The Instructor's Important Role in Supporting Mathematical Arguments in a K-5 Mathematics Specialist Program

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Introduction

Researchers have provided many examples of what it means to teach mathematics using an inquiry approach. This body of work addresses how student learning can be positively influenced when they have opportunities to explain and justify ideas, listen to others' explanations, ask clarifying questions when they do not understand classmates' ideas, represent their ideas using self-invented methods, engage in challenging problem solving activities, and so on (e.g., Ball & Bass, 2003; Cobb, Wood, Yackel, 1992; Hiebert et al., 1997; Lampert, 1990; Rasmussen & Marrongelle, 2006; Whitenack & Knipping, 2002; Yackel & Cobb, 1997). Further, these reports often highlight the fact that although teachers continue to facilitate discussions, the nature of these discussions is markedly different from those that take place in traditional mathematics classrooms. Teachers in inquiry classrooms ask different types of questions and highlight students' contributions for different purposes (Ball, 1993; Ball & Bass, 2003; Yackel & Cobb, 1997). Additionally, teachers in these types of classrooms take great care choosing representations, using students' ideas as starting points for advancing these discussions or capitalizing on ideas that underpin their students' strategies and methods. Often in these types of classrooms, the teacher must be sensitive to students' incomplete ideas and more informal methods and make decisions about how to use those ideas as the basis for exploring more formal ideas or practices that fit with the mathematical practices in the broader community (Ball, 1993; Ball & Bass, 2003; Yackel, 2002).

Yackel (2002) has continued to address the role of explanation and justification in her more recent work. She and her colleagues have provided new insights into how teachers (elementary school teachers and university instructors) support the development of mathematical arguments during whole class discussions (e.g., Stephan & Rasmussen , 2002; Whitenack & Knipping, 2002). Yackel, for instance, analyzed part of a university differential equations lesson to illustrate how a university instructor and his students established the plausibility of arguments about predator-prey problems. She argued that in this university classroom, the instructor focused discussions on "developing and supporting arguments for claims that were put forth but not on making claims per se" (p. 427). Once a claim was made, the instructor redirected the conversation so that students could explain and justify ideas. In fact, it was normative for the instructor and students to engage in these types of whole class discussions as they established plausible arguments. The instructor's role was important because he could direct the discussion so that students had opportunities to support claims using their understandings of the mathematical ideas at hand.

In our discussion we, too, address the instructor's role by highlighting one university course instructor's proactive role in establishing and maintaining normative practices for making mathematical arguments. Here, we make a case for how teachers' explanations and justifications in two different mathematics courses in a K-5 Mathematics Specialist program contributed in part to the mathematical arguments that were established during whole class discussions. To accomplish our task, we use examples from one lesson in an Algebra and Functions course and one lesson in a Rational Number and Proportional Reasoning course that are core courses in this endorsement program.

Following Yackel (2002), we use Krummheuer's (1995) ethnography of argumentation as an interpretive framework to reconstruct mathematical arguments in these two lessons. We use examples from the algebra lesson to illustrate how the instructor facilitated a discussion in which he coordinated two ensuing arguments. We use examples from the rational number lesson to illustrate how Krummheuer's framework can be used to hypothesize how the instructor could use a substantial argument as a starting point for engaging the teachers in proof making. By providing these two different types of analyses, we illustrate how this interpretive lens might be used to explicate the types of possible learning opportunities that occurred as teachers engaged in whole class discussions. We also illustrate the utility of conducting these types of analyses.

Theoretical Considerations

Krummheuer argues that by coordinating constructs associated with argumentation with ethnomethodology and interactionism, the researcher can understand the structure of an argument as well as how it is constituted and reconstituted by the participants during an ensuing whole class discussion. Argumentation as it unfolds in the classroom then is a social phenomenon; it has all the features that are usually ascribed to classroom discussions and other normative practices associated with face-to-face interactions. As Krummheuer (1995) states,

> Because of the emergent nature of social interaction, argumentations are usually accomplished by several participants. Such a case is called a collective argumentation…In addition, the development of a (collective) argumentation does not need to proceed in a harmonious way. Disputes in parts of an argumentation might arise that could lead to corrections, modifications, retractions, and replacements…The result of this process can be reconstructed and is called an argument. (p. 232)

So argumentation can be used as a way to describe ordinary classroom activity (Krummheuer, 1995). When part or an entire whole class discussion is reconstructed, in this way, we can develop a scheme of sorts that structures key parts of the discussion or argument that unfolded.

