Teaching Proof by Mathematical Induction: A Preliminary Report

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Abstract: This qualitative exploratory study examined two mathematicians' approaches to teaching proof by mathematical induction (PMI) to undergraduate pre-service teachers. Data considered in the study included classroom video across three weeks of PMI instruction for each professor, an interview with each instructor, focus group interviews of three students from each professor's class, and student solutions to common final exam PMI items. We report on the nature of the knowledge for teaching of PMI of the two instructors.

Background

As part of their preparation, pre-service secondary mathematics teachers take college courses such as discrete mathematics where they learn to make, test, and prove conjectures about mathematical patterns and relationships. In particular, they work with *proof by mathematical induction* (PMI). Such courses address the call by the National Council of Teachers of Mathematics' (2000) *Reasoning and Proof Standard* that K-12 teachers be prepared to help students create and validate (determine the truth of) logical arguments. Though several studies have looked at how learners might experience, understand, and use proof (Brown, 2008; Hanna, 2000; Harel, 2002; Selden & Selden, 2003; Tall, 1991; Weber, 2004), few look at how the teaching of proof, particularly PMI, is experienced by instructors who are mathematicians (e.g., Alcock, 2009). In this report, we have focused on the instructor. In a separate report, we have focused on the college learner (Davis, Grassl, Hauk, Mendoza-Spencer, & Yestness, 2009). In this exploratory study we asked the question: *What is the nature of mathematicians' perceptions*

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and teaching around the topic of proof by mathematical induction in an undergraduate discrete mathematics course for future teachers?

One approach to examining the teaching and learning of proof by mathematical induction is to consider the developmental, instructional, and learner aspects involved (e.g., Brousseau's (1997) ontogenic, didactical, and epistemological obstacles). We also have relied on the work of Harel (2002) regarding the role of intellectual necessity in the teaching and learning of proof by mathematical induction. Harel compared two forms of secondary school instruction for learning about inductive proof: traditional and necessity-based (see Table 1).

Table 1. Aspects of Traditional and Necessity-based PMI Instruction – Based on Harel (2002).

Traditional Instructional Approach	Necessity-based Approach
T.1. Teacher presents examples of how a formula with a single, positive, integral variable (like the sum of the first *n* integers) is generalized from observations and an observed pattern. **T.2.** Teacher talks about why examples are not enough to prove a proposition $P(n)$ is true for all positive integers *n*. **T.3.** Teacher demonstrates the principle of mathematical induction as a proof technique involving two steps: Step 1: Show that $P(1)$ is true. Step 2: Show $P(n)$ implies $P(n+1)$ for all *n*. **T.4.** Students practice applying the steps to mostly algebraic examples (e.g., formulaic and symbolic recursive relationships).	**N.1.** Students work with implicit recursion problems to develop pattern generalization skill; **N.2.** Students work with explicitly recursive relationships using quasi-induction as a method of testing conjectures. **N.3.** Teacher presents math induction as an abstraction of quasi-induction that meets students' felt need for a rigorous method of proof. **N.4.** Students make, test, and prove conjectures about a variety of mathematical statements using the language and procedures of mathematical induction.

Though both of the mathematicians who taught the discrete mathematics courses we observed used *both* traditional and necessity-based ideas, the balance of their use differed across the two instructors. In this sense, the study was informed by variety of didactical situations (and of didactical obstacles). The main result is about college teaching and the kinds of exposition and unpacking of conceptual restructuring that may be needed for an instructor to facilitate learning of proof by mathematical induction. That is, though the language/symbol set and procedure for proof by mathematical induction can be taken up and used by students in many successful ways, an explicit consideration by instructors of how students think about mathematics, particularly about what constitutes "problem-solving" and about the nature of "proof," may be necessary in coming to a rich and connected pedagogical content knowledge for teaching PMI.

