Finding a Suitable Alternative to a Potential Infinity Perspective: A Watershed Moment in the Reinvention of the Formal Definition of Limit

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Introduction

The concept of limit is fundamental to the study of calculus and to introductory analysis; this has been noted by many researchers (Bezuidenhout, 2001; Cornu, 1991; Dorier, 1995). Cornu (ibid) notes that limit "holds a central position which permeates the whole of mathematical analysis – as a foundation of the theory of approximation, of continuity, and of differential and integral calculus" (p.153). Indeed, limits arise in these and many other mathematical contexts, including the convergence and divergence of infinite sequences and series, applications related to determining measurable quantities of geometric figures, and mathematical descriptions of behavior of real-valued functions.

The formal definition of limit is foundational as students proceed to more formal, rigorous mathematics. The vast majority of topics encountered in an undergraduate analysis course, where students study the theoretical foundations of calculus, are built upon the formal definition of limit. Continuity, derivatives, integrals, and Taylor series approximations are just a few of the topics in an analysis course for which limit serves as an indispensable component. Further, the formal definition of limit often serves as a starting point for developing facility with formal proof techniques, making sense of rigorous, formally-quantified mathematical statements, and transitioning to abstract thinking. Tall (1992) notes that the ability to think abstractly is a prerequisite for the transition to advanced mathematical thinking, and Ervynck (1981) cites the definition of limit as an opportunity for students to develop the ability to think abstractly. For all of these reasons, the limit concept holds an important place in pedagogical considerations and as an object of research in mathematics education.

Literature Review

Though there are numerous ways to categorize the existing research on limit, I have chosen to separate the literature into two broad categories – *research on students' informal notions of limit* and *research on students' reasoning about limit in the context of the formal definition*. I define the former category as research that seeks to describe how students reason about the limit concept while focusing on the process of finding limit candidates via algebraic, graphical, or tabular methods. Such research does not have, as its focus, the ways in which students reason about the formal definition of limit. By *formal limit research*, I mean research that is focused on how students reason about or understand the formal definition of limit. The majority of existing research on students' understanding of limit consists of the former. These studies have concluded that informal treatments of limit often result in students developing misconceptions based on their interpretation of colloquial language used in the classroom to describe limits (Ferrini-Mundy & Graham, 1994; Monaghan, 1991; Tall, 1992; Williams, 1991). Other studies have shown that informal methods can also result in an over-reliance on simplistic examples used initially to introduce the concept (Cornu, 1991; Davis & Vinner, 1986; Tall & Vinner, 1981; Tall, ibid). For instance, in an effort to simplify the initial study of limits, students are often presented with continuous functions whose limit can be computed by simply evaluating the function at the limiting value. As a result, students often conclude that the limit of a function is simply the function value at the point of interest (Bezuidenhout, 2001; Cottrill et al., 1996; Davis & Vinner, 1986; Tall, 1992; Williams, 1991). Much of the literature emphasizes what students *do not* know about the concept of limit. However, some studies attempt to describe what students *do* understand about limits (Ferrini-Mundy & Graham, 1994; Oehrtman, 2003; Oehrtman, 2004; Williams, 2001). Rather than viewing student thinking from a deficient perspective, these researchers describe initial student thinking as entailing natural informal conceptions that might facilitate the development of strong conceptual understanding. For instance, as a departure from the misconceptions research discussed above, Ferrini-Mundy and Graham (1994) conducted clinical interviews in hopes of describing college calculus students' understandings of function, limit, continuity, derivative, and definite integral. Central to Ferrini-Mundy and Graham's research is the assumption that students make sense of tasks based on their own experiences.

> [S]tudents' constructions are rational and subject to explanation. We view the student's constructions not as errors or misconceptions to be eradicated and replaced with the 'correct' and publicly shared interpretations of major ideas, but rather as *expected phenomena* that are natural in the learning process (p.32). [italics added]

Ferrini-Mundy and Graham provide evidence that students' interpretations of informal language that is traditionally used may serve as a tool for developing eventual understanding. The idea that students' informal, and perhaps naïve, understandings should not be eradicated, but rather used as leverage for developing more sophisticated mathematics understandings was central to my research. While students' understanding of limit may not perfectly resemble the formal, mathematical understanding held by experts, it is my belief that embedded in their informal viewpoints are valuable ideas and constructs upon which new meanings can be built during the evolution of their understanding of the concept.

In contrast to informal limit research, relatively few studies have looked directly at how students reason about or understand the formal definition of limit. While some have suggested pedagogical approaches for *teaching* students about the formal definition of limit (Gass, 1992; Steinmetz, 1977), very little research exists regarding how students come to *understand* and *reason about* the formal definition. Existing research to date suggests that students, for a variety of reasons, struggle to understand and reason about the formal definition of limit (Cornu, 1991; Cottrill et al., 1996; Vinner, 1991; Gass, ibid; Larsen, 2001; Tall, 1992; Tall & Vinner, 1981; Williams, 1991), which is rich with quantification and notation. Cornu (ibid) suggests that the formal definition of limit is too cognitively sophisticated for first semester calculus students. Vinner's study (ibid) substantiates Cornu's claim, reporting that out of fifteen mathematically gifted calculus students who had spent significant time with the limit concept, only one was able to provide a formal definition for limit that might indicate "reasonably deep understanding of the concept" (p.78), and for this single student, the universal condition on ε was not explicit. Similarly, none of the students in the study conducted by Cottrill et al. (ibid) demonstrated the ability to progress to a point of reasoning formally about the limit concept.

Some scholars (Dorier, 1995; Tall, 1992) suggest that introductory calculus students' difficulties with the formal definition of limit might be attributable to an untimely introduction. For instance, Tall reports that formal definitions are not appropriate as cognitive tools for developing conceptual understanding:

> [F]ormal definitions of mathematics…are less appropriate as cognitive roots for curriculum development. Their subtlety and generality are too great for the growing mind to accommodate all at once without a high risk of conflict caused by inadvertent regularities in the particular experiences encountered (p.508).

