Task Design: Towards Promoting a Geometric Conceptualization of Linear Transformation and Change of Basis

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Reform-oriented mathematics education calls for instruction that assists students in developing from their current ways of reasoning into more complex and formal mathematical reasoning (e.g., Gravemeijer, 2004). Such instruction centers on a host of theoretical and pragmatic concerns, and our research explores many of these concerns in a first course in undergraduate linear algebra. Our work draws on the theory of Realistic Mathematics Education (Freudenthal, 1991) to design instructional sequences that build on student concepts and reasoning as the starting point from which more complex and formal reasoning develops. This emphasis on creating a classroom environment that fosters students' ability to reinvent advanced mathematical concepts served as inspiration for the development of an instructional sequence with the particular goal of fostering a geometric conceptualization of linear transformation and change of basis in R^2. This paper will focus on a particular task from the aforementioned instructional sequence, the Change of Basis Task, that was developed with the intent of leading to a student-driven reinvention of the well-known equation $A = PDP^{-1}$, for the case where A, P, P^{-1} and D are 2x2 matrices. In this paper, I will detail some of the student responses to and interactions with the Change of Basis Task, as well as elaborate upon what affordances the inclusion of the Change of Basis Task may provide in an upcoming semester-long classroom teaching experiment.

Larger Research Setting

The aforementioned instructional sequence is part of larger study that addresses the transition from informal to formal reasoning of students within the context of undergraduate linear algebra. The NSF-funded project, *Investing Issues of the Individual and Collective along a Continuum between Informal and Formal Reasoning*, is composed of two main research strands: a methodological and theoretical strand, and a linear algebra stand. Within the linear algebra strand, there are three primary goals: 1) to detail student conceptions of fundamental ideas; 2) to trace students' intellectual growth from informal to more formal ways of reasoning; and 3) to create instructional sequences and detail teacher moves to support student growth. This paper falls within the goals of the aforementioned linear algebra strand. In particular, the aim of the research reported in this paper is to detail student conceptions that were elicited during interaction with a specific task sequence so that this information may be used both to modify the task sequence and leverage it in such a way as to promote student intellectual growth from informal to more formal ways of reasoning.

However, research has shown that linear algebra poses a number of significant challenges for students in moving from informal to formal ways of reasoning. For instance, Hillel (2000) examines different modes of representation and students' difficulties with moving between them, whereas Carlson (1993) discusses possible reasons, such as a lack of connection to familiar concepts and a premature formalization of unfamiliar concepts, for the difficulties that students have in undergraduate linear algebra courses. Harel (1997) suggests ways to alter instruction so as to promote a higher level of student success. He suggests, for instance, to spend time trying to promote students' effective concept images of ideas in linear algebra, as well as to aim to create in students an intellectual need for the concepts that are to be learned. With the desire to promote students' view of themselves as active learners and participants in the practice of mathematics, our work draws on the theory of Realistic Mathematics education (RME) to design instructional sequences that build on student concepts and reasoning as the starting point from which more complex and formal reasoning develops.

In the fall semester of 2007, our research team conducted a four-week teaching experiment (Cobb, 2000) in an introductory linear algebra course at a large public university. The content for this teaching experiment was focused on a particular unit in the linear algebra curriculum—Eigen theory. Prior to this two-week unit, the research team conducted semi-structured interviews (Bernard, 1988) with individual students, with the aim of learning about how students conceptualize particular topics in linear algebra. Specifically, our questions focused on vectors, vector operations, matrix multiplication, linear dependence, and span (Larson & Smith, 2008). After the semester, we conducted a similar set of interviews, focusing on the same concepts as well as ideas from the Eigen Theory unit (Larson, Zandieh, & Rasmussen, 2008). As Larson et al. noted, familiarity with geometric interpretations of linear dependence and determinants afforded the students great insight and creativity when introduced to concepts of Eigen theory. Furthermore, upon reviewing the classroom video, we noted several instances in which the students used geometric interpretations of particular ideas as taken-as-if-shared (Stephan and Rasmussen, 2002) in both small group and whole class discussions. Because of this, we began to think about ways to foster other salient connections such as these in the upcoming semester-long teaching experiment (currently underway). We have been analyzing our previously collected data, focusing on particularly significant conceptualizations that could serve as underlying 'big ideas' that connect the ideas of the entire semester. For us, one such idea could the connection between algebraic and geometric ways of approaching the same task. We agree with Sierpinska (2000), who claims that geometric, arithmetic, and structural reasoning and the ability to move between these are fundamentally important in learning and understanding the core ideas of linear algebra. Thus, we aim to create tasks that build off of students' geometric understanding of the plane in order to develop rich conceptualizations of various linear algebra ideas, such as vector, matrix, linear independence, and basis.

Upon reflection of the teaching experiment data, I became interested in student conceptions of linear transformation. In particular, I was curious about how we could foster students' development of a connection between the matrix representation of a linear transformation and its geometric interpretation in R^2. In addition, I was curious about the possibility of fostering an intuition about the 'appearance' of a matrix and its corresponding linear transformation relative to some basis. For instance, can we look at any 2x2 matrix and say anything qualitatively about what the transformation it represents does to the plane? In the following section, I detail the instructional design theory from which I elicited a methodology for creating a task sequence to pilot in a small-group setting with university students. The task sequence was piloted in a setting most closely aligned with conjecture-driven design research (Confrey & Lachance, 2000), which is also detailed below.

