Prospective Elementary Teachers' Multiplication Schema for Fractions

JenqJong Tsay — University of Texas-Pan American — jtsay@utpa.edu

Shandy Hauk — University of Northern Colorado — shandy.hauk@unco.edu

Interviews with 12 prospective elementary and middle school teachers focused on computation, problem-posing, visualization for problem-solving, and eliciting from participants explanations of the connections among these, particularly explanations that would be accessible to children learning about multiplication. We analyze the nature of participants' conceptions and explanations and discuss implications for teacher preparation.

Background

In elementary schools, multiplication of two values is taught as a number of equivalent groups (*multiplier*) times the size of each group (*multiplicand*), with *multiplier* x *multiplicand* as the default order. (Harel, Behr, Post, & Lesh, 1994). Research with children up through middle school age, working with word problems, suggests that their choice-of-operation is strongly affected by the nature of the multiplier (Bell, Fischbein, & Taylor, 1984). For example, given two word problems with the same context but different number types may result in different choices: *Suppose peanuts cost $2 per pound. (a) What is the cost for 3 pounds of peanuts? (b) What is the cost for ½ pound of peanuts?* Students will often identify multiplication as the operation to use for (a) and division for use in (b) (Af Ekenstam & Greger, 1983). Fischbein and colleagues (1985) proposed an intuitive model to provide a theoretical account for this "non conservation of operations" (Greer, 1988): When the constraints of the underlying model are incongruent – for the learner – with the numerical data given in the problem, the choice of an inopportune arithmetic operation may occur. Though Marshall and colleagues (1989) proposed an instructional intervention based on semantic analysis to assist students in learning to match a situation to a useful schematic representation, researchers have expressed concern that such intervention may foster superficial strategies in solving world problems without helping learners to construct conceptual representations situated in the problems (Verschaffel & De Corte, 1993).

Some studies and teaching experiments have approached the learning of multiplication through *problem-posing* rather than problem-solving tasks (Fischbein et al., 1985; Lowrie, 2002). This work supposes that a generative connection to the task might support conceptual engagement during subsequent problem-solving. However, in some cases children's performance on such tasks improved only when the numbers were whole (not fractions). In other studies, researchers have examined how learners connect their solving of word problems to acting on manipulatives to solve problems, in terms of units of quantity (Behr et al., 1997). Such studies have challenged the dominance in school curriculum of the use of a context-independent interpretation of 1-unit (Steffe, 1988). To generate a mental construct, a learner needs to re-present the concept even in the absence of perceptual input (von Glaserfeld, 1982). Thus, neither students' capability in algorithmic calculation nor their competence in acting on manipulative aids cued by problem context reaches the utmost goal of constructing conceptually rich mental schema. Several researchers have worked to describe and explain the process of formation of mental constructions into object-like cognitive entities, such as *encapsulation* (Asiala et al., 1991, after Piaget), *reification* (Sfard, 1989), and *proceptual* thinking (Gray & Tall, 1994). In each case, the role of making connections among object-like cognitive entities is central.

Theoretical Perspective and Research Question

Action-Process-Object-Scheme (APOS) theory (Asiala et al., 1991) describes a hierarchical relationship among types of mental structuring (action, process, object-like entity, and schema) in which learner awareness, perception of totality, and coordination of aspects of a concept (identified by researchers through a genetic decomposition of the concept) are salient features. A style of teaching associated with APOS theory aims to assist students to move from one level to another and to gradually stabilize a developing mental construct. Reification and proceptual theories pay significant attention to the use of symbolic representation. According to Sfard (2000), in the process of reification, naming and symbolizing (creating a signifier) is no less important than the cognitive entity (signified). Gray and Tall (1994) have asserted that a flexible use of mathematical symbolism may compress action/process into object/concept, which in turn may liberate more capacity for cognitive activity and advanced mathematical thinking. Both perspectives, reification and proceptual thinking, value procedural skills in which manipulation on symbolic representations plays a central role in the process of stabilizing a mental construct into a conceptually rich understanding.

