the students attempted to define a state at infinity for the infinite iterative process in question by extracting patterns observed in certain global[4] properties

[4] Here the term "global" is used in reference to an intermediate state. A set's cardinality describes a global aspect of it, as it refers to all the elements in the set. Noting that a set contains only even numbers also refers to a global property of it. In contrast, "the set contains the number 5" is a local property as it does not make a statement regarding all the elements of the set.

of the sets obtained after finitely many steps and claiming that these patterns needed to be respected by the state at infinity.

Cardinality. Although in the tasks we used the objects produced at each intermediate step were sets, the participating students initially tended to focus on answering the question "How many?" instead of "Which elements?" when attempting to define a state at infinity for an infinite iterative process. We present two instances of this type of argument.

Episode 1. Consider the following excerpt from Tom's Pre-Test interview, during which he was working on a problem equivalent to the Original Tennis Ball Problem which also had a time index for the process (the Original 10 Marble Problem[5])

> Tom: Maybe if we have a sequence that for every term is an expression of the size…[…]. The way we defined the sequence in the end it goes to infinity…in the sense that it's monotonic increasing and it's not upbounded anywhere. So…I think there should be as many marbles in the jar as there are natural numbers. Clearly there aren't finitely many marbles in the jar at t=1 because if there were finitely many marbles that would mean our sequence a_n would converge to a particular number that would be the size of the number of marbles at t=1. But we already know that the sequence diverges. So now you know the size of the marbles is infinite.

Episode 2. Similarly, in the case of the Original Tennis Ball Problem, Max noticed that both bin A and bin B contained n balls after n steps had been performed, and for this reason claimed that both bins should have infinitely many balls after all the steps of the process had been performed.

Arguments of this type have already been documented in the literature in the context of these two specific problems (e.g. Dubinsky et al, 2005; Ely, 2007; Dubinsky et al, 2008; Mamolo & Zazkis, 2008), so our students' initial reactions to these tasks did not come as a surprise. However,

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[5] The Original 10 Marble Problem. Suppose there is a jar capable of containing infinitely many marbles and an infinite collection of marbles labeled 1, 2, 3, and so on. At time $t = 0$, marbles 1 through 10 are placed in the jar and marble 1 is taken out. At $t = 0.5$, marbles 11 through 20 are placed in the jar and marble 2 is taken out; at $t = 0.75$, marbles 21 through 30 are put in the jar and marble 3 is taken out; and in general at time $t = 1 - 0.5^n$, marbles $10n + 1$ through $10n + 10$ are placed in the jar and marble $n + 1$ is taken out. How many marbles are in the jar at time $t = 1$?

the variety of tasks used in our research allowed us to find instances of this type of argumentation in the context of other tasks as well, as described in the next episode.

Episode 3. Consider the case of the 1/2 Marble Problem, in which an infinite iterative process is defined such that the contents of a jar oscillate between {marble "1"} and {marble "2"}. The students were asked what was in the jar after all the steps had been performed. Max concluded that the jar contained exactly one marble whose label could not be determined because "you always have one marble in the jar, either 1 or 2".

The students' initial tendencies to extract a sequence of numbers (indicating the cardinalities of the sets representing the intermediary states of the process) from the problems, instead of focusing on the sequence of sets itself, might be an instance of an attempt to reduce the level of abstraction, in the sense of Hazzan (1999). Infinite numerical sequences are introduced in Pre-Algebra and Pre-Calculus courses in K-12, while the notion of the limit of an infinite sequence of numbers is treated by high school and college level Calculus courses. Mathematics majors are thus likely to have developed a familiarity with infinite numerical sequences, and there is evidence in the literature that some think of the limit of a numerical sequence as a "last element" (Mamona Downs, 2001). Thus, when students are asked to describe a final state for a sequence of sets, it is not entirely surprising that some choose to focus on an associated numerical sequence and its limit.

Generalizing other global properties from the intermediarye states to the final state. In some of the variations of the Original Tennis Ball Problem that we used, one easy way to describe the contents of the two bins (after an arbitrary finite step n) was through the means of a property shared by all the balls in a bin. For example, in the case of one of the post-test problems, after any finite step, one bin contained only balls with even labels, while the other bin contained only balls with odd labels. When such patterns were present in a task, our participants tended to claim that the observed patterns needed to hold for the final state.