Methods

Setting

The instructors in this study are both mathematicians. They each taught one section of discrete mathematics at the same 12,000-student doctoral-extensive university in the United States. At the time of the study, Dr. Isley, with a PhD in combinatorial algebra, had taught college mathematics for more than 20 years and was the author of the text used in the class.[1] He had taught discrete mathematics more than 20 times, and generally used lecture with occasional in-class activities. During the three weeks of PMI focus, Dr. Isley lectured 60% of the time and the class spent 40% of the time attending to in-class lecture presentations of inductive proofs (on overhead transparencies) by student teams. Before students presented, they met with Dr. Isley in his office, where he helped them validate their work. Dr. Vale, with a PhD in logic and model theory, had 10 years college teaching experience though this was his first time teaching proof by mathematical induction and the first time he had taught a discrete mathematics course. Dr. Vale used Isley's textbook, and developed additional activities for class, using a mix of traditional and necessity-based activities for instruction. During the observed lessons on PMI, Dr. Vale lectured about 35% of the time with the balance of about 65% of class time spent on students working individually and in groups to make, test, and prove conjectures by PMI about recursive and

[1] To protect the confidentiality of the instructors, we offer limited demographic information.

closed-form expressions. A notable distinction between the types of activities engaged in by students in Dr. Isley's class and those in Dr. Vale's class was that students in Dr. Vale's class had activities that included validating each others' inductive proofs during in-class group work and regularly had PMI proof-validation tasks where they analyzed potential proofs.

Most undergraduates in the two classes (65%) were planning on becoming secondary school mathematics teachers and some (about 25%) were planning to be primary school teachers with a specialty in mathematics. About half of the students in both classes had graduated from high schools within a 200-mile radius of the university and most had not encountered proof by mathematical induction before in a high school or college mathematics course. Like the U.S. secondary teaching population, the students were mostly from middle socio-economic status, majority culture, backgrounds.

Design and Analysis

We relied on information from four data gathering activities. First, we observed (in person or from a video recording) seven PMI-related class meetings for each instructor and completed related PMI textbook reading and activities (data set A). Second, at the end of the semester, we conducted 90- to 120-minute video-recorded interviews with focus groups of three students each – one group from each of the two discrete math classes (data set B). The third form of data (set C) was a 90- to 120-minute video-recorded interview with each instructor about mathematics, about proof by mathematical induction in particular, and about the teaching and learning of both. Finally, the fourth set of data (set D) was student work on two PMI-related common final exam items (n=49), one requiring students to generate a proof, one asking students to validate a purported proof. For the work reported here, we spent the greatest analytic effort on the instructor-related forms of data (set A – video of class instruction and set C – interviews with instructors). Figure 1 summarizes our iterative process for qualitative open coding for themes within data sets and axial coding for categories and sub-categories across data sets.