Theoretical Background

I grounded my research in the tenet that mathematics is essentially a human activity and in the instructional design heuristics of Realistic Mathematics Education (Freudenthal, 1991). RME theory is about knowledge construction, where experientially real contexts for the students can be used as starting points for progressive mathematization (Gravemeijer, 1999). This progressive mathematization is "embodied in the core heuristics of guided reinvention and emergent models. Guided reinvention speaks to the need to locate instructional starting points that are experientially real to students and that take into account students' current mathematical ways of knowing" (Rasmussen & Kwon, 2007, p. 191). In this section, I will discuss how the heuristic of guided reinvention was paramount in the creation of the task sequence, while the heuristic of emergent models will be addressed in the discussion section.

I chose the Italicizing N Task (see Figure 1) as an experientially real starting point for the task sequence because of the familiarity of the situation, as well as the existence of multiple potential entry points into the task. As I aimed to create subsequent tasks, I found myself repeatedly making use of the $A = PDP^{-1}$ equation (in order to create transformation matrices relative to particular bases) without really thinking about why or how that worked for my purposes. Soon I found myself asking, "*Wait, how could I have invented this?*" This sentiment, as I found later in my research, is one shared by others as well. Consider the following excerpt from Gravemeijer and Doorman (1999):

> Well-chosen context problems offer opportunities for the students to develop informal, highly context-specific solution strategies. These informal solution procedures then, may function as foothold inventions, or as catalysts for curtailment, formalization or generalization...The instructional designer tries to construe a set of context problems that can lead to a series of processes of horizontal and vertical mathematization that together result in the reinvention of the mathematics that one is aiming for. Basically, the guiding question for the designers is: *How could I have invented this*? (p. 117)

As I began to explore possible answers to that question, the task sequence began to evolve in such a way that the culmination of the task sequence could allow for the possibility of students reinventing the equation for themselves.

Methods

The Change of Basis Task was first piloted with members of our research team. This led to minor alterations in the task, such as wording and aesthetic issues. Next, the aforementioned instructional sequence, of which the Change of Basis Task is a part, was piloted with three different pairs of students. All of the students were mathematics majors that had previously completed a first course in linear algebra. All design experiments (Confrey & Lachance, 2000) were videorecorded and selectively transcribed for subsequent analysis. The experiment lasted from two to three hours, with the Change of Basis Task appearing in the second half. The first half was spent on two tasks—the Italicizing N Task and the Rubber Sheet Task—that emphasized the geometric interpretation of linear transformations of the plane, as well as on investigations of how the particular location and value of entries in a matrix influence the resulting transformation that the matrix represents.

As mentioned above, the process of piloting the task sequence with the student pairs is most closely aligned with the methodology of conjecture-driven design research (Confrey and Lachance, 2000). Confrey and Lachance define a conjecture in mathematics education as, "a means to reconceptualize the ways in which to approach both the content and the pedagogy of a set of mathematical topics" (p. 235). They enumerate two dimensions to a conjecture—the mathematical content dimension and the pedagogical dimension. For the design experiment discussed in this paper, I conjectured that developing geometric conceptualizations of linear transformations may potentially lead to students sensing a need for different basis systems (content dimension), and that aspects of this task sequence may contribute to creating such an environment (pedagogical dimension).

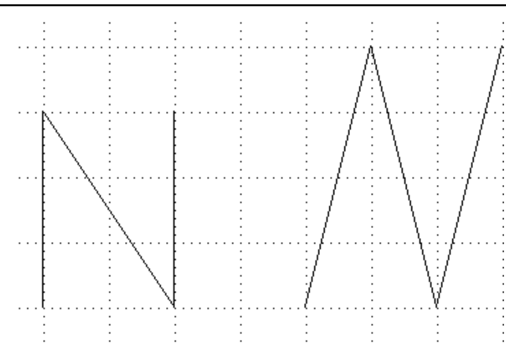
In the results section, I will focus only on the results of the design-experiment with Marissa and Sophie. Including relevant transcript and student work, I will discuss their interaction with each question in the Change of Basis Task, highlight some of the salient features were for them, and conclude with the details of the emergence of the $AP = PD$ equation. In the subsequent discussion section, I address my second research question, which explores how the conceptions elicited through interaction with the task sequence could be leveraged into more formal ways of reasoning about ideas fundamental to linear algebra. In particular, I use the RME heuristic of emergent models (Gravemeijer, 1999) as a tool for analyzing the potential of this task sequence to elicit a model of / model for shift in student thinking regarding matrices as linear transformations.

Results

In this section, I describe the three tasks that comprised the design experiment's task sequence. The first two tasks, the Italicizing N Task and the Rubber Sheet Task, will receive a cursory treatment, as a full description of the tasks and the accompanying student work is beyond the scope of this paper. The bulk of this section, therefore, will be spent detailing portions of Marissa and Sophie's work on each of the three questions in the Change of Basis Task. The purpose of doing so is in fulfillment of my first research question, asking what may be some of the student conceptions that are elicited during a guided reinvention of the equation $A = PDP^{-1}$.

The Italicizing N Task

The first task I asked Marissa and Sophie to engage with was The Italicizing N Task (see Figure 1). This task had been used the previous year in a semester-long teaching experiment (Cobb, 2000) and had proved to be a valuable resource for students in thinking about various aspects of linear transformation, such as order of operations and inverse.



Suppose the 'N' on the left is written in regular 12-point font. Find a matrix A that will transform 'N' into the letter on the right, which is written in italics in 16-point font.

Figure 1. The Italicizing N Task.

As seen above, the task prompt asked for a matrix *A,* but no specifications about *A* were given. However, Marissa and Sophie immediately agreed that they wanted a square matrix in order to be able to make use of determinants and inverses. They worked on this task for approximately 45 minutes, during which their approaches varied considerably. For instance, they first tried finding a matrix that sent the line segments of the first N (written in slope-intercept form) to the corresponding line segments in the second N. When they could not think of a way to coordinate this information into an equation in which *A* was a square matrix, Marissa said, "we could look at what happens to the points instead of the lines."