Episode 4. In session 4 the students worked on the Bin Swapping Tennis Ball Problem, which was defined such that after n steps (where n is an arbitrary natural number), one bin contained only the ball labeled 1, while the other contained the balls with labels 2, 3, …, n; however, the contents of the two bins were swapped at every odd step. Both Max and Tom reasoned that because of this observed pattern in the labels on the balls found in each bin after finitely many steps, "at the end" one bin must contain the ball labeled "1" while the other bin must contain all of the other balls (with labels in N\{1}). The fact that the process described by the problem caused the contents of the bins to be swapped at every odd step was not deemed to be of great importance by the students, who accounted for it by claiming that it was not possible to determine "which bin contains what" at the end, while still maintaining that a final state existed in the form of the partition (of the initial set of balls) mentioned above.

This type of argument resembles, to some degree, what was reported by Brown, McDonald & Weller (2008). In that study, students reasoned that because $\bigcup_{k=1}^{n} P(\{1, 2, 3, ..., k\}) = P(\{1, 2, 3, ..., n\})$, it must be true that $\bigcup_{n=1}^{\infty} P(\{1, 2, 3, ..., n\}) = P(N)$. The power set form of each intermediate state (of the infinite iterative process students used to make sense of the infinite union in this problem) is a global property of that state, and the students reasoned that the final state of this process needed to be a power set as well, despite evidence that the students knew and were able to apply all the theoretical definitions necessary to make sense of $\bigcup_{n=1}^{\infty} P(\{1, 2, 3, ..., n\})$. This provides supporting evidence that the students' tendency to employ "generalizing global properties" types of arguments may not be restricted to tasks set in pseudo-real world contexts (which are likely to trigger arguments based on real-life expectations and intuitions, as suggested by Mamolo & Zazkis, 2008), but influence students' reasoning on entirely abstract tasks as well.

6.1.2. The "reaching the limit" argument

Another type of argument that the students employed repeatedly surfaced in the context of geometrical construction tasks, in which a set of points on the real line or in the two-dimensional plane was defined by an infinite iterative process. In such situations, there were several instances in which the students claimed that besides the union[6] of all the sets representing the intermediate states, the final state needed to also contain the limit points of this union, if any.

Episode 5. In session 3 the students worked on the z^n Problem, in which z is a complex number of norm strictly less than 1 (and greater than 0) and an infinite iterative process is defined such that at step n the process defines a complex number $z_n = z^n$. The students were asked whether the set of points in the plane corresponding to the set of complex numbers produced by the completed process contained the origin, (0, 0) or equivalently, whether 0 belonged to the set of complex numbers produced by the completed process. Both students claimed that the sequence of complex numbers produced by the successive steps of the process converged to 0 and acknowledged that there was no natural number n such that z^n equaled 0. However, they could not agree on an answer to the main question of the problem. It is in this context that the following dialogue took place:

> Max: If we finish the process we're at the limit. That's the only way you finish the process.
> Tom: When I think of the set produced by the process, I think of every z^n where n is a natural number…
> Max: The only way you can finish the process is if you reach the origin. 'Cause if you're not, then you're not done yet!

Tom claimed that the final state for this process is the collection of all complex numbers of the form z^n (where n is a natural number), while Max believed that the set defined by the completed process" contained all the elements in $\{z^n \mid n \in N\}$ *and* the complex number 0, which is a limit point for $\{z^n \mid n \in N\}$. Later in that session Max commented that after all the steps of the process had been

6 In all the geometrical construction tasks, the process was defined in such a way that any intermediate state was included in the subsequent one (so $S_0 \subseteq S_1 \subseteq ... \subseteq S_n \subseteq ...$).

performed, “you’re sitting on top of the origin”, which he interpreted as 0 being “reached” by the process, and thus being part of the set of points produced by the process. Max’s inclination to use a “reaching the limit” type of argument here (and in the rest of the geometrical tasks) may be explained by the fact that in the case of a convergent (infinite) series $\sum_{n=1}^{\infty} a_n$, its sum can be conceptualized as the state at infinity of an infinite iterative process that at step n, adds a_n to the already computed $\sum_{k=1}^{n-1} a_k$, and this state at infinity *is* the limit of the sequence of numbers produced by the process. Thus, it is possible that the “reaching the limit” and “cardinality” (see 6.1.1) types of arguments are both manifestations of the same phenomenon, which is that students tend to work with sequences of sets as if they were sequences of numbers.