Dorier (ibid) points out that historically "less formalized tools were used to solve most of the problems [related to limits], while the 'ε-δ-definition' was conceived for solving more sophisticated problems and for unifying all of them" (p.177), yet at the outset of calculus and introductory analysis, students likely have difficulty understanding the importance of a definition designed to unify problems they have yet to encounter. The current research, then, suggests that the formal definition of limit may not be an appropriate starting point from which to build intuitive understanding about limit.

Regardless of when students should optimally encounter the formal definition of limit, the message seems clear – the formal definition of limit is difficult for students to understand. What is less apparent from the bulk of the literature, however, is how students might come to reason coherently about this difficult concept. Indeed, to date, very few studies (Cottrill et al., 1996; Larsen, 2001) have attempted to model such reasoning. I discuss these studies below, as they have had a profound impact on the development of my own research.

Cottrill et al. (1996) provide a genetic decomposition of how students might reason about the limit concept. This genetic decomposition describes the process a student might experience as he or she constructs a formal understanding of limit. Prior to collecting data, Cottrill et al. constructed an initial genetic decomposition for how students may come to understand the limit concept. This initial framework was subsequently revised based on analysis of the data collected in their study. Most notably, data analysis supported a more precise articulation of the initial steps in their seven-step decomposition, seen below:

1. The action of evaluating f at a single point x that is considered to be close to, or even equal to, a.
2. The action of evaluating the function f at a few points, each successive point closer to a than was the previous point.
3. Construction of a coordinated schema as follows.
   (a) Interiorization of the action of Step 2 to construct a domain process in which x approaches a.
   (b) Construction of a range process in which y approaches L.
   (c) Coordination of (a), (b) via f. That is, the function f is applied to the process of x approaching a to obtain the process of $f(x)$ approaching L.
4. Perform actions on the limit concept by talking about, for example, limits of combinations of functions. In this way, the schema of Step 3 is encapsulated to become an object.
5. Reconstruct the processes of Step 3(c) in terms of intervals and inequalities. This is done by introducing numerical estimates of the closeness of approach, in symbols, $0 < |x - a| < \delta$ and $|f(x) - L| < \varepsilon$.
6. Apply a quantification schema to connect the reconstructed process of the previous step to obtain the formal definition of limit.
7. A completed ε-δ conception applied to a specific situation. (Cottrill et al., ibid).

While the genetic decomposition offered by Cottrill et al. provides important insight into how students reason informally about limit, it remains unclear how students might develop the type of formal understanding described in the latter steps of the decomposition. The majority of Cottrill et al.'s analysis focused on students' reasoning at the first three steps in the genetic decomposition. Unfortunately, there was a lack of evidence in the study of students' thinking evolving to the point of having a formal conceptual understanding of limit.

The genetic decomposition suggested by Cottrill et al. (1996) served as a useful starting point for my study. Their research suggests that to develop a formal understanding of limit, one must merely formalize one's informal notions of the concept. In the decomposition outlined above, doing so amounts to formalizing the first three steps, specifically by reconstructing the coordinated schema described in Step 3c in terms of intervals and inequalities. I argue however, that the formalization process is not so straightforward – formal understanding does require one to think in terms of intervals and inequalities, but the transition to formal thinking is not merely a reconstruction of what is described in the first three stages of the genetic decomposition. Research by Larsen (2001) substantiates this opinion. Most students in Larsen's study did not make connections between their formal understandings and the rest of their concept image (Vinner, 1991), which was comprised mostly of informal conceptions described in the first three steps of the genetic decomposition. Larsen suggests that "the formal definition is structurally different from the dynamic conception as described by the first four steps of the genetic decomposition," thus making it "unlikely that a student could successfully interpret the syntax in terms of their dynamic conception" (p.29). In light of Larsen's findings, I offer the following distinction between informal and formal understanding of limit. In informal treatments of limit, the goal is generally to ***find*** a candidate for the limit. Formal understanding, on the other hand, typically addresses how one might ***validate*** the choice of a candidate. Finding and validating are two different processes[1]. In calculus courses, students are taught a variety of strategies for finding candidates for limits – direct substitution, algebraic manipulation, and tabular and graphical inspection. However, none of these satisfy the formal definition's requirement of validation. Research (most notably Cottrill et al., 1996) provides evidence that when students select a candidate for the limit of a function, they do so utilizing what I will refer to as an ***x-first perspective***. By ***x-first perspective*** it is meant that students focus their attention first on the inputs (x-values) and then on the corresponding outputs (y-values). The selection of a candidate is made based on what numeric value the y-values are getting close to as x-values get closer to a. In contrast, the validation of a candidate for a limit requires that one begin with a given candidate. Hence, the formal definition is dependent upon a candidate having already been selected. Validating a candidate, however, relies on one's ability to reverse his or her thinking. Instead of imaging what y-value *results* from a particular x-value, a student must first consider what is taking place along the y-axis, as Carlson, Oehrtman, and Thompson (2007) suggest: "In order to understand the definition of a limit, a student must coordinate an entire interval of output values, imagine reversing the function process and determine the corresponding region of input values." (p.160). Thus, the process of validating a candidate requires a student to recognize that his/her customary ritual of first considering input values is no longer appropriate. Instead the student must consider first a range of output values around the candidate, project back to the x-axis, and subsequently determine an interval around the limit value that will produce outputs within the pre-selected y-interval. Larsen's research (ibid) suggests that the intricacies involved in this y-first process are arguably far more cognitively demanding for students than merely formalizing

[1]Fernandez (2004) and Juter (2006) have also suggested that *validating* limits involves a process distinct from the process of *finding* limits. Their perspectives, in addition to Larsen's perspective discussed here, have assisted me in articulating my own thinking on the distinction between these two processes.

an *x*-first process, as Cottrill et al. (ibid) conjectured. The complex nature of the formal definition makes it highly unlikely that a student with a strong *x*-first perspective of functions would be able to conceive of a new concept in such a *y*-first manner, particularly when students typically learn first to *find* limits, not *validate* them.

