Folding Back and Connecting to Historical Mathematical Concepts

Kelli M. Slaten
University of North Carolina Wilmington
slatenk@uncw.edu

Abstract: The purpose of this preliminary study is to investigate and characterize undergraduate students' self-awareness of their growth of mathematical understanding within a history of mathematics course. The study utilizes a key feature of the Pirie-Kieren Dynamical Theory for the Growth of Mathematical Understanding (1994b), the process of *folding back,* in order to better understand what students learn from the historical development of mathematical concepts they study as undergraduate mathematics majors. Students enrolled in a history of mathematics course were prompted to write a reflection about their mathematical learning from the course. These reflections were analyzed for emergent themes as evidence that the process of folding back to their inner levels of knowledge within the context of historical development provided opportunities to deepen and extend their understanding of mathematical concepts.

Background

Undergraduate mathematics and mathematics education students experience much of the mathematics they learn as a formalized body of knowledge. They are seldom afforded opportunities to make connections or delve into concepts beyond algorithmic applications. Presenting mathematics as strictly based on axioms, theorems, and proofs assumes the logical structure of the presentation is sufficient for learning mathematics (Tzanakis & Arcavi, 2000). However, teaching mathematical concepts in such a formalized manner can result in disjoint, incomplete understanding (Katz, 2007; Martin, 2000; Pirie & Kieren, 1994a; Sfard, 1991). Avenues of instruction that provide opportunities for students to deepen their understanding of mathematical concepts are needed in order for students to gain a more complete knowledge of mathematics. The goal of this preliminary study is to investigate what undergraduate students learn from the historical development of mathematics based on their self-awareness of the growth of their mathematical understanding.

The use of the history of mathematics in the teaching and learning of mathematics is a not new idea, but there is a growing body of research concerning the role of the history of mathematics in mathematics education (Fauvel & van Maanen, 2000). For instance, studying mathematical concepts within the context of their historical development has been found to deepen student understanding, particularly when introducing a concept (Bagni, 2005; Katz, 2007; Liu, 2003; Man-Keung, 1997). Rather than introduce mathematical concepts in a strictly formalized manner, incorporating a historical perspective can help students better understand those mathematical ideas (Bartolini Bussi & Sierpinksa, 2000; Otte, 2007; Swetz, 1995; Tzanakis and Arcavi, 2000).

Many undergraduate students, particularly mathematics majors, struggle to understand the abstract mathematical concepts they are expected to learn in their undergraduate mathematics courses. Furthermore, the textbooks which accompany mathematics courses are a common resource of study for undergraduate students. Yet modern textbooks generally present the finished product of a mathematical idea which may have taken hundreds of years to develop (Liu, 2003). To help bridge this gap, the history of mathematics "can serve as a major pedagogical tool for the teaching of higher mathematics" (Swetz, 1995, p. 103).

Concerning the use of the history of mathematics as a pedagogical tool, there exists a plethora of classroom resources for teachers (i.e., Katz & Michalowicz, 2004; NCTM, 1998; Swetz, Fauvel, Bekken, Johansson, & Katz, 1995). However, there is currently a limited amount of research focused on *how* the historical development of mathematics contributes to students' growth of mathematical understanding, particularly when the content has already been presented to students through undergraduate coursework. The study of the historical development of these concepts provides opportunities for students to increase the depth of their mathematical understanding. This study is thus guided by the question: How does the process of folding back to explore mathematical concepts within the context of historical development contribute to undergraduate students' growth in mathematical understanding?

Theoretical Framework

The Pirie-Kieren Dynamical Theory for the Growth of Mathematical Understanding (1994b) serves as the overarching framework for this study. In particular, the feature of *folding back* guided the data collection and analysis of students' written reflections. The Pirie-Kieren theory describes eight embedded levels of understanding based on cognitive changes that occur during the processes of learning mathematics. These processes of learning are dynamic, recursive and non-linear. Folding back is a key feature of the theory and is an activity vital to the growth of mathematical understanding (Martin & Pirie, 1998; Martin, 2000; Pirie & Kieren, 1994b).