The model for an argument is comprised of four components: *conclusion*, *data*, *warrant*, and *backing*. These components comprise an *argumentation*, of which the conclusion, data and warrant make up the *core of the argument* (Krummheuer, 1995). In addition, in some cases, the argumentation must be reconstructed if, for instance, the data is invalid or is doubted by one's classmates. In this case, the participant must provide different data to support her conclusion. By way of contrast, if a classmate agrees with the conclusion but does not directly see the relationship between the data and the conclusion, the participant may be asked to provide additional information that warrants and/or provides backing to support her claims. In the second case, the participant need not develop a new conclusion-data pairing, but instead, she is obliged to provide evidence to support the conclusion that further clarifies the warrant. The result of this argumentative process is a mathematical argument that is socially constructed by the participants. As Krummheuer notes, a mathematical argument is the result of an argumentation. At times, an argument is part of a complex argumentation. Arguments can be embedded in other argumentations. This last point is particularly relevant to our discussion. One of our aims is to understand how the instructor facilitated a discussion in which several arguments surface and become part of a complex argument.

Another feature of Krummheuer's (1995) and Toulmin's (1969) theory relates to that of distinguishing among different types of arguments. Typically when one makes a mathematical argument, we are inclined to consider whether this argument aligns with the rules of logic that are often used to describe the work of mathematicians or those engaging in proof making. Krummheuer, following Toulmin, suggests that one need not characterize (or evaluate) an argument using the rules of deduction. For Krummheuer, an argumentation is socially accomplished as each participant tries to adjust his intentions to the contributions made by others. As a participant explains or justifies his ideas, the "meaning of the premise increases or changes" (p. 236). To distinguish this type of argument from the more traditional view of arguments, Toulmin and Krummheuer suggest that arguments can be either *analytic* or *substantial*. Analytic arguments are those that are made using the rule of deductive reasoning. Substantial arguments, by way of contrast, are arguments that are not held to tautological requirements. Contributions of this kind would not likely be part of an analytic argument, but could lead to a substantive argument. This said, outlining a substantive argument has merit in its own right. One can reconstruct an argument, and in doing so, develop a better understanding of the particular mathematical practices that emerge during whole class discussions (Stephan & Rasmussen, 2002).

So how might an interactionist perspective fit within this theory of argumentation? Here we draw on several recent works that have made use of a coordinated framework. In Yackel's (2002) retrospective analysis across teaching experiments, she draws on Krummheuer's (1995) work to extend normative practices to provide insight into the elementary teacher's and university instructor's roles in supporting the evolution of mathematical arguments. With regard to argumentation, we specifically focus our analysis around the instructor's role in facilitating and guiding the discussions as teachers explain and justify claims that they make. Like Yackel (2002), the focus of our discussion is around the process by which claims are substantiated rather than the process of developing or constituting a viable claim. Additionally, we draw on Yackel's ideas about when and for what purposes that teacher might contribute to an ensuing argumentation. In particular, does the instructor offer contributions in the form of warrants or backings when the teachers make omissions? Are there other situations in which the instructor might be obliged to make contributions?

We also draw on the work of Stephan and Rasmussen (2002) to further clarify the extent to which these ensuing argumentations become taken as shared. Stephan and Rasmussen offer two conditions that signal whether or not an argument is in fact agreed upon by the participants. They state,

> We contend that mathematical ideas become taken-as-shared when either (1) the backings and/or warrants for an argumentation no longer appear in students' explanations and therefore the mathematical idea expressed in the core of the argument stands as self-evident, or (2) any of the four parts of an argument (data, warrant, claim, backing) shift position (i.e., function) within subsequent arguments and are unchallenged. (p. 462)

So if the participants, as in our case, teachers and the instructor do not challenge supports for a data-conclusion pairing, we might infer that they have reached a consensus about the idea under consideration. Additionally, if part of an argument is used for a subsequent argument, we can also presume that the participants have established a shared understanding about an idea.