This concludes our discussion of the main types of initial arguments displayed by the students in response to our tasks. Table 1 on page 16 provides a summary of which type of argument was initially used by each student in response to each task[7] (the tasks are listed in chronological order).

The next subsection of the paper looks at the various ways in which the students employed references across tasks and examines how such references helped the students refine their conceptions of completed infinite iteration and states at infinity.

6.2. References to other tasks/mathematical contexts

The two students in this study often made references to problems or contexts other than the problem that they were working on at the time; these references involved comparing two or more problems in terms of what the students perceived as structural similarities or differences, and

[7] The Relabeled 10 Marble Problem, the $n \to n+1$ Marble Problem, and the Writer Problem were all isomorphic to the Original Tennis Ball/10 Marble problems. Our data suggests that the students’ normative reasoning on these three problems was largely due to the fact that they were placed after the Vector Problem in the task sequence. See section 6.2.2 for more details on the Vector Problem.

Tasks	Tom	Max
The Original Tennis Ball/10 Marble Pb.	CARD	CARD
The Vector Problem	NORM	NORM
The Relabeled 10 Marble Problem	NORM	NORM
The $n \to n+1$ Marble Problem	NORM	NORM
The ½ Marble Problem	Undecided	GEN-C
The 1+1/n Marble Problem	GEN-C, LIM	GEN-C, LIM
The Writer Problem	NORM	NORM
The Midpoint Problem	LIM	Undecided
The z^n Problem	NORM	LIM
The Triangle Problem	NORM	LIM
The Lamp Problem	Undecided	GEN-O
The Bin Swapping Tennis Ball Pb.	GEN-O	GEN-O

Table 1. Students' initial responses to each task

Legend

NORM – Normative solution

GEN-C – Generalizing the cardinality pattern observed among the intermediary states to the state at infinity (as discussed in section 6.1.1)

GEN-O – Generalizing patterns other than cardinality observed among the intermediary states to the state at infinity (as discussed in section 6.1.1)

LIM – "Reaching the limit" argument (as discussed in section 6.1.2)

were used for at least three different purposes: a) to reformulate the current problem using a new context, in hope that the new context will bring additional insight; b) to revisit a previously addressed problem, when working on the current problem provided insight that put the correctness of the solution given to the previous task under question; and c) to refute or support an existing argument for the current problem.

6.2.1. Reformulating the current problem

Episode 6. Let us consider again the Original 10 Marble Problem, which Tom worked on during the Pre-Test interview. His initial response to this problem was that there were infinitely many marbles in the jar at t=1, as mentioned in section 6.1.1. After the interviewer asked him whether he could name a specific marble found in the jar at t=1 and he started being troubled by the fact that he could not, he proposed the following:

> Tom: I'm thinking right now of an analogy. When I took Math 300… you know the infinity hotel? This is sort of like if you have rooms 1 through 10, and you move 10 people in there, put 1 person in each room, then person in room 1 moves out, right? So you just redefine the question in terms of hotel rooms. […] It makes me think of a wave – people are moving down the corridor of the hotel… but it's getting bigger as it slides across. If you have this increasing wave going down, that would be as you're approaching 1 [t=1], not when you're at 1. Cuz once you get to 1, the wave would be stopped. Because you're done with your operations, you're not performing anymore. So… it's sort of like you have this infinity…'cause the wave is increasing… so if you get to 1, then at that point you have infinitely many people in the hotel. But that infinity of people you would never find if you walked down the corridor. For any room, that person would have been taken out.

While Tom's reformulation of the Original 10 Marble Problem using the context of Hilbert's Hotel did not help him to entirely resolve the cognitive conflict he was experiencing, it did provide a new way for him to explain his reasoning. It also provided a context that potentially draws the solver's attention more towards the "which ones" question: the "sliding wave" metaphor stressed the fact that although the number of the people in the hotel was constantly increasing, they were also moving further and further away from an imaginary "beginning" of the hotel; with the ever-increasing pile of marbles, it may be easier to overlook the fact that the collection of marbles in the jar also "slides along N".