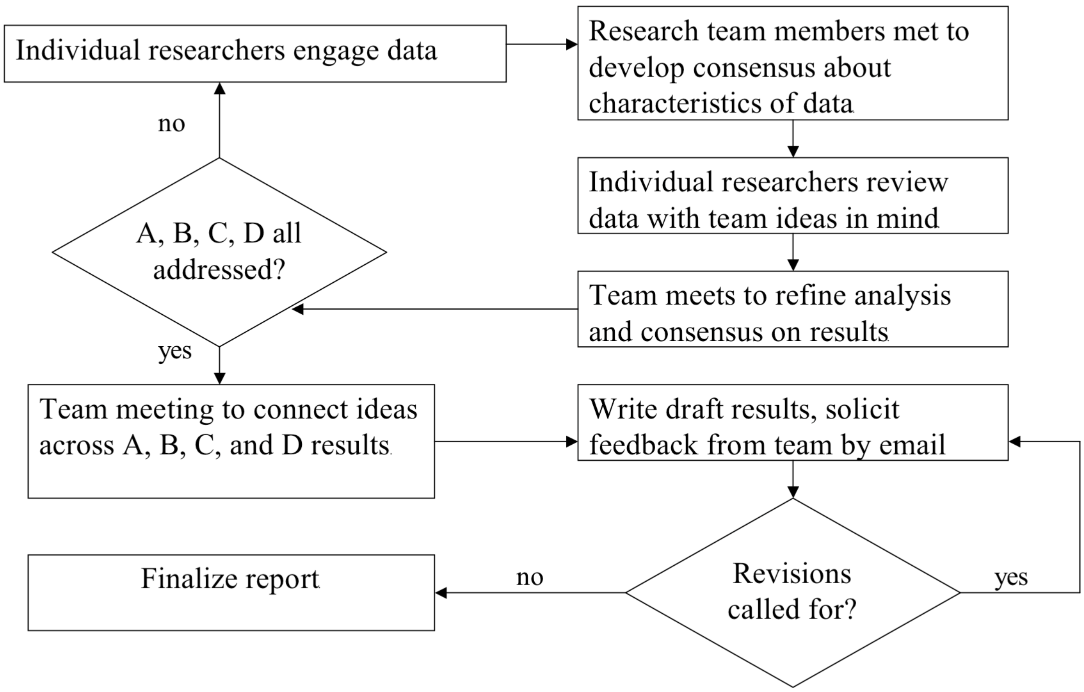


Figure 1. Flow chart of data analysis process.

A note on the interview protocol. Based on student focus group feedback, we chose a class-time video-clip from each professor's PMI instruction that exemplified something students found challenging in learning to work with proof by mathematical induction. We interviewed each instructor separately, asking questions about his perceptions of college teaching, of the inductive method of proof, and of his teaching of PMI in particular. During each interview, the instructor viewed the video-clip from his class. The clips helped frame and prompt discussion about teaching proof by mathematical induction.

We also conducted follow-up interviews with the two instructors, after the end of the term, that explored instructor satisfaction with student performance. We examined, with each instructor, students' solutions to two common final exam PMI items (one required generating an inductive proof, the other validating a PMI). We used constant comparative analysis within and across the two instructors' experiences.

Results

Perception of PMI as Proof Technique

Both instructors regarded PMI as a highly specialized kind of proof technique. Dr. Isley reported his perception of PMI as a three-step prescribed procedure, articulated the three steps of stating a base case, stating an inductive step, and carrying out the proof of the inductive step, concluding by saying, "I believe it works." He also considered PMI to be a "last resort" sort of proof technique. Dr. Isley noted that his main goal as author and instructor was to "get students to think logically and to follow the prescribed techniques" to use PMI.

Dr. Vale reported that, as a mathematician, his use of inductive proof depended on the problem context, particularly whether a recursive pattern was involved. He did not perceive PMI as being as powerful as deductive types of proof and said that for both teaching and doing mathematics he preferred proofs that "showed why" a statement is true. Dr. Vale connected PMI with the "very foundations of mathematics" and the successor relation in the generation of natural number. Nonetheless, he said, he was unsure how large a role PMI should play in a discrete mathematics course for future teachers. Dr. Vale saw the nature of discrete mathematics as more "about problem-solving than proving."

Perception of PMI as Course Content

Dr. Isley, one of the textbook's authors, discerned different roles for PMI as course content for different cohorts of students. He had several years experience teaching discrete mathematics to computer science students, who he said, "cared more about finding recursive pattern for algorithmic purposes" than about generating a proof or validating a mathematical statement. He was not fond of teaching PMI to this cohort. However, he noted that the concept of PMI was important for pre-service teachers, because "they might have to teach it and therefore they need to understand it." In addition to the three-step proof technique he outlined, Dr. Isley believed that his way of offering PMI content in the book and in his class provided students an opportunity to revisit a broad array of mathematics. These, he said, were the reason behind the problem set in the text of 37 statements to be proved by PMI.

Dr. Vale said he learned in teaching the course that it was serving the dual purpose of giving students experience in discrete mathematics and in proof techniques in general. He noted that the university did not offer a "bridges to proof" or formal proof introduction course. In this sense, Dr. Vale said it was understandable to have PMI as a special topic in discrete mathematics. However, he felt the textbook was organized in a way that presented "discrete mathematics content in service to proof by mathematical induction." Dr. Vale was concerned that it might lead students to overestimate the significance of PMI in the larger picture of all mathematics.

Intentions for Teaching PMI

Dr. Isley noted that his intentions were to teach PMI as a procedure, expecting that students would gain mastery of the steps during the course. In particular, on one hand, he hoped to count on students "to be able to state" the PMI procedure. On the other hand, he also said it was "probably too much to expect" students to "make sense" of PMI, to understand it, at the same time they were trying to master the formality of it. Dr. Isley expected students to gain conceptual understanding of PMI later, saying that his class was a first exposure in what he anticipated to be a multi-stage experience: (a) basic understanding in his class, (b) initial conceptual understanding in a next college class, and (c) "really understanding" when his students later taught PMI to their high school students.