In summary, the genetic decomposition offered by Cottrill et al. (1996) has served as a helpful framework from which to develop my own research. Specifically, their work provides evidence of how students reason about the informal/*x*-first process of finding limits (Steps 1-3 of their genetic decomposition); however, there is a dearth of data describing how students reason about the formal/*y*-first process of validating limits. Thus, it seems that more research is needed to elucidate the latter stages of their genetic decomposition. The overarching purpose of the research reported here was to generate such insights and to move toward the elaboration of a cognitive model of what might be entailed in coming to understand this formal definition. Specifically, the intent of the research was:

1. To develop insight into students' reasoning in relation to their engagement in tasks designed to support their reinventing the formal definition of limit, and;
2. To inform the design of principled instruction that might support students' attempts to reinvent the formal definition of limit.

The first objective above was set against the broader background goal of contributing to an epistemological analysis (Thompson & Saldanha, 2000) of the concept of limit of a real-valued function and its formal definition. Also, while other studies (e.g., Larsen, 2001; Fernandez, 2004) have sought to describe how students reason about limit as they *interpret* the conventional ε-δ definition of limit, my research sought to address this need by focusing on how students reason about limit in the context of *reinventing* a definition which captures the intended meaning of the conventional ε-δ definition. I theorized that interpreting a given formal definition might result in a very different type of reasoning than the reasoning that might arise while attempting to reinvent the definition. Hence, the use of an instructional trajectory designed to support students in reinventing the formal definition of limit was seen as a way to generate insights into students' thinking that would not be available through interpretation tasks. Indeed, the definition of limit constructed by Cauchy, and subsequently formalized by Weierstrass, was motivated by a need to specify the local behavior of functions in a precise manner. Neither of the mathematicians' definitions was a reformulation or an interpretation of the traditional formal definition. On the contrary, these mathematicians constructed their respective definitions in response to an inherent need to classify functional behavior. I felt, then, that I might learn a great deal about how students reason about the formal definition of limit if I engaged them in activities designed to foster their reinvention of the formal definition of limit.

Theoretical Perspectives

Ernst von Glasersfeld (1995), drawing on Piaget's genetic epistemology (1971, 1977), developed a psychological theory of knowing which is known as ***radical constructivism*** (RC). Two central tenets of RC are:

1. Knowledge is not passively received either through the senses or by way of communication, but is actively built up by the cognizing subject.
2. The function of cognition is adaptive, in the biological sense of the term, tending towards fit or viability and serves the subject's organization of the experiential world, not the discovery of an objective ontological reality (von Glasersfeld, 1995, p.51).

In this study, I drew on RC in a couple of important ways. First, RC functioned as a guiding framework methodologically, both in regards to the dynamic I aimed to create between participants in each of two teaching experiments I conducted, and in regards to how I selected participants for the teaching experiment phase. The sequence of instructional tasks I implemented in the two teaching experiments was designed to create a dynamic in which students might experience frequent perturbations, and thus, have the opportunity to make cognitive accommodations. Hence, the study's methodology was in line with the two tenets of RC: instructional activities were designed to motivate the cognizing subject to organize his or her experiential world and thus, actively build up knowledge. Also, participant selection included a criterion that participants be active seekers of viability and fit between their mathematical understandings. An important distinction is worth making, however. While I agree with the tacit assumption in von Glasersfeld's (ibid) theory that organisms are coherence-seeking beings, I also believe that in educative settings, some students are more coherence-seeking than others. This belief is reflected in the selection criteria I used for the study. Students selected to participate in this study had demonstrated a greater effort and desire, relative to other students, to consistently make sense of their experiential world as it relates to complex mathematical ideas.

Second, RC served as a lens through which I analyzed the data generated in the two teaching experiments which comprised the study. How one interprets the tasks he/she is presented is necessarily dependent upon one's prior experiences. As the students engaged with the instructional activities, their observable actions and behaviors provided evidence of how they might be interpreting said tasks. In a manner consistent with Steffe and Thompson's (2000) description of modeling students' interpretations, I compared my models of the students' interpretations with those targeted in instruction, so that I could make subsequent revisions for future iterations of the research cycle, and so that research findings could be cast as inferences about student reasoning given particular interpretations of instructional tasks. Also, given the impossibility of discovering ontological reality, the intention of data analysis was not to generate statements of fact about how students reason about or understand limits, but rather to generate *viable interpretations* of students' reasoning and understanding – i.e., interpretations that fit with their observed actions/behavior, in the sense that were they to reason in the ways I theorize, those ways might well express themselves in the observed behaviors.

In addition to the overarching perspective of RC, I briefly describe aspects drawn from the perspective of *developmental research* that guided the instructional design for my study. Gravemeijer (1998) describes the goal of developmental research as follows: “to design instructional activities that (a) link up with the informal situated knowledge of the students, and (b) enable them to develop more sophisticated, abstract, formal knowledge, while (c) complying with the basic principle of intellectual autonomy” (p.279). I view the goal of developmental research as being in line with the epistemological stance of RC, in that developmental research views knowledge as being constructed by individuals based on informal knowledge that is situated in their own experiences. A heuristic commonly associated with developmental research is *guided reinvention*. This well-established heuristic has been employed in numerous content areas of postsecondary mathematics education (see Larsen, 2004; Marrongelle & Rasmussen, 2006). Guided reinvention is described by Gravemeijer et al. (2000) as “a process by which students formalize their informal understandings and intuitions” (p.237). An important aspect of this process is the identification of plausible instructional starting points from which students might naturally formalize their informal understandings and intuitions. Traditionally, there have been two approaches for determining appropriate starting points for instruction – 1) Analyses of the historical evolution of the mathematical topic with an eye toward identifying motivating problems or contexts for conceptual development; and, 2) Examination of students’ informal strategies and interpretations of contextual problems that are directly related to the mathematical concept. Gravemeijer et al. (ibid) describe the history of mathematics and students’ informal interpretations as “sources of inspiration” for the researcher, who “tries to formulate a tentative, potentially revisable learning trajectory along which collective reinvention…might be supported" (p. 239). The first approach is intended to assist the researcher in formulating a learning trajectory in response to historical cognitive barriers and subsequent discoveries. While I did not analyze the historical evolution of limit with the aim of formulating a specific learning trajectory that would mirror the evolution of the concept, I did attempt to create an environment intended to mimic important aspects of the mathematical setting that Cauchy and Weierstrass experienced. That is, students selected for my study had no prior experience with the conventional ε-δ definition, and were posed with the challenge of characterizing local functional behavior using precise mathematical language. In this way, the historical evolution of the definition informed my selection of a starting point for instruction. The second approach aims to inform the researcher as to how he/she might provide students with authentic opportunities to experience perturbations and to make subsequent accommodations. My examination of existing research on students' informal reasoning about limit guided the design of initial instructional tasks for the teaching experiment phase of the study.