Note that in this case, the problem referenced by the student was not part of the task sequence used in this study, but one encountered in a prior mathematics class. Other instances in which the students reformulated the problem at hand involved using real-life contexts (e.g, comparing the situation in the Original 10 Marble Problem with a savings account in which one repeatedly deposits $10 and takes out $1 and at some point is left with nothing), or entirely abstract terms (e.g., claiming that the 1/2 Marble Problem was equivalent to asking what the limit of the numerical sequence "1, 2, 1, 2, 1, 2,…" was).

6.2.2. Revisiting a previously addressed problem from the task sequence

Episode 7. The Vector Problem (the first task in Session 1) was designed with the purpose of having students focus on the “which ones” question mentioned earlier. As pre-test data as well as previous studies (e.g. Mamolo & Zazkis, 2008) suggested that students’ real-life expectations and intuitions often triggered non-normative types of arguments, this task was formulated in more abstract terms than the Original 10 Marble/Tennis Ball problems.

The Vector Problem. Let $v = (1, 0, 0, ...) \in N^N$. You are going to “edit” this vector step by step.

- Step 1: $v = (0, 1, 2, 0, 0, ...)$
- Step 2: $v = (0, 0, 1, 2, 3, 0, 0, ...)$
- Step 3: $v = (0, 0, 0, 1, 2, 3, 4, 0, 0 ...)$

…………………………………..

This process is continued ad infinitum. Now assume ALL steps have been completed. Describe v at this point.

Upon starting working on the vector problem, within seconds the students commented on its similarity to the Pre-test marble/tennis ball problems and the Hilbert’s hotel formulation of the marble problem, noting that the string of non-zero entries “moves across the vector” in a manner similar to the wave of people from the hotel formulation of the Original 10 Marble Problem. Continuing to work on the vector problem, Tom commented that “for any entry, at some point it’s going to go to 0 and stay there, so if you’re done with your process it’s just going to be the 0 vector”. Max wondered momentarily whether the “final vector” wouldn’t contain “all the natural numbers between the zeros”, but then proceeded to explain that cannot be the case as there would be no specific position in the vector at which the string of natural numbers could start. Having agreed on the “0 vector” answer to the Vector Problem, the two students revisited the Original 10 Marble Problem:

> Max: If that’s true, then all the marbles are removed from the jar would be the right answer. […] ‘Cause you’re removing them 1, then 2, then 3, eventually you would remove them all if you finish the process. You couldn’t say any numbers that are in the jar, ‘cause you’re going to exhaust all the numbers. [The interviewer points out that

during the pre-test, not being able to name any marbles in the jar did not mean to him that the jar was empty.] Yeah, but I kind of changed my mind. [I: Why?] 'Cause even if you say that number is there at this step, at the next step it might not be there so if you do finish the process, there's not going to be nothing there. There can't be any numbers left, any natural numbers, so that can't happen.
Tom: If somebody asked me to write an argument for why there were no marbles in the jar at t=1 I think I could do it.[…] Whereas I don't think I could come up with one for why it should be infinity, because I have no idea where they are, because they're not in the rooms of the hotel.[…] My first intuition was that there were infinitely many marbles, but where are they, I mean which ones are they**,** I think would be the question. Because if you say marble 2371 is part of the infinity, no it's not because it's been removed. You can say that about any marble. But I still have this intuition that if it's getting larger there should be infinitely many…

The abstractness of the Vector problem, together with a formulation that had students focus on individual elements/positions as opposed to the cardinality of a set, appears to have helped the students move towards a more logical approach to this set of problems and resolve the cognitive conflict by choosing the solution for which they felt they could write a convincing argument, while at the same time acknowledging that the "logical" argument seemed counterintuitive.