After a bit more work, Marissa and Sophie found the correct matrix requested in the task, the matrix $A = \begin{bmatrix} 1 & 1/3 \\ 0 & 4/3 \end{bmatrix}$. The next thing I asked them to consider was whether or not they could tell if their answer "made sense."

MW: What my question is, is there a way we can just look at this and tell if it is right? And by that I mean, you know, if you are doing a physics problem and you're trying to find the height of the building, if you actually think about it and you get, like, negative twenty-seven as the height of the building, it's wrong, right? [Girls: Right.] You can just tell it is wrong. So is there any way we can look at this matrix and be like, yeah, that actually makes sense. I bet that actually works.

This statement prompted a 10-minute exploration and discussion into what we could tell about transformations from their matrix representations. For instance, we began to discuss the identity matrix and how it does not 'affect the plane.' This led to an informal discussion of the standard basis vectors[1] and how we could use what we know about what happens to them (under the transformation) in order to determine what happens to the entire plane. The girls conjectured about what influence particular matrix components had on a transformation, determining that the upper diagonal entry (on a 2x2 matrix) stretched the horizontal direction of an image, the lower

[1] It is important to note that the term 'basis' was not used at all during the entire interview. A conscious effort was made to use familiar notions, such as 'coordinate system,' as starting points.

diagonal entry stretched the vertical direction, and the upper off-diagonal entry 'pushed' the image to the left or right.

The Rubber Sheet Task

The sequence's next task, the Rubber Sheet Task (see Figures 2 and 3), was presented to Marissa and Sophie one question at a time (it is condensed here for ease of presentation).

Imagine that you have a thin rubber sheet. You (and your friend, if you need the help) stretch the rubber sheet in one direction and keep it fixed in another direction.

In the $y = -3x$ direction, you pull the sheet by a factor of three.
In the $y = x$ direction, you keep the sheet fixed.

1. Try to sketch a relatively accurate "before and after" shot of this rubber sheet. Qualitatively sketch what you predict would occur.
2. Use the given image (see Figure 3a) to try and be as accurate as possible.
3. Predict (using Figure 3b) what will happen to $\mathbf{v}_1 = \begin{bmatrix} 2 \\ 0 \end{bmatrix}$ and to $\mathbf{v}_2 = \begin{bmatrix} -1 \\ 1 \end{bmatrix}$.
4. Develop a method for checking your prediction.

Figure 2. The Rubber Sheet Task

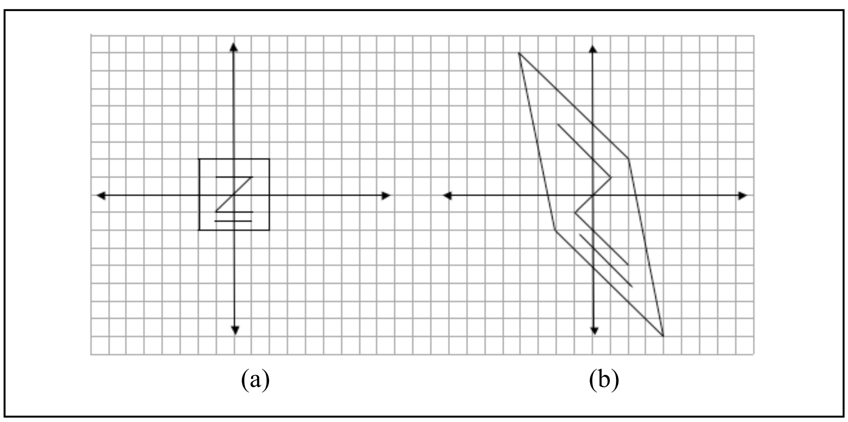


Figure 3. The original and transformed rubber sheet

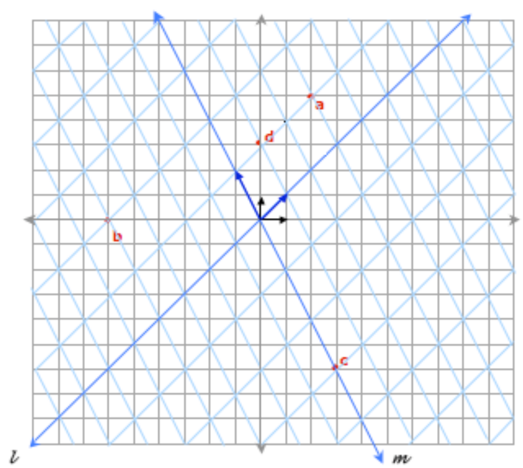
Note that here, rather than giving the graphical inscription representing the image, the task gave a verbal description and asked the students to derive the graphical interpretation. The intention of this was to bring to the foreground the idea that the entire space (rather than just a region, such as the letter ‘N,’ or particular vectors) moved according to the transformation, as well as to begin to introduce ‘stretch factors’ along particular lines. The Task overall was challenging, but question 2 seemed especially difficult, most likely because the line $y = -3x$ has no obvious visual correlation with the *Z* image. Furthermore, Marissa expressed that it would have been nice is they could have used something other than the standard coordinate system. Thus, I argue that the Rubber Sheet Task, together with the Italicized N Task, helped to begin cultivating a need for the notion of change of basis. It is to Marissa and Sophie’s informal exploration of change of basis that we now turn.

The Change of Basis Task

Marissa, Sophie, and I spent our last hour together working on the Change of Basis Task (see Figure 4). Because I did not assume the students to have prior knowledge of basis, this title was not given to them; rather, their paper was entitled, “Location, Location, Location.” I introduced the task by connecting to a sentiment Marissa had verbalized while working on the Rubber Sheet Task. Marissa had expressed that in the Italicizing N Task, they could tell their answer for matrix *A* was correct because the columns of the transformation matrix *A* represented the transformed vectors <1, 0> and <0, 1>, and it was easy to see this correspondence in the geometric representation of the transformation. In the Rubber Sheet Task, however, knowing what happened to those two vectors was not very helpful when thinking about the transformation.