For a prospective teacher, the mathematical knowledge needed to *teach* is more than the knowledge needed to *do* mathematics (Shulman, 1986; Ball & Bass, 2000). Also important is a facility in packing and unpacking object-like understandings in order to supply explanation, and make sense of students' thinking, both in planning for instruction and in-the-moment-of-teaching (Hill, Ball, & Schilling, 2008). In particular, we started from the hypothesis that understandings of multiplication by prospective K-8 teachers would be foregrounded if they were asked to describe a particular kind of connection: that between doing multiplication in response to: (a) a symbolic statement (decontextualized) and to (b) a word problem (contextualized).

The study reported here sought to gain insight into the following questions: (1) What are the ways in which prospective grades K-8 teachers may perceive the isomorphic relationship between abstract structures (decontextualized mathematics problems) and concrete structures (contextualized or story problems) for fractions in simple multiplication? (2) What roles in problem-posing and problem-solving might fraction as APOS-object, and fraction multiplication as APOS-object (Asiala et al., 1991) play in understanding multiplication?

Design and Setting

The 12 women in this study were prospective grades K-8 teachers who had completed the first 2 of 3 semesters of teacher-preparatory mathematics at a comprehensive U.S. university. According to the instructors and textbook authors (Bennett & Nelson, 2000), the courses aimed to teach mathematics with conceptual understanding. One task-based interview with each participant (60 to 100 minutes each) formed the primary data for the study. The interview was framed in a preparing-for-mathematical-teaching context and was designed to bring to the surface participants' understandings of multiplication. Specifically, each participant worked with four numerical prompts,

$$\text{(a) } 4 \times 3 \quad \text{(b) } 4 \times \frac{5}{6} \quad \text{(c) } \frac{3}{4} \times 6 \quad \text{(d) } \frac{3}{4} \times \frac{2}{5}$$

in interviews that followed five steps: (1) computation, (2) problem-posing, (3) visualization of problem-solving, (4) sketch for visualization, and (5) comparison of ideas and material generated in Steps 1 and 4. All interviews were audio and video recorded and transcribed. Analysis was phenomenological, using constant comparative methods. In recording researcher observations about participants' interactions with tasks and in analyzing their responses we relied on Pirie and Kieren's (1994) method for diagramming a person's progressions through, and folding back among, layers of understanding (e.g., facets of action, process, object, and schema activity). We used these Pirie-Kieren models for participants' problem-posing and associated problem-solving interactions so that the dynamic patterns emergent from the interviewees' efforts could be classified into categories based on the nature of object-like entity understandings.

Results

During Step 1 of the interview, 8 of the 12 participants made no errors in computing numerical prompts. However, 4 of 12 confounded “multiply across” (e.g., $a/b \times x = a/b \times x/1 = ax/b$), with “cross multiply” (e.g., $a/b \times x = a/b \times x/1 = bx/a$, or $a/b \times x = a/b \times x/x = ax/bx$). Note that at the time of the interviews, all were enrolled in the third semester of their mathematics sequence and studying proportions, including “cross multiplying” to find the unknown value x, in a proportional equation like $a/b = x/d$. Though analysis of computational error was not the purpose of this study, participants’ procedural skills with decontextualized symbolic prompts in Step 1 may have mediated their efforts to identify isomorphisms in Step 5 of the interview. Subsequently, in Steps 2 through 5 of the interviews, five categories of object-like entities in concept building appeared to be problematic for participants: multiplier-as-operator, fraction-as-multiplicand, fraction as only a part-whole-relation, fraction-as-multiplier, and fraction multiplier acting on a fraction multiplicand.

Multiplier-as-operator. All 12 participants posed a complete story and described in words or using a sketch the process of solving for prompt (a) 4×3. However, the nature of their understandings varied. In the following excerpt, the interviewer (denoted *Int*) and Ann (all names are pseudonyms) negotiated the personality of multiplier to act on the multiplicand.

> *Ann*: [4×3] probably means that I have four pieces of candy and I have three friends. If I were to say pretend you had three candies, or three piles of candy with one in each pile, so you would have three candies and I want to add four candies to each pile, then I would add 1,2,3,4, 1, 2, 3, 4…then I would end up with the same answer [as 12 candies] and they would represent the same thing both three candies and four candies they are. I am representing candies all the way across. But in the problem

that I gave I said that you have three friends and you are giving them each four pieces of candy so the numbers represent different things.

Int: So in this case, what does the number three represent?

Ann: kids, friends, people.

Int: Four is four pieces of candy, and four candies times three people is?

Ann: Twelve.

Int: Candy or people? Have you ever thought about it?