Dr. Vale said his intention was for students to start understanding the infinite iterative process behind PMI while they built some ability in PMI procedures and in working with recursive pattern. He believed that once students had a sense of the potential relationship between closed form and recursive descriptions of a pattern, they would have less difficulty in transferring and connecting between the procedure and formality of PMI. To support students in building what he called a "basic overall understanding," Dr. Vale's intentions were to have students experience the "ideas behind proof by mathematical induction" in several ways. He said he thought that the future high school teachers in his class would benefit from experiencing alternative styles as learners (e.g., individual and group work, lecture). He expected them to someday use a variety of instructional methods when they became school teachers. Overall, he believed that it was his students' responsibility to construct their own personal understanding of PMI. As an instructor, he said, he was "only a facilitator."

PMI Instructional Implementation

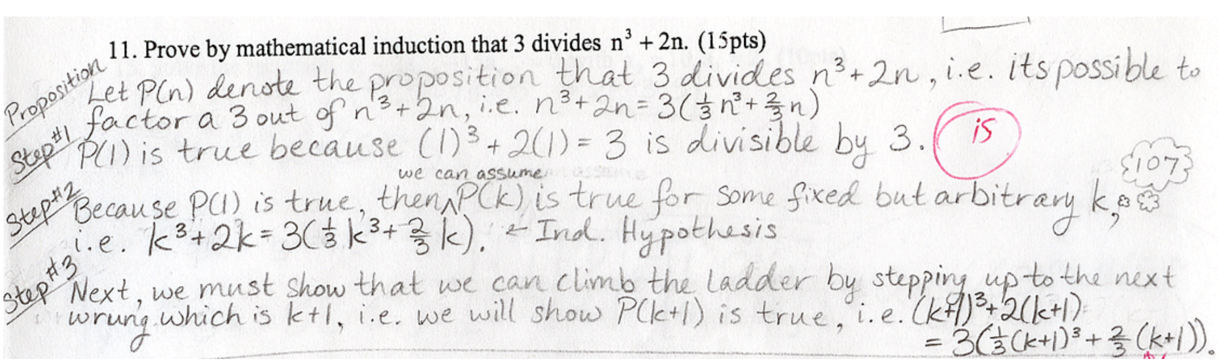
Of the 42 instructional meetings in the course, Dr. Isley's class spent the better part of 7 of these on PMI. Dr. Isley said that his experience with students' difficulties in learning PMI had led him to create the culminating set of 37 statements to be proved by induction. It was from this set that students chose for their in-class presentations. In addition to lecturing, Dr. Isley assigned homework where students were expected to practice generating inductive proofs. Dr. Isley also said he had his students “teach” their proofs by presenting them to class, in a hope that his students would gain confidence both in teaching and in understanding PMI through the dual activities of watching others present inductive proofs and showing PMI to other students. Dr. Isley also felt that having students present their work was a good way for him to assess their mastery of the procedure, particularly through his interactions with students in his office before they presented. Before presenting proofs, students met with Dr. Isley in his office, where he helped them validate their work mainly by checking if they followed the format of prescribed procedure. He said would not allow students to present their work if it was incomplete or incorrect. According to Dr. Isley, he expected students to have interplay with the presenters, especially to raise critical questions by pointing out problematic or incorrect components in their proofs. However, Dr. Isley’s own careful co-validation with presenters in his office meant that during the in-class presentations students mainly raised clarification questions, requesting details or information about the problem-solving aspects of the proof. Because all knew that the instructor had previewed presentations, Dr. Isley suggested that students might not want “to stick their neck out” to question the presentation. He considered that students had devoted “medium effort” to critiquing each other. Dr. Isley did not directly prepare his students for what he expected in terms of critiquing. This observation, confirmed by Dr. Isley, indicates an opportunity for expanding explicit instruction about the substance both in students’ PMI performance and the classroom social norm of critical interplay.