MW: So the next one here, we’re going to work off of what Marissa just said...And so Marissa said that it would have been more helpful if we could look at what happened to the, along that line [the line of pulling] as a column. [Girls nod.]

I then handed the girls the graph and the first question of the Change of Basis Task (see Figure 4). Note that there are three questions for this task. As in the Rubber Sheet Task, the girls saw one question at time. After taking a few moments to examine the graph provided, we discussed how we could think about describing point locations other than with the 'standard' labeling system. On the graph in Figure 4, there are gridlines corresponding to the standard *x, y* coordinate system—in the task, this was called the 'black coordinate system.' There is a second coordinate system, which was in blue on the girls' task sheets. The 'blue coordinate system' corresponds to the coordinate system that has $y = x$ and $y = -2x$ as its axes. Finally, note that each of the four axes has a 'component vector' for that direction.



1. Write the coordinate of each of the above points relative to both the blue and the black coordinate systems.

2. Devise a method for switching from one labeling system to another.

3. Imagine there is a transformation T that stretches the plane by a factor of 4 in the *l* direction and keeps it fixed in the *m* direction. Devise a method that takes the vector **v** = <2,1> relative to the blue coordinate system and finds its transformed components, relative to the black coordinate system.

Figure 4. The Change of Basis Task

Question 1: Naming points according to both systems. In Figure 5, we see Sophie's work on Question 1. The left-hand column represents each point in the black coordinate system and the right-most column represents the same point's 'name' in the blue coordinate system.

a: $\begin{bmatrix} 2 \\ 5 \end{bmatrix} \to \begin{bmatrix} 3 \\ 1 \end{bmatrix}$

b: $\begin{bmatrix} -6 \\ 0 \end{bmatrix} \to \begin{bmatrix} -4 \\ 2 \end{bmatrix}$

c: $\begin{bmatrix} 3 \\ -6 \end{bmatrix} \to \begin{bmatrix} 0 \\ -3 \end{bmatrix}$

d. $\begin{bmatrix} 0 \\ 3 \end{bmatrix} \to \begin{bmatrix} 1 \\ 1 \end{bmatrix}$

Figure 5. Sophie's written work on Question 1.

Note that Sophie wrote each point as a vector. This notation is consistent with Marissa and Sophie's work on the Italicizing N Task, in which they began to write points of interest (often the vertices of the original letter N) as vectors that got transformed into other vectors. I include this information about the girls' work on Question 1 in order to facilitate discussion of their work on the subsequent questions. In the next section regarding Question 2 of the Change of Basis Task, I detail what I see as two significant episodes in order to draw attention to two different ways in which Marissa and Sophie interacted with the notion of an inverse matrix.

Question 2: Switching between naming systems. The next part of the task involved developing a method for switching between the two labeling systems (see Figure 4). As soon as Marissa heard the task, she stated, "So it's making one big matrix with all this." The girls then discussed the problem, deciding to take all the vectors in black to their counterparts in blue first (they had said they could also go from blue to black). Marissa then noticed a problem with their approach of using all four vectors in their matrix.

Marissa: I think we can get it—Oh, wait. Can we get a determinant? Can we get the inverse? Because it's going to be—
Sophie: Oh, yeah. Huh.
Marissa: It will be 2 by 4.
Sophie: 2 by 2. We can do 2 by 2.

Sophie then explained that they could pair vectors *a* and *b* to make a “little square.” After a quick statement that this fix would allow them to find a determinant, the girls discussed which vectors to pair together.

Marissa: So which one we going to do? *ab* and *cd*?
Sophie: *ab* together, and c*d* together?
Marissa: Because I think it should be the same answer. If we do *ab*, *ac*, *ad*. [Sophie: yeah] Any combination should be the same because it’s the same...I mean, we should try more than one to see if it’s the same.

The girls decided that Marissa would work with the pair of vectors *a* and *b*, that Sophie would work with the vectors *c* and *d*, and that they should get the same answer in the end. The girls’ written work in setting up this situation is given in Figure 6. Note that they had a matrix *A* (which they hypothesize will be the same matrix *A* for both of them) acting upon 2 x 2 matrices. These 2 x 2 matrices are composed of column vectors that represent the vectors *a-d* in the black system (see Figure 5). The resultant matrices (the matrices on the right side of the equation) are composed of column vectors that represent the corresponding vectors *a-d* in the blue system.

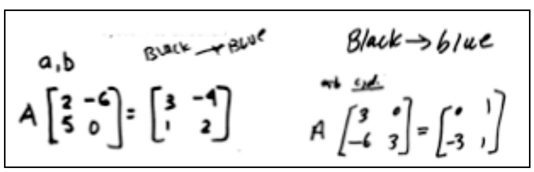


Figure 6. Marissa (left) and Sophie’s work (right) on finding the matrix *A*.

The girls worked independently to calculate the values for the matrix *A* and arrived at the same answer, $A = \begin{bmatrix} 2/3 & 1/3 \\ -1/3 & 1/3 \end{bmatrix}$. Both girls used the same approach to solve for matrix *A*, which was consistent with their approach in both the Italicizing N Task and the Rubber Sheet Task. Marissa’s work is given in Figure 7.