Ann: Yeah, I have never thought of that. Your answer is candy, you end up with 12 pieces of candy. I guess your three doesn't matter like I thought it would. Okay then, the three would represent where you are putting them, like how you are separating them so how many times you need to use them like when you, often times when you use multiplication you would say four candies and I need to give them to three people, it is just like adding four three times, in relation to addition.

Int: Three groups of four?

Ann: Exactly. So your three is your groups, your four is your number within those groups.

The participant's perception of multiplier 3 went from "three friends" to "three candies" and then to "three groups" (of four candies). Her struggle in identifying the nature of multiplier concurs with Steffe's (1988) observation about the nature of unit in such contexts. The multiplier 3 is not just for 3 one-units as a number of groups, but also for 3 units (groups) of 4 one-units (candies). In the interview excerpt, Ann's understanding of multiplier-as-operator with natural number may be seen as moving from shaky *process* toward *object*.

Fraction-as-multiplicand. In working with (b) 4 x 5/6, two of the participants did not recall an appropriate property of positive integer as multiplier. Beth's story for the prompt 4 x 3 was, “John has grouped four groups of three marbles in each group.” Here, the multiplier 4 played an explicit role as operator. However, Beth did not *conserve* the operation (Greer, 1988) in going from 4 x 3 to prompt (b) 4 x 5/6. In Step 3, visualizing her problem-solving, for (b), Beth changed representations and rewrote the whole number multiplier as a fraction, 4 x 5/6 = 4/1x5/6 = 20/6 = 3 1/3, and altered its personality as multiplicative operator (see Figure 1).

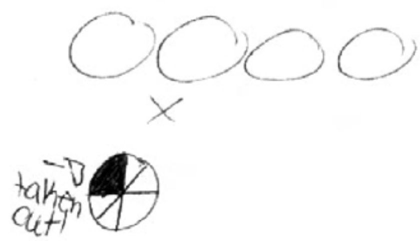

Figure 1. Beth’s Step 3, visualizing problem solving of 4 x 5/6.

Beth: It helps to think about the four as a fraction, so like four over one. That helps because it puts it in, both in the same context.

Int: So now we have, an integer times a fraction, any idea where to start?

Beth: Um, well you have four wholes. So I am just going to go ahead and draw them here, four wholes. Then we have five [out of six], almost a whole.

Int: Okay, now I would like you to compare your sketch to its numerical calculation. The number one and the number five-sixths, which one is bigger?

Beth: One. So five-sixths is less than a whole, duh. Let’s try that one again. So this [sketch] is not good, we are just going to forget about that.

Beth: Okay. We still have four wholes, correct?

The interviewer drew Beth’s attention to her story for 4 x 3 and asked her to pose a story for (b) in an analogous way. When the multiplier is a whole number, the multiplication-as-repeated-addition model can also work for a fractional multiplicand.

Beth: Oh, could you have four groups of five-sixths, does that work? You would have four of these, so these are all like five-sixths?

Int: Does it make more sense?

Beth: I think so but I don't know how to explain it…

Int: I have four groups. What is inside each group?

Beth: Not even one, a part of one.

Int: A part of one. How many ones? How many sixths…

Beth: Hey, that works.

As proposed by Fischbein et al. (1985), one of the values in 4×5/6 is incongruent to 4×3. In terms of unit types, the numerical prompt 4×5/6, thought of as 4(5(1/6(1))), is one more layer than 4×3, or 4(3(1)). We suspect Beth may not have chosen a useful arithmetic operation because of this incongruence. She may have had, at the time of the interview, a schema of fraction multiplication that was repeated *action*-based, essentially additive, and was the only schema she recalled in the moment.

Fraction as exclusively a part-whole-relationship. Most of the participants contextualized and visualized a fraction numerical prompt. However, 7 of the 12 participants' understandings of fraction seemed to be confined to the part-whole-relationship personality. Kieren (1980) differentiated part-whole-relationship personality from measure personality for the rational number x/y. In the former, some whole is split up into y parts and x of these parts are taken. The latter sees $1/y$ as a unit to be used repeatedly to determine an x/y quantity. In the following excerpt, Cher's conception of 5/6 included five out of six pieces but the idea of 5(1/6), that is of five units of size one-sixth did not appear to come to mind for her.