Though he started with mostly lecture presentation and short in-class activities on PMI, Dr. Vale eventually organized about 65% of PMI class time around students working in groups to make, test, and prove conjectures about recursive and closed-form expressions. According to Dr. Vale, his plan was “going into it, show some examples, and then have students try something.” He observed the trouble students were having with the first day’s activity and felt “students were caught by the formality” of the algorithmic nature of PMI on the first day. Consequently, in the following class, Dr. Vale demonstrated his own thinking as he analyzed closed form and recursive representations of a discrete mathematical pattern and pointed to corresponding aspects of a formal proof by mathematical induction. In particular, Dr. Vale said he had done this in a “hope that students could generalize” the idea of recursive pattern to the infinite iterative inductive implication: $P(1)$ implies $P(2)$, $P(2)$ implies $P(3)$, etc., to $P(k)$ implies $P(k+1)$. While students appeared to be busy in the subsequent in-class activity, they also appeared to be struggling to work with inductive implication. One student noted to another that having more steps could be good but it also seemed “even more overwhelming.”

As Dr. Vale had intended, he provided students with faulty proofs. In particular, he introduced the idea of invalid proof by appearing to use PMI to “prove” that all horses have the same color. His purpose was “somehow to provide conflict for students to resolve.” After his presentation, he perceived his students were not ready to resolve this conflict, and that instead it might have led to students having even more frustration about PMI. In the student focus group interview, two of three students confirmed Dr. Vale’s perception about their negative feeling about the horses example. In the following week, Dr. Vale adjusted his teaching to have students “teach” by having them do an extended activity in groups that included reading a paper about teaching recursion and PMI in high school (Allen, 2001), generating a worksheet for high school students, and later, completing several tasks where they validated purported proofs. From the class video and focus group interview, students seemed uncomfortable with this shift from lecture to student-driven group work. Dr. Vale said students had “already marched to the edge of a cliff” by experiencing his introduction of PMI as a procedure. He chose the Allen article and created an activity around it to “pull them back from the edge of the cliff” – to help his students to get a sense of recursive pattern first and to build a reason (intellectual necessity) for understanding PMI. However, he was aware that such activities, since they were so different from the initial “here’s the problem, use this algorithm to solve it” approach, caused discomfort for his students. Dr. Vale said he had underestimated the difficulty of PMI and did not yet have “an efficient way of teaching it.” Nonetheless, he felt some success because his students did “suspect” PMI and he believed that “students should doubt anything, including PMI.”

Instructor Perception of Student Performance

Students who experienced Dr. Isley’s largely traditional approach felt a procedural competence in asserting the framework for proof by mathematical induction (for an example of the detail with which students presented the procedure in writing a proof, see Figure 2).



11. Prove by mathematical induction that 3 divides $n^3 + 2n$. (15pts)

Proposition Let P(n) denote the proposition that 3 divides n^3+2n, i.e. its possible to factor a 3 out of n^3+2n, i.e. $n^3+2n=3(\frac{1}{3}n^3+\frac{2}{3}n)$

Step #1 P(1) is true because $(1)^3+2(1)=3$ is divisible by 3. is

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Step #2 Because P(1) is true, then we can assume P(k) is true for some fixed but arbitrary k, i.e. $k^3+2k=3(\frac{1}{3}k^3+\frac{2}{3}k)$. ← Ind. Hypothesis

Step #3 Next, we must show that we can climb the ladder by stepping up to the next wrung which is k+1, i.e. we will show P(k+1) is true, i.e. $(k+1)^3+2(k+1) = 3(\frac{1}{3}(k+1)^3+\frac{2}{3}(k+1))$.

Figure 2. Student proof by mathematical induction, from Dr. Isley’s final.

In Dr. Vale's class, students had in-class tasks where they were meant to validate each other's proofs, but students reported not always being sure that the group had created a valid proof unless it was approved by Dr. Vale. Additionally, Dr. Vale regularly provided faulty proofs with structural and syntactic/symbolic errors for students to validate (see Figure 3). Dr. Vale noted in his interview that he would likely use student proofs with "authentic errors" the next time he taught PMI.