Method

The study was conducted over the course of seven months (May-December) during 2007, with four students (one female and three males) from a large urban university. Participants for the study were selected based on the following criteria: 1) strong informal understanding of limit; 2) no prior experience with the formal definition of limit, be it in high school or other calculus courses taken at the university level; and, 3) demonstrated ability to communicate their reasoning freely and without hesitation. All four participants had been students in both my Calculus I course during the Fall of 2006 and Calculus III course during the Spring of 2007, and three of the four participants had been students in my Calculus II course during the Winter of 2007. Thus, through my interactions with these students over the course of the academic year, I had ample data on which to base my selection of these participants.

The study consisted of two separate teaching experiments (Steffe & Thompson, 2000) – one for each pair of students. Each teaching experiment included ten sessions, or teaching episodes, with approximately one session per week. These sessions were videotaped and each lasted approximately 60 to 100 minutes. Each session proceeded in a similar format – the students responded to written and verbal tasks I presented, taking on the roles of both *conjecturer* and *refuter*[2]. The second teaching experiment was based on revisions made to the instructional trajectory as a result of analyzing the data produced during the first teaching experiment.

Instructional Trajectory

Following some initial attempts at defining *limit at a point*, I asked the first pair of students to define *limit at infinity*. My decision to have them first pin down a precise definition of *limit at infinity* was based on my conjecture that the formal definition of *limit at infinity* is cognitively less complex than the formal definition of *limit at a point*. In the case of *limit at infinity*, one is only required to describe closeness along the *y*-axis, whereas in the case of *limit at a point*, one must describe closeness along both axes. I anticipated that this sequence would provide a natural progression allowing students to use their definition of *limit at infinity* in reinventing the definition of *limit at a point*. Thus, based upon this conjecture, the following tasks, listed in sequential order, formed the structure of the first teaching experiment:

- Attempts to motivate the need for a rigorous definition of *limit at a point*
- Generation of examples and counterexamples of *limit at a point*

[2] At the outset of each teaching experiment, I encouraged the respective pair of students each to take on both of these roles at various times for the duration of the teaching experiment. I told them that the role of *conjecturer* entailed proposing thoughts and ideas they were having, even if those thoughts and ideas were not fully formulated. I told them that the role of *refuter* meant taking on a contrarian role wherever warranted, seeking logical inconsistencies in each other's ideas, so that ideas might be refined.

- Initial attempts to precisely define *limit at a point*
- Generation of examples and counterexamples of *limit at infinity*
- Reinvention of the definition of *limit at infinity* through a process of refinement
- Reinvention of the definition of *limit at a point* using the definition of *limit at infinity* as a foundation and motivation for continued refinement

A post analysis of the first teaching experiment pointed to particularly useful elements of the instructional trajectory, as well as pedagogical aspects that were less fruitful. This analysis influenced the structure of the second teaching experiment. Further, I viewed the second teaching experiment as an opportunity to explore "other paths to success" – in particular, I wondered if the second pair of students could reinvent a definition of limit capturing the intended meaning of the conventional definition without first defining limit at infinity. Thus, the following tasks, listed in sequential order, formed the structure of the second teaching experiment:

- Generation of examples and counterexamples of *limit at a point*
- Initial attempts to precisely define *limit at a point*
- Attempts to define *close* incrementally in an increasingly restrictive fashion
- Reinvention of the definition of *limit at a point* using the definition of *close* as a foundation for describing *infinite closeness*

In sum, the central instructional goal for each teaching experiment was for the students to generate a precise definition of *limit at a point*. The students' generation of examples and counterexamples of limit served as a starting point from which to proceed in reinventing the formal definition. The students used these examples and counterexamples as a source of motivation for refining their definition throughout the respective teaching experiments.

Data Analysis

The analytic approach I utilized is consistent with grounded theory methods (Glaser & Strauss, 1967), wherein data analysis is a cyclic process in which hypotheses about students' reasoning are generated, reflected upon, and subsequently refined until increasingly stable and viable hypotheses emerge. The analysis of data occurred at a variety of levels. As each teaching experiment was unfolding, I conducted ***ongoing*** analysis, which informed my decisions about subsequent sessions within the same teaching experiment. Ongoing analysis consisted of: 1) transcribing each session; 2) constructing a *content log*, which contained descriptive notes characterizing what I was asking the students to do, inferences about the students' interpretations of what I was asking them to do, and conjectured potential conceptual entailments of students' reasoning about limit in the context of reinvention; and, 3) composing an 8-10 page document outlining my instructional goals and conjectured tasks for the upcoming session, as well as my rationale for those tasks. Following the completion of each teaching experiment, I conducted a ***post*** analysis of the data generated by each pair of students. This provided me an opportunity to analyze each data set more deeply, so as to begin to develop themes present throughout the data set. Post analysis consisted of reviewing the videos and transcriptions of all ten sessions, highlighting noteworthy excerpts, and making conjectures about thematic elements of student reasoning. Finally, following the completion of both teaching experiments, I conducted a ***retrospective*** analysis (Cobb, 2000), in which I was able to analyze the entire corpus of data[3] at a deeper level than the preceding analyses. Retrospective analysis consisted of reviewing the post analyses of both teaching experiments, comparing and contrasting student reasoning between the four students. Doing so led to a refinement of my description of thematic elements present in student reasoning. At all three levels of analysis, I was engaged in frequent discussions with other mathematics educators who had intimate knowledge of the study in an effort to reach consensus about the data.

[3] The data corpus for analysis consisted of twenty two videotaped sessions each lasting 60-100 minutes.