Cher: 5/6… [means] something is divided into six portions, there is five remaining [of 6]

Int: How about 6/5?

Cher: 6/5, um there were two something that were divided into six [five] pieces, the remainder of what is left is one full one and one of the six [five]…

An understanding of fractions that is exclusively part-whole could be a challenge for participants in tracing the connections between transformation of units in their problem-posing and problem-solving visualization efforts – these multiplication algorithms are mainly based on measure personality. We saw some additional evidence to support this result in Daisy's interview.

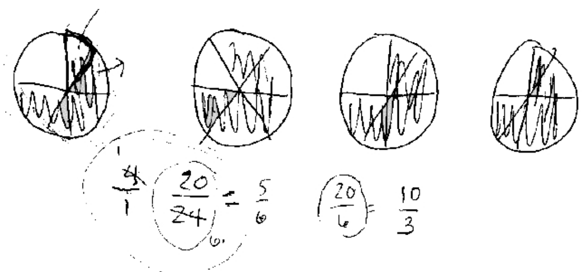

Figure 2. Daisy's visualizing problem solving of 4 x 5/6.

Daisy: [For 5/6,] there are six pieces so five of them are colored in because it is five of the six pieces and that is where the 5/6 comes in, and then I did four of them for the four times the 5/6. So I know the 20 of the 24 pieces, are colored which would equal 5/6, which doesn't help me out though because I am stuck at – Oh, what if I did four of them, one, 5/6 then I am left with 20/6 which could be reduced to 10/3 but then I don't know that leads me to so...

Daisy's understanding of fraction as part-whole-relation led her to see 4x5/6 as (4x5)/(4x6), or 20 out of 24, which was inconsistent with her computation of 4x(5/6)=20/6=10/3. Mathematically, her computational procedure and sketch matched perfectly. Psychologically, she did not perceive the personality of 6 in 5/6 and in 20/6 as one type of measuring unit, 1/6, derived from partitioning one into six equal parts, or as units of measure 1/6.

Fraction-as-multiplier. Several participants called on the symmetric property (axb=bxa) and said prompts (b) and (c) were "exactly the same." The request to create a story where a fraction was a multiplier acting on a whole number, challenged participants' belief in the symmetric property and was, ultimately, not fruitful during interviews. However, prompt (d) 3/4x2/5 involved only fractions, so fraction-as-multiplier was required in some way. In posing a story and/or visualizing problem-solving for (d), 5 of the 12 participants used addition. Elda immediately posed a story for the prompt 4x5/6, but had five unsuccessful attempts on 3/4x5/6. Elda's first attempt was: "If Megan was making a pasta dish and it asked for 3/4 cup of milk and 2/5 cup of salt. How many cups are needed to make the pasta dish?" She immediately realized what she posed involved the operation of addition rather than multiplication. She tried again:

Elda: Megan's pasta dish called for 3/4 cup of milk but she put in 2/5. How much did she forget to put in?

Int: Did you mean to put 2/5 *of the 3/4 cup*?

Elda: Like, she put in 3 tablespoons instead, how much did she put in?

Later, she contextualized 2/5 as "2/5 cups," though Elda may have tried to express 2/5 *of* the 3/4 cup, in which "3 tablespoons" was about 2/5 of 3/4 cup in her sense. What Elda said might mean a mental structure where 2/5 acted on 3/4 to get "3 tablespoons," or it could be her second attempt was another additive one (in this case, subtraction). In drawing the visualization of her problem solving, Elda tried three times, but each time her strategy involved addition only and Elda seemed to be aware that her attempts did not use multiplication.

Fraction multiplier acting on a fraction multiplicand. Flora created a story for prompt (d) 3/4x2/5 involving the concept of group size and number of groups where 3/4 was group size. However, instead of 2/5 as two groups of measure one-fifth, of something else (i.e., 2((1/5(1)) where the inner 1 represents one whole group of three-fourths), she used “two out of the five” groups. In her drawing, she had five groups of 3/4. She circled two of them and added them together to get 3/2 and said,

> *Flora:* So here are my five groups of three-fourth. And – I want to add these two [groups of 3/4] together. So is, that’s the same as three and a half [3/4 x 2/1 =6/4 = 3/2] – No. – Yeah. So that is it. So I need to say two-out-of-the-five somewhere.