6.2.3 Referencing another problem to refute or support an existing argument

Episode 8. As already discussed in section 6.1.2, Max and Tom disagreed on the z^n Problem (with respect to whether the complex number 0, or equivalently the (0, 0) point, belonged to the set of points defined by the completed process): Max strongly believed that it did, while Tom argued the opposite. It is in this context that the following exchange took place:

> Max: then how can you tell me this equals 1 [referring to 1+1/n Marble Problem[8]], and you're trying to tell me that 0 [does not belong to the set]?
> Tom: Yeah, it is kind of inconsistent reasoning. Because when I gave that example with the balls, I was kind of thinking…
> Max: When we had this one, we ended up saying at the end we had 1 on the ball. Well, I say we have 0 right now. We're AT the origin!
> Tom: Yeah, I mean, yeah, it depends…if we use that reasoning, which I recall being what we agreed on, then I guess you'd have to say that the origin, the zero vector is in your set.

[8] The 1+1/n Marble Problem was formulated by Tom in Session 1. It involves an ω-type set of marbles in which the n^{th} marble was labeled with "1+1/n" (so the sequence of labels was 1+1/1, 1+1/2, 1+1/3, …). The infinite process in this problem involved starting with the first marble and placing it in a jar, then replacing it with the 2^{nd} one, which at its turn is replaced by the 3^{rd} one, and so on. Both students strongly believed that after all steps had been completed, the jar contained a marble labeled "1".

> But on the other hand I am not really sure I agree with that [the solution given to the 1+1/n problem] anymore.

Interestingly, while both students acknowledged some type of structural similarity between these two problems, they reacted in different ways to the acknowledgement: Max believed even more strongly in his "0 belongs to the final set" position on the z^n Problem, while Tom started questioning the validity of the previously agreed upon solution for the 1+1/n Marble Problem and continued to argue for "0 not part of the final set" in the case of the z^n Problem. Although in this episode the two students reacted differently to the reference made to another task, it is important to note that both Max and Tom displayed concern for reasoning in a consistent manner across tasks.

6.3 Charting the flow of ideas

As the data discussed in 6.2 suggests, the students' progress through the task sequence designed for this study was far from linear. There were numerous instances when previously addressed tasks were referenced or revisited, and problems or mathematical contexts from the students' mathematical background were brought up as the students strived to develop a conception of completed infinite iteration that was, in their view, consistent across tasks. Figure 1 contains a representation of our view of the "web of connections" built by Max and Tom through the course of the pre-test and the first five of the problem-solving sessions. As this diagram indicates, the "web of connections" entangles all but one of the problems in our task sequence. Back-references to five of the tasks led to solution changes (for the referenced tasks) on the part of either one or both students.

In the final session, the first author adopted a more teacher-like role with the purpose of engaging the students in reflection with respect to the strategies they used in determining whether a task was similar or different from another one, and consequently whether the reasoning used for one task was applicable to another. This intervention, as well as the two students' performance on the post-test, is discussed in the next section.