Discussion/Results

Analysis of the data generated in the two teaching experiments point to two central findings: 1) students are likely to employ an *x*-first perspective in their initial attempts to define limit and to view the utilization of a *y*-first perspective as counterintuitive; and, 2) students are prone to reason initially from a *potential infinity* perspective during the reinvention process, and may experience difficulty in finding a suitable alternative to such a perspective. The first of these findings is detailed in an earlier paper (Swinyard, 2008a). The results presented in this discussion focus primarily on the second finding.

A Suitable Alternative to a Potential Infinity Perspective

A considerable hindrance to the students' efforts to reinvent the definition of limit was their struggle to find a suitable alternative to the ***potential infinity perspective*** they initially utilized during the reinvention process. Tirosh (1991) describes *potential infinity* and *actual infinity*, in relation to the history of mathematical development, as follows – "[T]he two competing ideas of infinity were *potential infinity* in which a mathematical process can be carried out for as long as required to approach a desired objective, and *actual infinity* in which one contemplates the totality of infinity, through, for example, conceiving the totality of *all* natural numbers at one time" (p.200). Evidence in both teaching experiments suggests that students are prone to reason initially from a potential infinity perspective during the reinvention process, and may experience difficulty in finding a suitable alternative to such a perspective. Both pairs of students' initial definitions focused heavily on the act of carrying out an infinite process. Below are two such examples from the first teaching experiment.

Pair #1, Definition #2: If you could zoom forever and always get closer to *a* and *L*, then you have a limit. **(Session 4)**

Pair #1, Definition #3: A function has a limit *L* at *a* when zooming in FOREVER both horizontally and vertically yields no gaps that have length > 0 AND that it looks like it approaches a finite number *L*. **(Session 5)**

For the second pair of students, a potential infinity perspective persisted throughout the teaching experiment, as is evidenced by their final definition of limit.

Pair #2, Definition #9:

1) Come up with a guess, *L*.
2) Determine a closeness interval $L \pm z$ around your guess.
3) If there exists an x_1<a such that L+z>f(x)>L-z is true for all x between x_1 and a AND an x_2>a such that L+z>f(x)>L-z is true for all x between x_2 and a then shrink your closeness interval and try again. If you can't shrink your interval anymore, then L is your limit.
If not, then L is not your limit. **(Session 10)**

To be clear, then, both pairs of students appeared to initially follow the same reasoning trajectory in regards to issues related to infinity – both pairs began by reasoning from a potential infinity perspective, with their focus on describing the incremental completion of the infinite limiting process. Both pairs of students subsequently recognized the limitations of such a perspective, noting in distinct ways the impossibility of completing an infinite process in a finite amount of time. The following excerpt illustrates this realization for one of the students in the first pair.

Amy[4]: I don't know, it seems like we keep dancing around some kind of concept that we have to talk about in a series of…analogies or hypothetical situations, you know? Like…the hypothetical situation in which you are doing something forever….I guess like the first thing that leaps to mind for me is that we're trying to parse out what we mean by, by these impossible processes that we're describing for…whether we have a limit.

Craig: And you're saying impossible there why?

Amy: Because you can't zoom in forever…[Y]ou can't do something an infinite number of times….[A]ll you can do is find…the level of examination which disproves your idea but you can't ever get to where you can conclusively prove it through the methods we've been discussing…I have a hard time getting too worked up over the language about what it means to zoom and what we're looking for when we zoom when we have lurking in the back this presupposition that whatever that means to zoom,…we have to repeat that process an infinite number of times.

[4] To provide anonymity, all student names presented in this report are pseudonyms.

Likewise, the second pair of students expressed dissatisfaction with a potential infinity perspective.

> Jason: And what we're trying to do with all this confusing language is describe in words what that function does in the vicinity of *a*.
>
> Chris: As *x* gets closer to *a*,…I don't want to say close, because how close is close?
>
> Jason: Fantastic point…You could always get closer…[T]here's an infinite amount of closers between close and close enough….I'm raising an objection now to the idea of "close enough". I don't think that there is "close enough," because of the idea that there's always a closer. So if there's an infinite number of closers between close and close enough, how can close enough even exist?

This dissatisfaction with describing the incremental completion of an infinite process led both pairs of students to seek a suitable alternative perspective. At this point, the two pairs diverged in their reasoning. During the seventh session, the first pair of students spontaneously employed the notion of *arbitrary closeness* to encapsulate the infinite limiting process, which ultimately led them to the following definition.

> **Pair #1, Definition #9:** $\lim_{x \to a} f(x) = L$ provided that: given any arbitrarily small # λ, we can find an (a±θ) such that $|L-f(x)| \leq \lambda$ for all *x* in that interval except possibly x=a. **(Session 9 – Final Definition)**

The decision to operationalize *infinite closeness* via the notion of *arbitrary closeness* marked a watershed moment in the first pair's reinvention of the formal definition of limit. In contrast, the second pair of students did not utilize the notion of *arbitrary closeness* and instead swept the cognitive issues of a potential infinity perspective "under the rug," in the sense that they superficially resolved the cognitive dilemma of imagining the carrying out of an infinite process by simply accepting that the end of the process must somehow mysteriously happen. The reader will note that the second pair's final definition (shown previously) is expressed from a potential infinity perspective and does not adequately address how the infinite limiting process might be encapsulated. Further, when asked to interpret the first pair's final definition of limit, the second pair of students interpreted the notion of *arbitrarily small* as representing a ***single*** definition of closeness, as opposed to ***all*** definitions of closeness. This interpretation ultimately hindered the second pair in reinventing a definition of limit which encapsulated the infinite limiting process.

Operationalizing Infinite Closeness

A related finding bears mentioning. Defining *closeness* prior to defining *infinite closeness* proved to be a critical moment in both teaching experiments. Albeit under different circumstances, both pairs of students defined *closeness* outside of the context of *limit at a point* and subsequently used that definition to operationalize *infinite closeness* in the context of *limit at a point*. The first pair of students defined *closeness* in the context of *limit at infinity*, while the second pair of students defined *closeness* completely outside of the context of limit[5]. In both cases, defining *closeness* in an incrementally restrictive fashion (i.e., 10, 2.5, 1.5, .5, etc.) appeared to initiate important cognitive shifts for the students. First, the iterative nature of this defining process gave the students a way to imagine how one might define *closeness* at any level of desired specificity, thus allowing them to think of *infinite closeness* as a notion that can be characterized in a hypothetical manner (i.e., as *closeness* at any level of desired specificity). While only the first pair of students subsequently encapsulated the limiting process by utilizing the notion of *arbitrary closeness*, operationalizing *infinite closeness* by first defining *closeness* appeared to support both pairs of students in making significant and profound refinements to their respective definitions of limit.