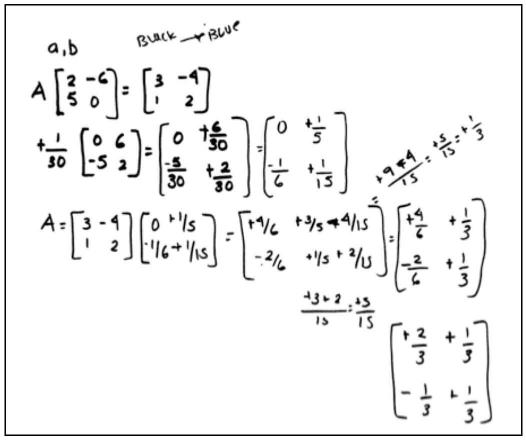


Figure 7. Marissa's written work for #2.

In the first row of Marissa's work (Figure 7), she set up her equation. In row two, she calculated the inverse of the matrix that contained the vectors *a* and *b* from the black system, the matrix $\begin{bmatrix} 2 & -6 \\ 5 & 0 \end{bmatrix}$. In the third line, she utilized this new inverse matrix in such a was as to isolate *A* on the left side of the equation and to have the resultant matrix $\begin{bmatrix} 3 & -4 \\ 1 & 2 \end{bmatrix}$ times the inverse matrix on the right side of the equation. The result of this calculation was the matrix *A*.

Note in the above segment the way in which Marissa and Sophie interacted with the notion of inverse matrix. They used an inverse matrix as a sort of 'calculational tool' that assisted them in solving for the matrix *A*. That is, in every situation in which they were solving for a matrix *A*, they found matrices *B* and *C* from the data and set up the equation $AB = C$. They then calculated B^{-1} from the algorithm $\begin{bmatrix} a & b \\ c & d \end{bmatrix}^{-1} = \frac{1}{ad - bc}\begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$. Note that this algorithm was not introduced or discussed in the interview—Marissa and Sophie brought this knowledge to the interview. Finally, to solve for *A*, they multiplied CB^{-1}. When interacting with inverse matrices, the girls never explicitly stated what the conceptual meaning of an inverse matrix was for them. They interacted with inverse matrices as if they were tools to use in the process of "solving for

A,” much analogous to a common approach of solving by x in $3x = 6$ by dividing by 3 on the right side. We cannot be sure if they were aware of how this solution method works, as they were not explicit. However, when Marissa and Sophie worked on devising a method that took vectors relative to the blue system and renamed them according to the black system, the notion of inverse surfaced in quite a different manner.

Marissa: From blue to black. Well, if we do the same letters we only need to change these two [holds a finger over the input, one over the output, and then flips her fingers over], right? [S: Yes.]...Because it is the same thing, but I am doing the inverse of the other one.
Sophie: You should get the same.
MW: The exact same answer?
Marissa: Yeah.
Sophie: It should be, but we’ll see.
Marissa: Maybe the order...
Sophie: Of the columns may change or something.

As the above transcript indicates, Marissa and Sophie hypothesized that this new matrix, going from blue to black, should either be the exact same as the matrix A they found for going from black to blue (see Figure 7), or be the same but with the columns switched. The girls worked for a few minutes independently, each using the same vectors as before—Marissa with the vectors a and b, and Sophie with the vectors c and d. After a bit of work, they check in with each other.

Sophie: You got the same or different answer?
Marissa: No, I’m not getting the same answer.
Sophie: I’m not getting the same, either.

After a few more seconds, Sophie asked Marissa if she got fractions in her answer, but she was not finished with her calculations. Sophie then turned to me and asked, “Did you get fractions?” I contend that Sophie was concerned about her answer not containing fractions because that would invalidate both of the guesses as to what this matrix might be. Recall they had guessed that either the matrix here (going from blue to black) was the same at the one going black to blue, or that the columns were switched. Because the matrix *A* in Figure 7 had fractions in it, Sophie's results for this new matrix invalidated their conjectures.

Sophie spent a minute checking her answer and comparing it with mine, and then checked on Marissa's progress. Seeing that Marissa was still calculating, Sophie began to examine her own answer, which was the matrix $\begin{bmatrix} 1 & -1 \\ 1 & 2 \end{bmatrix}$.

Sophie: I don't know why but I have a feeling—well, I do actually know why...that's, oh...Oh! ...That kind of...Ah! [Marissa looks up at her.] Go ahead, you go, go, keep on going. 'Cause I'm not sure yet, though, so...

Sophie worked quietly for 34 seconds, before she looked up at me and said, "So, I was going to say, this looks kind of like the inverse of that one, in a way, so I was trying to get the determinant of that one." The hesitation in this statement was because her calculations weren't giving her what she expected. We then spoke about her work, and she found an error in her calculation of the determinant.

Sophie: Yeah, that works out then. [Does a few more calculations under her breath.] So this one [points to the blue to black matrix] became the inverse of that one [points to the black to blue matrix]...Funny!

After Marissa completed her calculations, I asked Sophie about her new insight.

MW: And then, Sophie, can you explain what you were just doing?
Sophie: Well, I look at it, and you know, I was like it's funny, it seems that this is the same as that, this is the same as that, times three, right? But these two switch.
Marissa: So it has to be the, the-
Sophie: Right. It has to be—
Marissa: The inverse.
Sophie: So it has to also be maybe the inverse.
Marissa: And the signs also switch.
Sophie: Right. [M: Hmm.] So I got the determinant of this one...[continues explaining her procedure.]
MW: So does it make sense that they are inverses?
Sophie: Well, we inversed the operation—
Marissa: Hmm-mm. We changed this. In the beginning this [the new output vector] was in here [as the input vector], so we changed the places, so it makes sense. It has to be the inverse.

MW: And that's because in the equation, the input and output switch places?
Both: Yes.