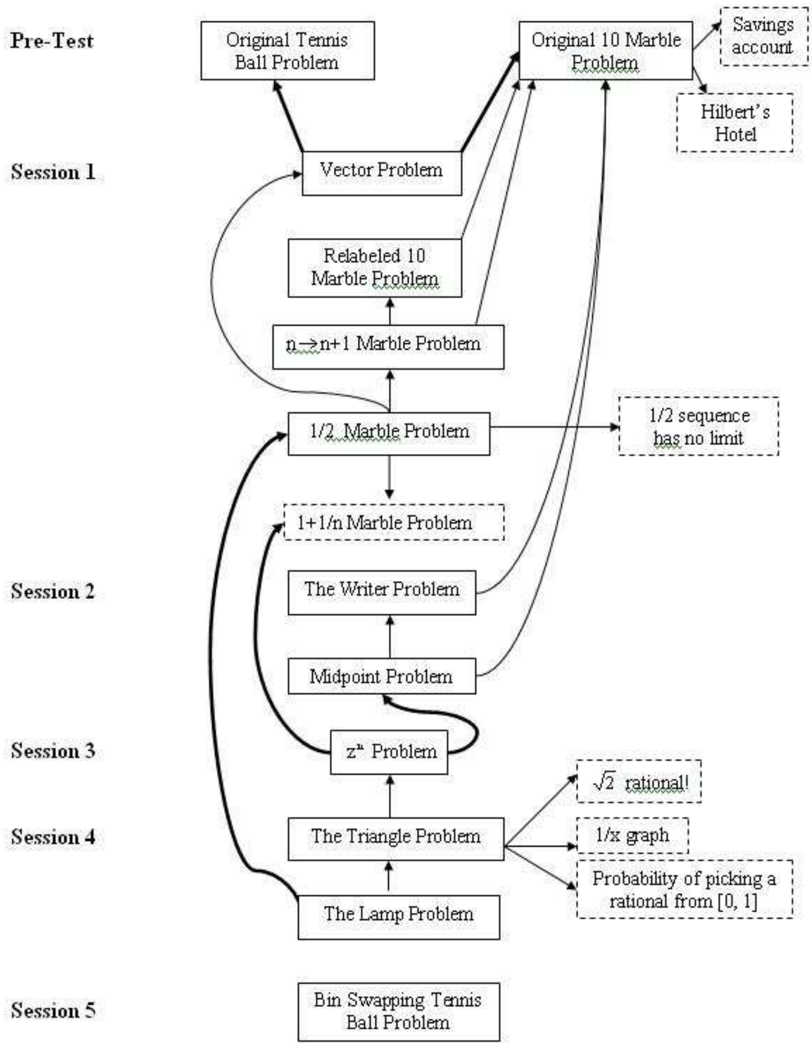

Figure 1. Student references across tasks

Legend
$x \rightarrow y$: While discussing x, problem y was referenced
Bold arrow: the reference caused one or both of the students to change their solution to the referenced problem
Dashed line box: the problem in the box was proposed by one of the students

6.4. Our intervention and post-test results

At the end of five problem-solving sessions, both students displayed changes in their reasoning about infinite iteration, compared to their initial responses to our tasks. In the case of problems such as the Original 10 Marble Problem and the Original Tennis Ball Problem, both students had moved from non-normative to normative solutions. With respect to the geometrical construction problems, Tom had settled on normative solutions after originally being the one to propose the “reaching the limit” type of argument, while Max was increasingly drawn towards “reaching the limit” arguments as the study progressed. Finally, on “no final state” tasks both students exhibited “generalizing global properties” tendencies, although Tom displayed more discomfort with them than Max.

In the final session, the first author drew the students’ attention back to the Original Tennis Ball Problem (which they had solved normatively by the end of the first session) and pointed out that the solution they had agreed upon implicitly used the “continuity assumptions” described in section 2. The students were then encouraged to revisit the other problems in the task sequence and investigate whether the final state they had defined in each case was consistent with these assumptions.

The ensuing reflection on the task sequence with the continuity assumptions in mind affected the students’ conceptions of infinite iteration differently. For Tom, it led to normative solutions to all the tasks in the sequence. In contrast, Max accepted the continuity assumptions as sensible for the subclass of problems isomorphic to the Original Tennis Ball Problem, but claimed that they were “not applicable” or “incomplete” in the case of each task where considering them seemed to point to a different answer than the one Max believed to be correct. Throughout this final discussion he continued to display a concern for consistent reasoning across tasks, but was unable to formulate a set of assumptions that would produce a set of final states matching the ones he claimed were “the correct ones” for the whole task sequence.

Table 2 summarizes each student’s final position on each task used in the problem solving sessions, as well as on the Post-Test tasks (“$x \approx y$” means “problem x is isomorphic to problem y”).

Tasks	Tom	Max
The Original Tennis Ball/10 Marble Pb.	NORM	NORM
The Vector Problem	NORM	NORM
The Relabeled 10 Marble Problem	NORM	NORM
The $n \to n+1$ Marble Problem	NORM	NORM
The ½ Marble Problem	NORM	GEN-C
The 1+1/n Marble Problem	NORM	GEN-C, LIM
The Writer Problem	NORM	NORM
The Midpoint Problem	NORM	LIM
The z^n Problem	NORM	LIM
The Triangle Problem	NORM	LIM
The Lamp Problem	NORM	GEN-O
The Bin Swapping Tennis Ball Pb.	NORM	GEN-O
Post-Test Problems		
Pb. ≈ The Original Tennis Ball Pb.	NORM	NORM
Pb. ≈ The Bin Swapping Tennis Ball Pb.	NORM	GEN-O
Pb. ≈ The Midpoint Problem	NORM	LIM