When a student folds back to an inner, localized level of understanding, not only is the current, more formalized level of understanding extended, but the inner level has also changed because it has been informed by that outer level. This process "thickens" the inner level of understanding and thus, a student gains a broader and deeper understanding of a concept. Furthermore, the process of folding back can allow for the reconstruction of inner levels of understanding (Pirie & Kieren, 1994b). Through the study of the historical development of mathematics, students are provided opportunities to fold back and thicken their inner levels of understanding. This process of folding back allows the inner levels of understanding to be informed by their outer levels of understanding resulting from their formalized experiences with the mathematics.

According to Martin and Pirie (1998), effective folding back is more likely when a student is given the opportunity and time to work with their inner levels of understanding in order to enrich it. A student who needs to fold back must re-evaluate his or her current knowledge instead of just recalling memorized ideas (Martin, 2001). By providing undergraduate mathematics majors the opportunity to fold back and learn about the historical development of concepts they know as general, abstract mathematical concepts, they are given the opportunity to gain a deeper understanding of these concepts. Indeed, by encouraging students to fold back via the historical development of mathematics, the pedagogy of mathematics education is further informed:

> Learning is not a uniformly continuous process…when we are aware of where in our history the major shifts occurred, we will be more sensitive to the students' problems of trying to grasp the material. If we can understand how mathematicians accommodated a radical idea when it was first developed, we can better plan the strategy to help the student learn and accommodate it" (Lehmann, 1995, p. 161).

In other words, the cognitive difficulties encountered during the development of mathematical ideas can inform and guide the pedagogical use of the history of mathematics and subsequent opportunities for the growth of students' mathematical understanding.

Methodology

The participants of this study are 14 junior and senior level undergraduate students enrolled in a semester-long course, The Historical Development of Mathematics. The course is a requirement for secondary mathematics education majors, but is an elective course for mathematics majors. Nine of the participants were secondary mathematics education majors and the remaining five were mathematics majors. Primary data were taken from students' written reflections collected during the semester. For the purposes of this study, the final reflection assignment was analyzed for emergent themes concerning students' thinking about and awareness of the concepts they learned during the course. The use of reflection supports the idea that the processes of reorganizing one's own understanding during the process of folding back require a degree of self-awareness (Martin, 2008). The emergent patterns of folding back are framed through students' awareness of their own growth in mathematical understanding. These written reflections were completed at the end of the course and were based on the following prompt:

> At the beginning of the course, you were asked why you think it is important to learn the history of mathematics. The general consensus was that students would learn mathematics did not come to us as a formulated and complete product. Write a paragraph reflecting and expanding on your ideas now that you've finished the course. What other aspects of the history of mathematics contributed to your understanding of the mathematics covered in the course?

Results

Students' written reflections were analyzed for emergent themes and patterns about their self-awareness of their growth of mathematical understanding. The anticipated result of students reflecting on their growth of content-specific understanding did not surface as often as expected. Instead, the theme of learning based on the historical development of mathematics in general was elaborated through the students' reflections. Within this theme of historical development, two broad categories were consistently found throughout the data: people and content.

People

Students in this study explained how they gained a deeper understanding of mathematics through learning about the thought processes of the mathematicians who developed these concepts. Several examples of student responses follow:

> **S1**: Now that I understand how some of these findings were thought about in the beginning is really helpful.

> **S2**: Seeing how these people thought about math help me understand why they did things the way that they did.

> **S3**: If we can look at the thought process that many mathematicians have had over the years we can better understand how to do things and what does not work.

Content

Students also explained their mathematical learning and how this learning fit into their overall understanding of mathematics:

> **S4**: Learning about how mathematical concepts were developed through history has also deepened my personal understanding of these concepts.

> **S5**: It also gave me a chance to see where it all came from, outside of a formal generalized proof.

> **S6**: This course has definitely linked all the seemingly random fields and theorems of math that I have been bombarded with over the years.

> **S7**: I was able to use what I learned in class and apply it to some of my other classes.

> **S8**: I did not know what Euclidean geometry was and that it was based on a set of axioms and definitions.