C. Describe at least one "structural" error in the purported proof by induction then suggest at least one other thing the proof writer might do to begin developing a correct proof.

I want to show that $2 + 4 + 6 + \ldots + (4n - 2) = 2n^2$ is true for all integers $n \geq 1$.
I'll assume that $2 + 4 + 6 + \ldots + (4k - 2) = 2k^2$.
$2 + 4 + 6 + \ldots + (4(k-1)-2) + (4k - 2) - (4k-2) = 2k^2 - (4k - 2) =$
$2k^2 - 4k + 2 = 2(k^2 - 2k + 1) = 2(k - 1)^2$
By the Principle of Mathematical Induction, $2 + 4 + 6 + \ldots + (4n - 2) = 2n^2$ is true for all integers $n \geq 1$.

C) 1. One thing is describe the sum as a recursion! $b_n = 4(n-2) + b_{n-1}$. Also at least check one value.

2. Clearly write where k+1 is introduced, I'm not sure where it is.
$2+4+6+\dots+(4k-2)+(4(k+1)-2) = 2k^2+4(k+1)-2$

Figure 3. Student validation, graded as completely correct by Dr. Vale.

Just under half of the students in both courses wrote a complete and correct proof by mathematical induction on the final. Also, about half (not necessarily the same students) in both classes completely and correctly validated a proof by mathematical induction on the final exam (see Figure 4).

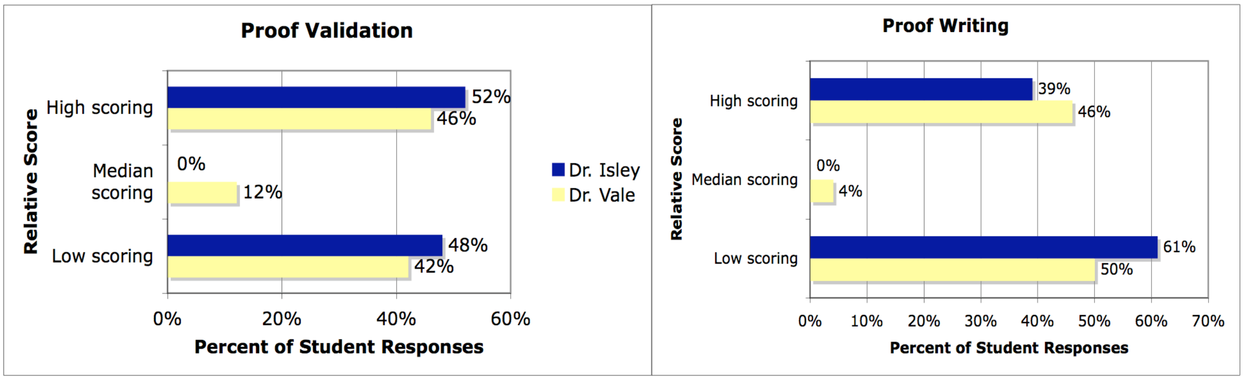

Figure 4. Instructor scoring of student performance on final exam tasks.

Infinite Iteration

A potential category of epistemological obstacle[2] emerged from what we had, at first, seen as an issue of didactical obstacle. Coding of student work on the common final exam items led us to identify two challenges, one associated with each instructor. In Dr. Isley's case, the use of non-standard terminology in-class appeared to be associated with idiosyncratic language/symbol use in communicating proofs. For example, in Figure 2 the student asserts "Next, we must show that we can climb the ladder by stepping up to the next wrung." While Dr. Isley and his students saw the ladder analogy as useful in learning about PMI, such a statement on a proof in another context (e.g., in another instructor's class) might not have had much meaning. In Dr. Vale's class, students worked a great deal with recursive representations of relationships (e.g., defining a_k in terms of a_{k-1} – note that in Figure 3 the student suggests defining the relationship recursively). The symbolic foundation of working with recursion appeared to be associated with errors in some students' proofs, such as several students writing or validating an inductive step based on showing $P(k)$ implies $P(k\text{-}1)$ rather than $P(k)$ implies

[2] Here we refer to an aspect of PMI that is challenging and yet it is necessary to tackle it in the restructuring of understanding that is required to build a valid concept definition for PMI.