Although Marissa had stated that in order to go from blue to black they were "doing the inverse of the other one," neither Marissa nor Sophie connected this idea to that of inverse matrix. Thus, I argue that the object of "inverse matrix" and the procedure of "inverse operation" were not strongly connected for Marissa and Sophie. They did suspect that some relationship would exist between their two answers, but Sophie (on more than one occasion) expressed genuine surprise when she discovered the matrices were in fact inverses of each other. Also note that the way in which she realized this relationship was by visual inspection—she could 'see' hints of the determinant, etc., but the connection to the input and output switching places (a more conceptual meaning of inverse) only came upon reflection.

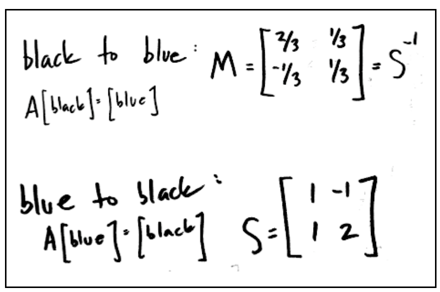


Figure 8. Interviewer's recording of the girls' work on Question 2.

Finally, I initiated a summary of their answers to Question 2. The first matrix they found transformed vectors relative to the black system into vectors relative to the blue system (matrix *A* in Figure 7). In the interview, we called this process "renaming." The second matrix they found took vectors in the blue system and "renamed" them in the black system. Furthermore, Marissa and Sophie found that this second matrix is the inverse of the first (see above transcript). As we summarized this, I wrote down the matrices the girls had found and asked for help in naming the matrices (see Figure 8). I called the second matrix "*S*" because Sophie had found it first. When I asked what we should name the other one, Sophie said, "M for Marissa," while Marissa said, "*S* inverse because they are inverses." Thus, both labels were recorded. Take note of the bottom matrix, *S*, as it will play an integral part in Marissa and Sophie's solution in Question 3.

Question 3: Renaming and stretching. For Question 3 of the Change of Basis Task (see Figure 4), Marissa and Sophie spent a bit of time discussing what approach they should use.

Marissa: What we can do, I was thinking, <2, 1> in the blue one, so, it will be [draws the vector], it will be here. And so, then, as we did with the previous one to—

Sophie: We find out where it is with respect to the blue one and then write it in the black one.

Marissa and Sophie work on this task together for quite awhile, sharing moments of both difficulty and insight. For instance, Sophie suggested that they write the given vector into its component parts $\mathbf{v} = \begin{bmatrix}2\\1\end{bmatrix} = \begin{bmatrix}2\\0\end{bmatrix} + \begin{bmatrix}0\\1\end{bmatrix}$ (see Figure 9). Because they were able to see that the first component vector < 2, 0 > coincided with line *l*, which was one of the stretching directions, they agreed that they could stretch this vector by a factor of four. As creative and insightful as this was, there still existed a bit of difficulty. For instance, Marissa got a resultant of < 10, 0 > after stretching < 2, 0 > by four because she was visualizing adding four lengths onto <2, 0 >, rather than seeing < 2, 0 > as one of the four lengths she will have in the result.

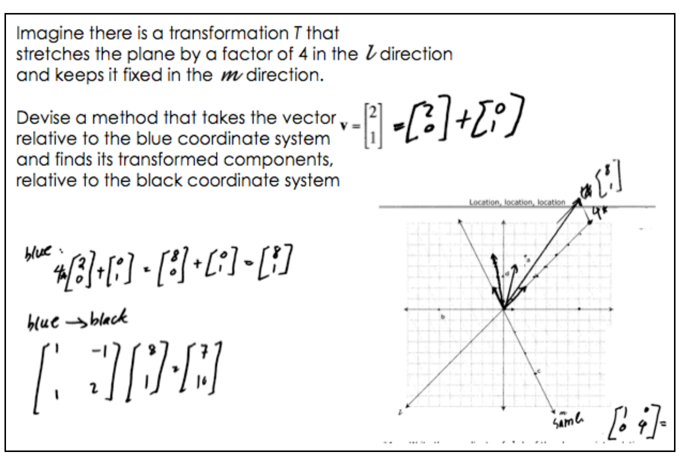


Figure 9. A portion of Sophie's written work on #3.

As Figure 9 shows, Marissa and Sophie did stretch the vector < 2, 0 > correctly to < 8, 1>. In order to find how to express < 8,1 > relative to the black basis, the girls used the matrix *S* (seen in the lower left corner of Figure 9) that they had developed in Question 2. Also note that there is a version of the transformation matrix relative to the blue coordinate system in the lower-right corner of Figure 9. This matrix is seen again in the following section, in which I press them to determine the transformation matrix relative to the black coordinate system.

Pushing further: Trying it the other way. Because of Marissa and Sophie's rather successful and insightful encounter with Question 3, I decided to push them a bit further. When they were originally discussing an approach for Question 3, they had mentioned that they could also rename the vector according to black first and *then* perform the pulling transformation. However, this challenging route was never pursued.

MW: And so, I think one thing that was interesting and very helpful is that you went ahead and said let's just pull it according to the blue answer first [girls nod]. And it was really easy for you to see how to pull it by four and leave it fixed in the *y* direction [girls nod]. What would have happened, just for instance, if you had switched it to black first and then tried to figure it out?

Sophie: Um...that, well, the *m* [the line labeled *m*]...that would be kind of hard, wouldn't it? [M: Um-hmm.] Because you have to look at *m* with this coordinate [lays her hand out like the standard *y*-axis], the black coordinate [lays her hand out like the standard *x*-axis], because it was easier to just look at the *m* as the, um, actual coordinates rather than looking at it as a line in another coordinate.

Marissa: I think it would be harder because we're, remember how um...when we have a matrix when it was simple [referring to the Italicizing N Task] like a number here than zero, no, the other way. Zero number, number. This is kind of like the blue one. But then when we change it to the black one, we will have also number over here. [Sophie: Right]. So I think that's harder to see.