Discussion

The purpose of this preliminary study is to provide insight into the ways undergraduate mathematics majors grow in their understanding of mathematical concepts when explored within the context of the history of mathematics. The analysis of the growth of understanding is grounded in the students' written reflections based on their self-awareness of their own knowledge. When asked to reflect on their growth of understanding from studying the historical development of mathematics, the students in this study addressed what they learned from historical mathematicians' thought processes as well as how their overall knowledge of mathematics was deepened and extended. Future research is planned through the continued collection of data and subsequent analysis to refine the emergent themes found in this study. The goal of elaborating and using the concept of folding back is to discover the actions that lead to students' continued growth in mathematical understanding (Martin, 2008). While research has not yet found what those actions might be, the results of this study reveal the potential of using the historical development of mathematics as an action to deepen and extend the formalized mathematical knowledge students learn as undergraduate mathematics majors.

References

Bagni, G. T. (2005). Infinite series form history to mathematics education [Electronic Version]. *International Journal for Mathematics Teaching and Learning*. Retrieved October 17, 2008 from http://www.cimt.plymouth.ac.uk/journal/.

Bartolini Bussi, M. G., & Sierpinska, A. (2000). The relevance of historical studies in designing and analysing classroom activities. In J. Fauvel & J. van Maanen (Eds.), *History in mathematics education: The ICMI study* (pp. 154-161). Dordrecht, The Netherlands: Kluwer.

Fauvel, J., & Maanen, J. v. (Eds.). (2000). *History in mathematics education: The ICMI study*. Dordrecht, The Netherlands: Kluwer.

Katz, V. (2007). Stages in the history of algebra with implications for teaching. *Educational Studies in Mathematics, 66*, 185-201.

Katz, V. & Michalowicz, K. (Eds.). (2004). *Historical modules for the teaching and learning of mathematics*. Washington, DC: Mathematical Association of America.

Lehmann, J. P. (1995). Converging concepts of series: learning from history. In F. Swetz, J. Fauvel, O. Bekken, B. Johansson & V. Katz (Eds.), *Learning from the masters* (pp. 161-180). Washington, DC: Mathematical Association of America.

Liu, P. H. (2003). Do teachers need to incorporate the history of mathematics in their teaching? *Mathematics Teacher*, *96*(6), 416-421.

Man-Keung, S. (1997). The ABCD of using history of mathematics in the (undergraduate) classroom. *Bulletin of Hong Kong Mathematical Society, 1*(1), 143-154.

Martin, L. C. (2000). *Folding back and growing mathematical understanding.* Paper presented at the 24th Annual Canadian Mathematics Education Study Group, Montreal, Canada.

Martin, L. C. (2001). *Growing mathematical understanding: Teaching and learning as listening and sharing.* Paper presented at the 23rd Annual Meeting of the North American Chapter of the International Group for the Psychology of Mathematics Education, Snowbird, UT.

Martin, L. C. (2008). Folding back and they dynamical growth of mathematical understanding: Elaborating the Pirie-Kieren Theory. *Journal of Mathematical Behavior, 27,* 64-85.

Martin, L. C, & Pirie, S. (1998). *"She says we've got to think back": Effective folding back for growth in understanding.* Paper presented at the 22nd Annual Meeting of the International Group for the Psychology of Mathematics Education, Stellenbosh, South Africa.

National Council of Teachers of Mathematics. (1998). *Historical topics for the mathematics classroom.* Reston, VA: National Council of Teachers of Mathematics.

Otte, M. (2007). Mathematical history, philosophy and education. *Educational Studies in Mathematics, 66,* 243-255.

Pirie, S., & Kieren, T. (1994a). Beyond metaphor: Formalising in mathematical understanding within constructivist environments. *For the Learning of Mathematics, 14*(1), 39-43.

Pirie, S., & Kieren, T. (1994b). Growth in mathematical understanding: How can we characterize it and how can we represent it? *Educational Studies in Mathematics, 26*, 165-190.

Sfard, A. (1991). On the dual nature of mathematical conceptions: Reflections on processes and objects as different sides of the same coin. *Educational Studies in Mathematics, 22*, 1-36.

Swetz, F., Fauvel, J., Bekken, O., Johansson, B., & Katz, V. (Eds.). (1995). *Learn from the masters.* Washington, DC: Mathematical Association of America.

Tzanakis, C. & Arcavi, A. (2000). Integrating history of mathematics in the classroom: An analytic survey. In J. Fauvel & J.v. Maanen (Eds.), *History in mathematics education: The ICMI study*. Dordrecht, The Netherlands: Kluwer.