$P(k+1)$. In both cases, we suggest that the underlying issue for learners was the meaning evoked by the notation/language used to make sense of the iteration and infinity encapsulated by the inductive step.

Discussion

Our interviews with the two instructors suggested that a key difference in their perceptions of, and interactions with, teaching of PMI was connected to their epistemological stances. Both professors referred to "how people learn" and both communicated their epistemologies in part using common educational research ideas (e.g., "information processing," "constructivist," and "positive reinforcement").

Dr. Isley relied on conjectures he made from student products, like proofs, and from observing students' behavior in his office when getting ready to do their presentations. Dr. Isley reported shaping students' performance in PMI in terms of Skinnerian operant conditioning. He both positively and negatively "reinforced" knowing the three-step PMI procedure: for example, by congratulations to presenters after their presentations and by putting on the board the initials of the 14 students (out of 26) who he sensed "got it" after a PMI assessment. For the presentations, Dr. Isley specified a sequence for students to follow, otherwise he would not allow them to present their work: picking a PMI question, attempting to prove it, seeing the instructor for his critique and expansion (usually a couple of times during his office hours), and then writing it up. During the interview, Dr. Isley expressed his disappointment that some students were "unable to follow the procedure and came up with something new." He regarded this situation as "normal" and noted that there was not much he could do, except continuing to reinforce the desired behavior. Dr. Isley also emphasized the importance to him of students coming to talk with him during his office hours and observing their peer's presentations. He believed this practice in working with others would build up students' confidence and reduce their fear and anxiety, as well as set up self-regulation (here, an aspect of constructivist epistemology appeared for Dr. Isley) for "being responsible" for their proving behavior.

Dr. Vale's original strategy was mainly based in constructivism with some aspects of cognitive information processing. In fact, he mentioned both of these in discussing his views about how people learn. He intended to have his students become familiar with the iterative nature of PMI and with its procedural formality by first "learning vicariously through observing" his demonstration and then he expected students to build their own mental structures and procedures for PMI through activity-based group work. Dr. Vale said that he did not see evidence in the questions students asked, or in the group work they did, to suggest that students were trying to understand. He felt, instead, they "were trying to pass!" So, he shifted his strategy toward a different kind of construction, one motivated by a need to understand. This pedagogical shift was met by students with responses ranging from "Thank God!" to "Oh, man, and I thought I didn't understand it *before*." Dr. Vale offered two kinds of goal-driven activities. One was the multi-day activity, based on Allen's (2001) short article, to leverage a potential intellectual need students might feel for the skills a high school mathematics teacher uses to teach PMI. A second set of activities provided false/purported proofs for students to "grade" – though Dr. Vale used this word in the interview, in-class he referred to "validating" the purported proofs and evidence from the focus group interviews suggests that students did not connect "validating" to "grading" as part of their future work as teachers.

Both instructors perceived themselves to be partially successful and both said there were ways they would seek to improve instruction the next time they taught the course. Dr. Isley perceived himself to be successful in getting about half the students through the first of the three stages of interaction with PMI he anticipated they would need. For Dr. Isley, future improvements in teaching would be focused around the idea of students being responsible for regulating their learning. Dr. Vale perceived himself successful as a teacher in that for most students he saw evidence of student analysis of the structure of PMI on multiple levels and in exploring the idea of infinite iteration. For Dr. Vale, future improvements in teaching PMI would come from the lessons he had learned about creating intellectual necessity for PMI before introducing it as a proof technique and before formalizing the procedural framework for PMI.

Future Research. Our ultimate goal is a research-to-practice piece about teaching PMI at the college level. Future work includes following Drs. Isley and Vale into their future instructional forays in PMI. We would like to find ways to facilitate instructors' awareness of how student thinking, in particular college students' mental constructs around inductive proof, relates to student learning; then, study the nature of such awareness and instructor response and/or follow up teaching activities, given that awareness. We also would like to find ways to facilitate instructional design related to intellectual-need-driven PMI instruction, and to study the nature of that design and its implementation, including students' perceptions of these (e.g., the received curricular value).

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