MW: Oh, I completely agree. Can we actually figure out what that matrix would be for the blue one?

Marissa: For the blue one?

Sophie: When you do the stretching? [*MW*: Hmm-mm.]

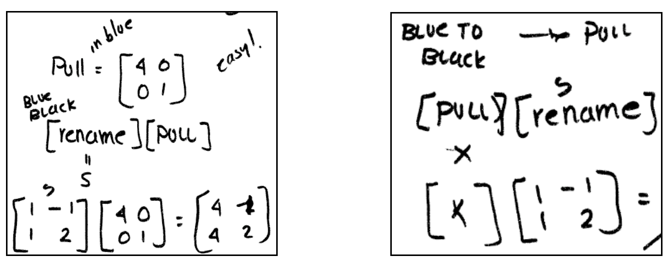
Marissa: [After a few seconds] Four, zero, zero, one.

MW: Well, I'm going to add that to a list of things. Ok, so we're saying the transformation, we're going to call that the action of pulling it. In the blue, we're going to say that I pulled it by four in one direction and by one in the other direction. [S: Yes.] So that could be some matrix. And the black—

Marissa: You have to have numbers instead of zeroes.
MW: That one is hard right? [S: Yes.] We don't know what that one is yet...Ok, a minute ago we said you could either rename it and then pull it, or we could pull it and then rename it. [Girls nod.]

Marissa and Sophie began to work together, discussing ideas and asking questions, for instance, about 'order of operations' for matrices—that is, if we want to 'pull' and then 'rename,' the matrices are written in that order from right to left.

Marissa: We can do this...This [Figure 10a] is the blue to black, right? So when we multiply it, this one [the order of the matrices] has to be the other way. Pull times rename, right? And the one we draw now [Figure 10b] is this one, so I'm going to put an *X*.
Sophie: So it's actually switched [the order of the matrices], isn't it?
Marissa: Yeah, so we switch it. When multiplying it we have to switch it, right?
Sophie: Ok. Yeah.
Marissa: So we know this one [in Figure 10a], this is yours, is *S*, is your name. This one [in Figure 10b] is also *S*, and we don't know this one [in Figure 10b]. This one [in Figure 10a], the pull, we figured out was this one, right? Four, zero...[S: Yeah.] So we can multiply these two...



(a) (b)

Figure 10. A portion of Marissa's work on trying to find the transformation matrix for the black system.

Marissa's work in Figure 10b continues in Figure 11. Note, in Figure 11, that Marissa set the matrix composition from Figure 10b equal to the result in Figure 10a, which is $\begin{bmatrix} 4 & -1 \\ 4 & 2 \end{bmatrix}$. In the next line in Figure 11, Marissa multiplied matrix by $\begin{bmatrix} 2/3 & 1/3 \\ -1/3 & 1/3 \end{bmatrix}$, which is the inverse of $\begin{bmatrix} 1 & -1 \\ 1 & 2 \end{bmatrix}$ (see Figure 8). Note that Marissa did not recalculate the inverse here—she recalled the relationship of *S* and S^{-1} and used her previous result. Finally, she multiplied these two matrices together in order to find the transformation matrix relative to the black coordinate system, the matrix $\begin{bmatrix} 3 & 1 \\ 2 & 2 \end{bmatrix}$.

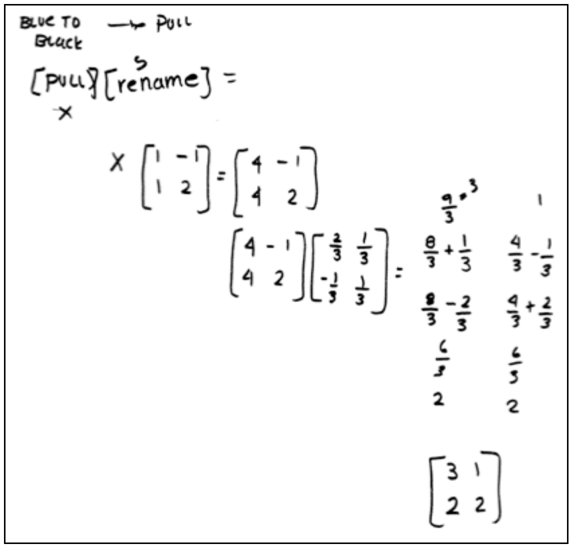


Figure 11. Marissa finding the transformation matrix for the black system.

Finally, we discussed how the transformation—pulling along line *m* and staying fixed along line *l*—is not obvious from the new matrix transformation. This connected to Sophie's previous sentiment that it was "easier to just look at the *m* as the, um, actual coordinates." Recall that Marissa had predicted that the transformation matrix would "have numbers instead of zeroes," and she was correct. In their responses, I hold that Sophie seemed to be thinking about the actual *action* of the transformation and how a convenient coordinate system would make the geometric transformation easier to visualize, whereas Marissa seemed focused on the matrix *representation* of the transformation and how a convenient coordinate system would make the matrix easier to ascertain.

The above section details how Marissa and Sophie were able to reinvent an informal version of the matrix equation $AP = PD$, and how Marissa—in her manipulation of this equation in order to solve for matrix A (called matrix X in Figure 11)—developed the equation $A = PDP^{-1}$

en route to her solution. This use of the equation can be seen in Figure 12, which contains my written recordings of our closing conversation.

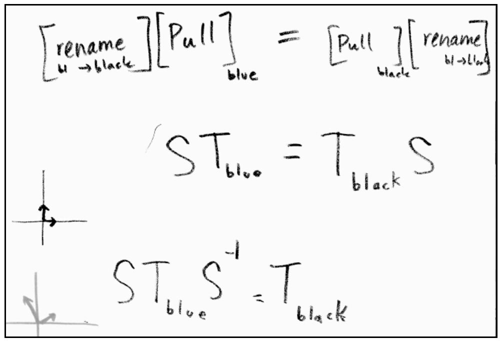


Figure 12. Interviewer's written record of summary conversation.