Contributing to an Epistemological Analysis

To *contribute to an epistemological analysis* (in the sense of Thompson & Saldanha, 2000) is to gain insight into what is entailed in coming to understand a particular mathematical idea in

[5] i.e., in response to the prompt, "[F]or every single one of its x-values, how would you write out what it means for that function to be ***close*** to a pre-determined value L?"

relation to engagement in instruction designed to support the development of that understanding. The central objective of this study was to develop insight into students' reasoning in relation to their engagement in instruction designed to support their reinventing the formal definition of limit. This objective was set against the broader background goal of contributing to an epistemological analysis of the concept of limit and its formal definition.

The genetic decomposition offered by Cottrill et al. (1996) can be thought of as a conjectured model of how students may come to formalize their understanding of limit. In this sense, a genetic decomposition can be thought of as a contribution to an epistemological analysis. Previously, I noted the ways in which the genetic decomposition proposed by Cottrill et al. lacked empirical evidence that could inform the latter stages of their model of student reasoning. The aim of my research was to help elucidate those latter stages. In Figures 1 and 2, I provide a portion[6] of my own genetic decomposition, based on data gathered during the two teaching experiments which formed this study[7]. A few details are worth noting. First, unlike the genetic decomposition presented by Cottrill et al., the genetic decomposition presented here focuses only on the transition from informal to formal reasoning (i.e., stages 5-7 in the genetic decomposition offered by Cottrill et al.). Thus, this genetic decomposition is based on the assumption that students already have an informal understanding of limit. Specifically, this means that students are able to:

1) Discuss when a limit does exist and why
2) Discuss when a limit does not exist and why
3) Determine limits for both finite and infinite situations
4) Sketch graphs satisfying given conditions related to both finite and infinite limits

[6] The complete genetic decomposition I propose also addresses students' transition from an *x*-first to a *y*-first perspective. See Swinyard (2008b) for the details of the full genetic decomposition.
[7] It is worth noting that the methodology employed in this study was different than that utilized in the study conducted by Cottrill et al. (1996), in that student reasoning about limit in my study was in the context of reinvention, as opposed to interpretation, of the formal definition. Hence, the genetic decomposition presented here was based on data collected in an experimental setting distinct from that experienced by the students in the Cottrill et al. study.

5) Provide an informal definition of limit that demonstrates viable conceptual understanding

Second, I choose to split the genetic decomposition into two parts. Part 1 characterizes student reasoning prior to the instructional intervention of encouraging the students to define *closeness*. Conversely, Part 2 characterizes students' reasoning subsequent to this instructional intervention. Third, unlike the genetic decomposition proposed by Cottrill et al., the one presented here is not in a strict numeric stage format, but instead is presented in the form of a flow chart. This was done to maximize the explanatory power of the cognitive model. In particular, this form allows for the complete genetic decomposition (see Swinyard, 2008b) to capture multiple cognitive difficulties being experienced by the students simultaneously. A description of each part of the genetic decomposition follows.

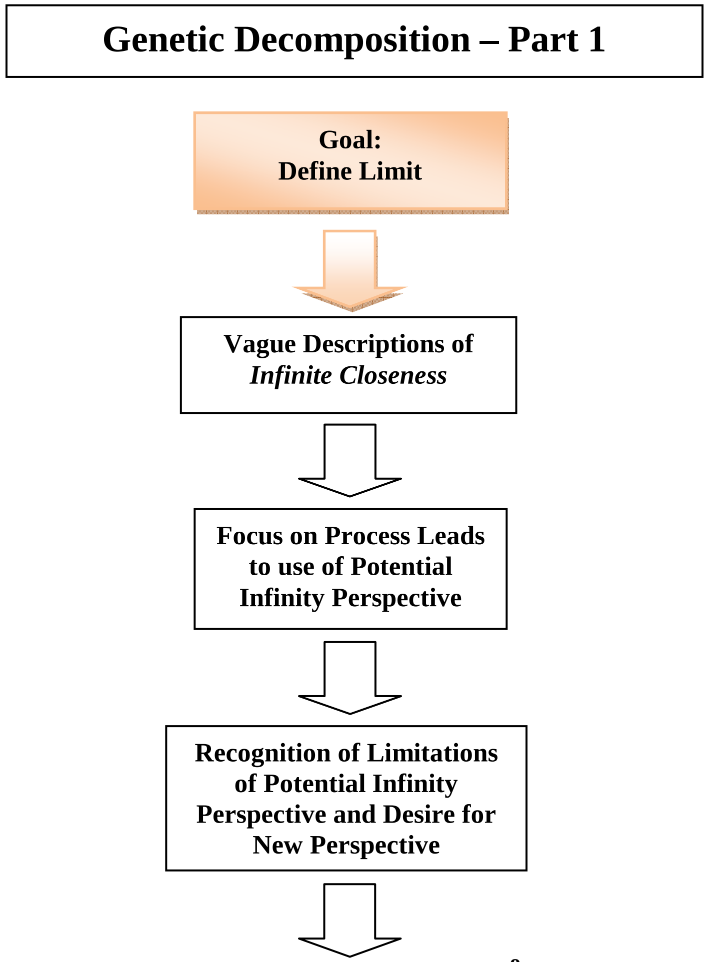

Figure 1 – Genetic Decomposition[8] – Part 1

The first part of the genetic decomposition can be summarized as follows: Evidence from this study suggests that in response to being charged with the task of defining what it means for a function to have a limit L at $x=a$, students' initial characterizations are likely to include vague descriptions of *infinite closeness*. The first definition provided by Amy and Mike is one such example: f has a limit L at $x=a$ provided as x-values get closer to a, y-values get closer to L. Upon recognizing that vague descriptions of *infinite closeness* mischaracterize particular functions as having limits at x-values for which no limit exists (e.g., functions with jump discontinuities),