Flora saw that 2/5 was not an operator that acted on three-fourths but also clearly articulated understanding of fraction as part-whole: “I need to say two-out-of-the-five somewhere.”

Gina also used part-whole relationship to assign context to the multiplier 2/5.

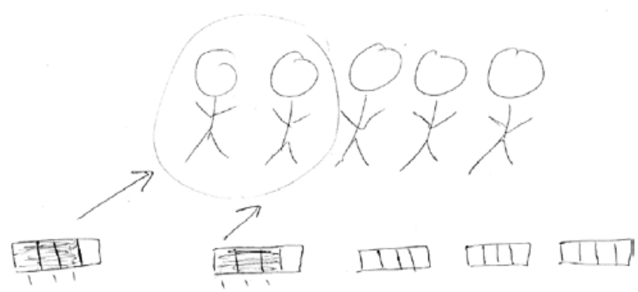

Figure 3. Gina’s visualizing problem solving of 3/4 x 2/5.

Gina had five candy bars with four pieces in each candy bar and 20 pieces total. Instead of 2/5 acting on 3/4, she had 2/5 act on 5 and 3/4 act on 4 1-units, i.e., 2/5(5(3/4(4(1)))), to have 6 out of 20 pieces as the contextualization for 3/4x2/5. By doing so, there was no need to conceptualize a transformation of units like 1/5 (or 1/4), 1/20, or to form different unit types.

Conclusion

One can perform actions on an object, physically or mentally, as in Daisy’s sketch for visualizing problem solving of 4x5/6, and adopt symbols to represent it like “4x5/6=20/6=10/3,” without bringing to mind some properties in the process. An encapsulation of the incomplete process into a sort of *pseudo object* may lead to a "pseudostructural" conception (Sfard, 2000). We hypothesize that all of the 12 participants, like Elda, had experienced *action* of fraction multiplication. But they may not have had awareness of all key properties in the *process*, and therefore did not perceive the process as a totality. A pseudostructural object of fraction multiplication appeared to be sufficient for many to choose an appropriate operation for solving a given word problem. However, the visualization task called for de-encapsulation or unpacking of both *number* and *operation*, from object back to process and action.

De Corte's (1988) empirical study supported the claim of Fischbein et al. (1985) that children's difficulties in solving multiplication word problems may arise when their underlying models are incongruent with the numerical data given in the problem. This study suggests that for adult prospective teachers, a complex version of incongruity is at work. In unpacking both understanding of number and of operations, several sites for incongruity emerge. Cognitively, for the same operation (multiplication) the fraction multiplicative structure is not congruent with whole number multiplicative structure. Teaching with emphasis either on identifying key features from word problems and procedural skills or on concrete experience is, although necessary, not sufficient for learners to construct complexly connected cognitive objects that can be untangled from multiple potential incongruities. This suggests that a richly connected and "unpackable" understanding of multiplication for positive rational numbers may require an equally complex constellation of ways to identify and respond to incongruity. That is, we suggest that this study offers empirical support for the assertion of many that mathematical discourse incorporating procedural skills, problem posing, visualization, and identifying isomorphic corresponding relationships can all play valuable roles in arousing learners' awareness of actions and process, in reifying and encapsulating mental constructs into object-like entities, unpacking or de-encapsulating the same, and using symbolism flexibly to advance mathematical thinking.

Finally, our experience in interviews with prompt (c) and its challenge to participants' belief in the symmetric property leads us to the following suggestion for teacher-educators. In working with prospective teachers, consider working with the abovementioned constellation of activities in the context of multiplication of two fractions (as in prompt (d)) before situations with one fraction; and then address a similar constellation of activities in connecting and unpacking the ideas of *fraction of* and *out of* to move into the context of fraction as multiplier acting on whole number multiplicand (e.g., problems like prompt (c)).

References

Af Ekenstam, A., & Greger, K. (1983). Some aspects of children's ability to solve mathematical problems. *Educational Studies in Mathematics, 14*, 369-384.

Asiala, M., Brown, A., Devries, D. J., Dubinsky, E., Mathews, D., & Thomas, K. (1991). A framework for research and curriculum development in undergraduate mathematics education. In J. Kaput, A. H. Schoenfeld, & E. Dubinsky (Eds.), *Research in collegiate mathematics education II* (pp. 1-32). Providence, RI: American Mathematical Society.