Marissa and Sophie's success in re-inventing the equation $A = PDP^{-1}$ in a way grounded in their ways of reasoning and in a way meaningful to them and is inextricably dependent upon various aspects of how they thought to interact with the tasks. For instance, their dependence upon the matrix equation $AB = C$ and the corresponding solution method $A = CB^{-1}$ afforded them a way to manipulate matrices in a flexible way. Each student's mathematical experience is unique, and what happened here is no exception. As previously mentioned, I interviewed two other pairs of students, and their approached sometimes varied greatly from Marissa and Sophie's. For instance, Sammy and Billy relied heavily on solving systems of linear equations in order to solve for a particular matrix. Further documentation of other student work regarding the task sequence is beyond the scope of the paper, but this information is invaluable in informing how these tasks may be of use in a classroom setting. In the discussion section, I close by positing one possible route for further development if implemented in some form in a classroom.

Discussion

In the results section, we saw that the task sequence—composed of the Italicizing N Task, the Rubber Sheet Task, and the Change of Basis Task—elicited a variety of student conceptions fundamental to linear algebra. We also saw that these conceptions brought about various affordances and limitations along the actual trajectory of reinventing the equation $A = PDP^{-1}$. Finally, I argue that ideas from the task sequence could be leveraged as starting points for other mathematical ideas in linear algebra. In particular, I argue that reasoning with linear transformations could change from functioning as a *model of* correlating matrix representations with geometric transformations to functioning as a *model for* reasoning about basis. Furthermore, I argue that through developing the idea of basis, a notion of the necessity for the concept of change of basis could be fostered simultaneously.

The second heuristic of the RME theory is known as *emergent models*, which function to the bridging function between a student's informal understandings and a more formal way of knowing. Gravemeijer (1999) states that, "The shift from model of to model for concurs with a shift in the students' thinking, from thinking about the modeled context situation to a focus on mathematical relations" (p. 163). In order to detail this more precisely, Gravemeijer enumerates four types of activity through which the model of/model for transition can be witnessed. These four types of activity are: Activity in the Task Setting, Referential, General, and Formal (p. 164). It is through an explicit description of these four levels that I now discuss the potential I see for how the ideas embodied in this task sequence could be leveraged from a model of reasoning about specific linear transformations of the plane into a model for reasoning about basis and its applications.

Activity in the Task Setting. This level of activity involves students working through problems in a particular experientially real setting. In the task sequence discussed in this paper, both the Italicizing N Task and the Rubber Sheet Task serve as an experientially real setting in which students are presented with the opportunity to develop a sense of how matrices correspond to linear transformations in the plane, particularly the transformations of stretch and shear.

Referential Activity. This level of activity, as described by Rasmussen and Zandieh (2008), "involves models-of that refer (implicitly or explicitly) to physical and mental activity in the original task setting" (p. 12). Taking the entire task sequence as one 'task setting,' the Change of Basis Task provided Marissa and Sophie with many instances in which they were able to reference their activities in the first two tasks. For instance, in Marissa's explanation about why figuring out the pulling matrix according to the black coordinate system would be difficult, she referenced both the Italicizing N Task and the Rubber Sheet Task, calling attention to how the matrix representation of the geometric transformation was easy in the former but difficult in the latter.

The following two forms of activity, general and formal, were not witnessed during Marissa and Sophie's engagement with the task sequence. This is partially due to the nature of the task sequence. Given that the very last activity in which they engaged witnessed the reinvention of the equation $A = PDP^{-1}$, the girls were not given any immediate opportunity to exhibit their reasoning about linear transformation or their informal ideas about what experts know as basis. However, I argue that there would be the opportunity for these two levels of activity to be actualized if a derivation of the task sequence were utilized and developed further in the classroom. Thus, I will conclude by conjecturing how the task sequence could be leveraged into developing general and formal levels of activity regarding basis and change of basis.

General Activity. This type of activity involves a shift away from dependence on the original task setting and a shift to models-for a focus on independent mathematical interpretations and formalizations. As stated above, my time with Marissa and Sophie ended just after the reinvention of $A = PDP^{-1}$ for the plane. We did not spend much time summarizing, reflecting, or interpreting. However, I argue that this would be the next natural level of progression, and that this would constitute a general level of activity. That is, reasoning with matrices as linear transformations could begin to function as a model for reasoning about basis. The task sequence, at least for Marissa and Sophie, elicited a strong need for the ability to describe linear transformations in the most convenient 'coordinate system.' This desire to express the transformation in the simplest way possible drives the need for the development of the notion of basis, as well as change of basis. Upon reflection on Marissa and Sophie's work, the opportunity to formalize various ideas related to basis seems apparent. For instance, the next level of investigation could explore, for instance: what is a coordinate system? If I can have more than one, when would I prefer one coordinate system to another? If I can have more than one, is there a way to translate between the systems? Therefore, I argue that by developing the idea of basis through a need to simplify the expression of linear transformations, a necessity for the concept of change of basis could be fostered simultaneously.

Formal Activity. Finally, this fourth type of activity "involves students reasoning in ways that reflect the emergence of a new mathematical reality and consequently no longer require support of prior models-for activity" (Rasmussen & Zandieh, 2008, p. 12). As stated about in the general activity level, I contend that ways of thinking and reasoning about linear transformations in light of their geometric interpretations and matrix representations could serve as a strong foundation for developing the notion of basis and change of basis. The next natural step, then, would be for students to develop the ability to think about basis as an 'entity' in its own right, free of the contextual confinement from which it was first indirectly explored in an experientially real, accessible way.

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