[8] Shaded boxes and arrows denote noteworthy instructional interventions, and thus, are not, strictly speaking, part of the genetic decomposition. However, given the dialectic between student reasoning and instruction, it is reasonable, given the study's methodology, to include the initial task which situated student reasoning, and, in Part 2 of the genetic decomposition, the instructional intervention which initiated the resolution of students' cognitive difficulties. The un-shaded boxes and arrows in this diagrammatic representation represent the students' ways of reasoning in the context of reinventing the formal definition of limit, and thus, constitute the core of this genetic decomposition.

students attempt to flesh out what they mean by x-values getting closer to a and y-values getting closer to L. In their attempts to describe *infinite closeness* with greater precision, students' focus appears to turn to describing the limiting process. Attempts to summarize the infinite limiting process appear to lead students to subsequently utilize a potential infinity perspective. The inability to describe the completion of an infinite process in a finite amount of time appears to raise students' awareness of the limitations of a potential infinity perspective, and in turn, motivate the students to seek a new perspective. However, despite the motivation to adopt a new perspective, all four students in this study had difficulty finding a suitable alternative to the potential infinity perspective they initially employed.

Evidence from the study suggests that in response to the difficulties described in the preceding paragraph, students may benefit greatly from being asked to define *closeness* in a concrete and increasingly restrictive manner. Part 2 of the genetic decomposition, shown in Figure 2, illustrates the continued evolution of student reasoning about limit in the context of reinvention.

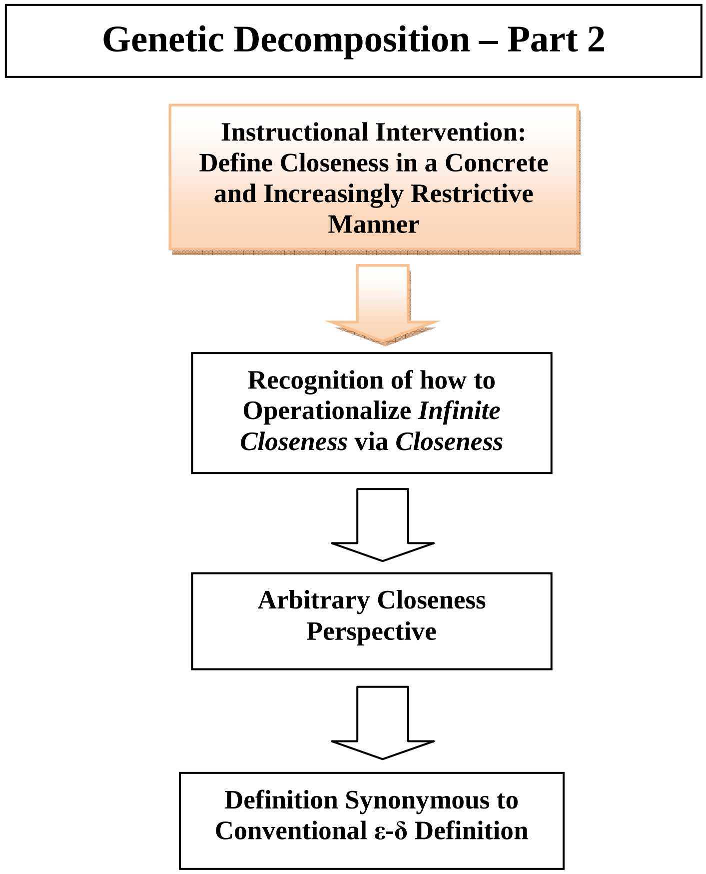

Figure 2 – Genetic Decomposition – Part 2

Defining *closeness* appears to initiate a significant cognitive shift in student reasoning. Defining *closeness* in a concrete and increasingly restrictive manner appears to lead students to recognize how to operationalize *infinite closeness*. Whereas prior to the instructional intervention students in this study expressed frustration over trying to define *infinite closeness*, the act of defining *closeness* in a concrete and increasingly restrictive manner appeared to allow them to momentarily set aside the challenge of having to actually complete the infinite limiting process. Shifting their attention away from the insurmountable task of describing the incremental completion of an infinite limiting process appeared to provide the students a suitable mental environment for recognizing that they could use the notion of *arbitrary closeness* to encapsulate the infinite limiting process. The adoption of an arbitrary closeness perspective appears to support students in reinventing, and reasoning coherently about, a definition synonymous to the conventional ε-δ definition.

Pedagogical Implications

The findings presented here inform pedagogy in a couple of important ways. First, evidence from both teaching experiments underscores the value of having students define *closeness* in a concrete and increasingly restrictive manner. In their attempts to define limit, the students in this study became paralyzed by the prospect of characterizing what it means to be infinitely close. However, when they were able to set aside the cognitive dilemma of incrementally completing an infinite process, and were asked only to define what it means to be close (in a concrete and finite sense) to a particular *y*-value, *L*, the students were then able to recognize how they might operationalize *infinite closeness* by use of their definition of *closeness*. Having students define what it means to be close to some pre-determined value *L*, either in the context of limits at infinity or in the context of, say, a step function, may support them in reasoning coherently about infinite closeness. Second, evidence from the study suggests that students may not interpret the phrase “an arbitrary small number” as representing ***all*** small numbers, but rather may view “arbitrary” as a referent to a ***single***, fixed small number. Such an interpretation clearly has an adverse effect on one’s ability to interpret the conventional formal definition of limit. It appears, then, that pedagogical interventions designed to support students in developing coherence with the “arbitrary” construct would be constructive.

References

Bezuidenhout, J. (2001). Limits and continuity: Some conceptions of first-year students. *International journal of mathematical education in science and technology, 32*, 4, 487-500.

Carlson, M. P., Oehrtman, M. C., & Thompson, P. W. (2007). Key aspects of knowing and learning the concept of function. In M. P. Carlson & C. Rasmussen (Eds.), *Making the connection: Research and practice in undergraduate mathematics* (pp. 150-171). Washington, DC: Mathematical Association of America.