Table 2. Students' final solutions on each task (including post-test tasks)

7. Discussion

The main findings of this study are:

- Students' initial metaphors for the final state of an infinite iterative process were strongly influenced by generalizing global properties from intermediary states to the final state (for the ball/marble-manipulation problems) and the "reaching the limit" approach in the case of the geometrical construction problems
- References across tasks were used spontaneously by the students for a variety of purposes and at times led to changes in the students' reasoning on one or all of the tasks involved in the comparison
- Each student displayed a concern for reasoning consistently across tasks and constantly adjusted his arguments to the given tasks to achieve what he perceived to be consistent reasoning in light of the structural similarities and differences identified among the tasks

As noted earlier, our students' initial responses to the Original 10 Marble/Tennis Ball problems (episodes 1 and 2) are in agreement with what has been reported by other studies of college students' reasoning on these tasks (Dubinsky et al, 2008; Mamolo & Zazkis, 2008). Additionally, our study examined students' reasoning on "no final state" tasks and geometrical construction tasks, and identified the main types of arguments employed by the students in these contexts (episodes 3-5).

Our analysis suggests that the initial metaphors created by the students for states at infinity are highly context dependent, which echoes the findings reported by Tirosh & Tsamir (1996) regarding the effect of task context and task representation on student reasoning (in the context of tasks involving size comparison of infinite sets). As discussed in episode 7, the Vector Problem led the students to reason in a normative manner without any external intervention, while the Original Tennis Ball Problem (an isomorphic problem) did not; additionally, geometrical contexts seemed to trigger student conceptions of the state at infinity that were greatly influenced by the limiting behavior of the points constructed by the process (episode 5). This data supports the conjecture that students *do have* the ability to reason normatively about states at infinity when the task context or task representation evoke the right schema, normative solutions that can then be used as "building blocks" in the development of normative conceptions of infinite iteration at large.

Considering our students' conceptual journeys through the course of the study, we propose that one way in which students can learn to reason normatively about infinite iteration is to refine their initial conceptions of the state at infinity by working through a complex class of related infinite iteration tasks. The students in our study did so by spontaneously employing perceived similarities or differences among the presented tasks to constantly adjust the types of arguments used for each task in order to maintain consistent reasoning across tasks (see episodes 6-8); this, in turn, changed the way the students perceived structural similarity among tasks, which is in agreement with Wagner's (2006) claim that the gradual formulation of a set of abstract principles applicable to a class of problems is intimately connected with the process of determining structural similarity. This learning environment, coupled with the first author's initiative to focus the students' attention on a normative type of reasoning and encouraging them to relate it to the other types of arguments displayed by the students, proved to be extremely effective in Tom's case, and partially effective in Max's case (see Table 2, in comparison to Table 1).

Regarding Lakoff and Nunez's (2000) basic metaphor of infinity, we acknowledge that the initial types of reasoning displayed by the students in our study produced topical (task dependent) metaphorical final states that were grounded in embodied experience (as described in episodes 1-3). However, a good number of these topical metaphors corresponded to non-normative final states, which is why we agree with Dubinsky et al. (2008) and Brown et al. (2008) that BMI does not accurately explain how one may reach normative conceptions of infinite iteration. Our data suggests that the refinement of the collection of topical metaphors into a global, normative one can be facilitated by providing the students with an "anchor" in the form of a task that did trigger normative reasoning in students, and encouraging them to relate and refine their metaphorical final states to other tasks in relation to the "anchor" line of reasoning (as described in section 6.4).

Lastly, our findings suggest that by focusing on analyzing student reasoning on infinite iteration in the context of only one task, empirical studies conducted from an APOS perspective (such as Brown et al. 2008) risk overlooking a possible learning path towards normative understandings such as the one proposed by this study. Furthermore, explaining a student's non-normative reasoning on an infinite iteration task by the lack of a certain type of mental construction that APOS posits to be necessary for a normative conception of infinite iteration seems to suggest a deficit approach that risks to ignore what students do "bring to the table" – initial topical metaphors of final states that, normative or not, represent the students' current view of what a state at infinity is and should be taken into consideration and built upon by any instructional intervention, not avoided for fear of future misconceptions.

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