Ball, D. L., & Bass, H. (2000). Interweaving content and pedagogy in teaching and learning to teach: Knowing and using mathematics. In J. Boaler (Ed.), *Multiple perspectives on the teaching and learning mathematics* (pp. 83-104). Westport, CT: Ablex.

Behr, M., Harel, G., Post, T., & Lesh, R. (1994). Units of quantity: A conceptual basis common to additive and multiplicative structure. In G. Harel & J. Confrey (Eds.), The development of multiplicative reasoning in the learning of mathematics. (pp. 123-180). Albany, NY: SUNY Press.

Behr, M., Khoury, H. A., Harel, G., Post, T., & Lesh, R. (1997). Conceptual units analysis of preservice elementary school teachers' strategies on a rational-number-as-operator task. *Journal for Research in Mathematics Education, 28*, 48-69.

Bell, A.W., Fischbein, E., & Taylor, G. (1984), Choice of operation in verbal arithmetic problem: The effects of number size, problem structure and context. *Education Studies in Mathematics, 15*, 129-147.

Bennett, A., & Nelson, L. (2000). *Mathematics for elementary teachers: A conceptual approach* Madison, WI: McGraw Hill.

De Corte, E., Verschaffel, L., & Van Coillie, V. (1988). Influence of number size, problem structure and response mode on children's solutions of multiplication word problems. *Journal of Mathematical Behavior, 7,* 197–216.

Fischbein, E., Deri, M., Nello, M. S., & Marino, M. S. (1985). The role of implicit models in solving verbal problems in multiplication and division. *Journal for Research in Mathematics Education, 16*, 3–17.

Gray, E. M. & Tall, D. O. (1994). Duality, ambiguity and flexibility: A proceptual view of simple arithmetic. *Journal for Research in Mathematics Education, 25*, 115–141.

Greer, B. (1988). Nonconservation of multiplication and division: Analysis of a symptom. *Journal of Mathematical Behavior*, *7*, 281-298.

Harel, H., Behr, M., Post T., & Lesh R. (1994). The impact of the number type on the solution of multiplication and division problems: Further considerations. In G. Harel & J. Confrey (Eds.), *The development of multiplicative reasoning in the learning of mathematics* (pp. 365-388). Albany, NY: SUNY Press.

Hill, H.C., Ball, D.L., & Schilling, S.G. (2008). Unpacking "pedagogical content knowledge": Conceptualizing and measuring teachers' topic-specific knowledge of students. *Journal for Research in Mathematics Education.*

Kieren, T. E. (1980). *Recent research on number learning*. Washington, DC: U.S. Department of Education. (ERIC Document Reproduction Service No. ED212463)

Lowrie, T. (2002). Young children posing problems: The influence of teacher intervention on the type of problems children pose. *Mathematics Education Research Journal 14*(2), 87-98.

Marshall, S. P., Barthuli, M. A., Brewer, & Rose, F. E. (1989). *Story problem solver: A schema-based system of instruction*. San Diego, CA: San Diego State University.

Pirie, S. E. B., & Kieren, T. (1994). Growth in mathematical understanding: How can we characterize it and how can we represent it? In P. Cobb (Ed.), *Learning mathematics: Constructivist and interactionist theories of mathematical development.* Boston: Kluwer.

Sfard, A. (2000). Symbolizing mathematical reality into being: How mathematical discourse and mathematical objects create each other. In P. Cobb, K. E. Yackel and K. McClain (Eds.), *Symbolizing and communicating, perspectives on mathematical discourse, tools, and instructional design* (pp. 37-98). Mahwah, NJ: Erlbaum.

Shulman, L. S. (1986). Those who understand: Knowledge growth in teaching. *Educational Researcher, 15*(2), 4-14.

Steffe, L. (1988). Children's construction of number sequences and multiplying schemes. In J. Heibert and M. Behr (Eds.), *Research agenda for mathematics education: Number concepts and operations in middle grades* (pp. 119-140). Reston, VA: National Council of Teachers of Mathematics.

Verschaffel, L. & De Corte, E. (1993). A decade of research on word-problem solving in Leuven: Theoretical, methodological, and practical outcomes. *Educational Psychology Review, 5*(3), 1-18.

von Glasersfeld, E. (1982). The role of figural patterns in the development of numerical concepts. *Archives de Psychologie*, *50,* 191-218.