Cobb, P. (2000). Conducting teaching experiments in collaboration with teachers. In R. Lesh & A. E. Kelly (Eds.), *Research design in mathematics and science education* (pp. 307-333). Hillsdale, NJ: Erlbaum.

Cornu, B. (1991). Limits. In D. Tall (Ed.) *Advanced Mathematical Thinking*. Dordrecht, The Netherlands: Kluwer Academic Publishers, 153-166.

Cottrill, J., Dubinsky, E., Nichols, D., Schwinngendorf, K., Thomas, K., & Vidakovic, D. (1996). Understanding the limit concept: Beginning with a coordinated process schema. *Journal of mathematical behavior, 15*, 167-192.

Davis, R., & Vinner, S. (1986). The notion of limit: Some seemingly unavoidable misconception stages. *Journal of mathematical behavior, 5*, 281-303.

Dorier, J. (1995). Meta level in the teaching of unifying and generalizing concepts in mathematics. *Educational studies in mathematics, 29*, 175-197.

Ervynck, G. (1981). Conceptual difficulties for first year university students in the acquisition of limit of a function. *Proceedings of the Psychology of Mathematics Education, 5* (pp. 330-333), Grenoble, France.

Fernandez, E. (2004). The students' take on the epsilon-delta definition of a limit. *Primus, 14*, 1, 43-54.

Ferrini-Mundy, J., & Graham, K. (1994). Research in calculus learning: Understanding of limits, derivatives, and integrals. In E. Dubinsky & J.J. Kaput (Eds.) *Research issues in undergraduate mathematics learning: Preliminary analyses and results*. Washington D.C.: MAA, 31-45.

Gass, F. (1992). Limits via graphing technology. *Primus, 2*, 1, 9-15.

Glaser, B., & Strauss, A. (1967). *Discovery of Grounded Theory*. Chicago: Aldine.

Gravemeijer, K. (1998). Developmental research as a research method. In J. Kilpatrick & A.Sierpinska (Eds.), *Mathematics Education as a Research Domain: A Search for*

Identity (ICMI Study Publication) (Book 2, pp. 277-297). Dordrecht, The Netherlands: Kluwer.

Gravemeijer, K., Cobb, P., Bowers, J., and Whitenack, J. (2000). Symbolizing, modeling and instructional design. In P. Cobb, E. Yackel and K. McClain (Eds.), *Symbolizing and Communicating in Mathematics Classrooms*, Erlbaum, Mahwah, NJ, pp. 225–273.

Juter, K. (2006). Limits of functions as they developed through time and as students learn them today. *Mathematical thinking and learning, 8*, 4, 407-431.

Larsen, S. (2001). Understanding the formal definition of limit. Unpublished manuscript, Arizona State University.

Larsen, S. (2004). Supporting the guided reinvention of the concepts of group and isomorphism: A developmental research project (Doctoral dissertation, Arizona State University, 2004) Dissertation Abstracts International, B65/02, 781.

Marrongelle, K., & Rasmussen, C. (2006). Pedagogical content tools: Integrating student reasoning and mathematics in instruction. *Journal for research in mathematics education, 37,* 5, pp. 388-420.

Monaghan, J. (1991). Problems with the language of limits. *For the learning of mathematics, 11,* 3, 20-24.

Oehrtman, M. (2003). Strong and weak metaphors for limits, Proceedings of the 27th Conference of the International Group for the Psychology of Mathematics Education held jointly with the 25th Conference of PME-NA, Vol 3, (pp. 397-404). Honolulu, HI.

Oehrtman, M. (2004). Approximation as a foundation for understanding limit concepts. *Proceedings of the Psychology of Mathematics Education North American Chapter, 23* (pp. 95-102), Snowbird, UT.

Piaget, J. (1971). *Genetic epistemology*. New York: W. W. Norton.

Piaget, J. (1977). *Psychology and epistemology: Towards a theory of knowledge*. New York: Penguin.

Steffe, L. P., & Thompson, P. W. (2000). Teaching experiment methodology: Underlying principles and essential elements. In R. Lesh & A. E. Kelly (Eds.), *Research design in mathematics and science education* (pp. 267-307). Hillsdale, NJ: Erlbaum.

Steinmetz, A. (1977). The Limit can be understood, *MATYC Journal 11,* 2, 114-121.

Swinyard, C. (2008a). Students' Reasoning about the Concept of Limit in the Context of Reinventing the Formal Definition. Electronic Proceedings *of the 11th Annual Meeting of the*

Research on Undergraduate Mathematics Education, Special Interest Group of MAA. San Diego, CA: San Diego State University.

Swinyard, C. (2008b). *Students' Reasoning about the Concept of Limit in the Context of Reinventing the Formal Definition*. (Unpublished doctoral dissertation, Portland State University, 2008).

Tall, D.O. (1992). The transition to advanced mathematical thinking: Functions, limits, infinity, and proof. In D.A. Grouws (Ed.) *Handbook of research on mathematics teaching and learning*. New York: Macmillan, 495-511.

Tall, D.O., & Vinner, S. (1981). Concept image and concept definition in mathematics with particular reference to limits and continuity. *Educational studies in mathematics, 12*, 151-169.

Thompson, P. W. & Saldanha, L. (2000). Epistemological analyses of mathematical ideas: A research methodology. *Proceedings of the Twenty-second Annual Meeting of the International Group for the Psychology of Mathematics Education*. Tucson, Arizona.

Tirosh, D. (1991). The role of students' intuitions of infinity in teaching the Cantorian theory. In D. Tall (Ed.) *Advanced Mathematical Thinking*. Dordrecht, The Netherlands: Kluwer Academic Publishers, 199-214.

Vinner, S. (1991). The role of definitions in the teaching and learning of mathematics. In D. Tall (Ed.), *Advanced Mathematical Thinking* (pp. 65-81). Boston: Kluwer.

von Glasersfeld, E. (1995). *Radical constructivism: A way of knowing and learning*, Falmer Press, London.

Williams, S. (1991). Models of limit held by college calculus students. *Journal for research in mathematics education*, *22*, 3, 219-236.

Williams, S. (2001). Predications of the limit concept: An Application of repertory grids. *Journal for research in mathematics education*, *32*, 4, 343-36.