Our hope is that as we conduct these analyses, we contribute to the continuing conversation about what might be possible when mathematics instruction aligns with reform efforts in the university mathematics classroom. Additionally, we offer an account of how the instructor might facilitate discourse that takes seriously teachers' important role in making mathematical arguments.

Methodological Considerations

Our examples are taken from two different whole class discussions, one in an algebra course and one in a rational number and proportional reasoning course. These two courses are part of a 3-year graduate degree program designed to prepare teachers to become K-5 Mathematics Specialists. The rational number and proportional reasoning course is the third course in the sequence; the algebra course is the fifth and final mathematics course in the program.

Course participants entered the program with a range of experiences. Some had several years of teaching experience whereas others had taught fifteen or more years. About one-third of the participants were already serving in some type of leadership role in their school buildings. The instructor who led both of these two whole class discussions, *Instructor*, is a research mathematician who has been actively involved in providing professional development opportunities for teachers for over 20 years. The courses were the second and the third courses for which he had served as the primary instructor in the mathematics specialist program for the same cohort of teachers. As such, over time, he and the teachers had established a rapport in which they had mutual respect for one another.

Instructor regularly used the teachers' contributions to facilitate discussions. He encouraged teachers to represent their ideas, even if their ideas were partially complete as they supported claims or if they were not certain of the correctness of their answer. As teachers did so, *Instructor*, for his part, highlighted ideas, asked clarifying questions, and so on. As a consequence, *Instructor* and the teachers together established that teachers' ideas were particularly valued. More generally, using constructs related to argumentation, by engaging in these types of discussions, teachers developed arguments as they provided warrants and backings to support the claims that they made. *Instructor*, for his part, also played an important role in what and how contributions were highlighted or used to support the teachers' claim. So norms for engaging in argumentation were the result of *Instructor's* and the teachers' mutually orienting interactions during discussions (cf. Yackel, 2002).

The classroom data is taken from classroom data corpus that includes observation notes of the lessons, videotape recordings of small group and whole class discussions, digital recordings of participant interviews and small group discussions, digital photos of participants' work during whole class discussions and participants' individual work. As we observed these and other lessons, we noted if we needed to revisit particular lessons during our analysis process. We began our analysis process by conducting a preliminary analysis of each lesson. As we did so we realized that we needed to transcribe these two lessons to conduct a more thorough microanalysis of the entire lesson.
To conduct microanalyses, we first viewed the videotape and the transcription of the small group and whole class discussions. As we watched the videotaped lesson, we identified the mathematical ideas that surfaced and clarified each of the participant's contributions. We then re-watched the lessons and coded the participants' contributions using argumentation constructs (data, conclusion, warrants and backings).

Using Ethnography of Argumentation to Analyze Whole Class Discussions

We first use the framework to reconstruct two arguments that emerged in the algebra course and how *Instructor* facilitated claims, warrants and backings that were simultaneously established as evidential supports in both arguments. As we present our analysis from the algebra course, we will also highlight *when Instructor* offered warrants or backings, and *for what purposes* (cf. Yackel, 2002). Secondly, we use the framework use an example from the rational number course to illustrate how the instructor might use a substantial argument to as an opportunity to introduce teachers to formal proof making. In particular, we hypothesize how *Instructor* and the teachers might have developed a more general argument for the density property of the rational numbers.

The Algebra Lesson

To illustrate how we might use this interpretive lens, we reconstruct part of a whole-class discussion as the participants discussed the following task: *Explain what happens to [r + s + (1/s)] as s increases from some very small positive number (e.g., 0.001) to some very large positive number (e.g., 1,000)* (Schifter, et al., 2008). Previously *Instructor* and the teachers had made arguments for why *r/s* decreased as *s* increased as well as why *1/s* approached 0 as s increased. We refer to the former as Argument 1 and the latter as Argument 2.

As the discussion continued participants established yet a third argument, Argument 3. Previously, Teacher B and Teacher C provided the conclusion and data pairing for this new argument: r + s + 1/s increased because 1/s approached 0. Neither gave specific examples to support this claim, however Teacher C stated that she had developed a table of values that became larger as she substituted larger values for s. After Teacher C gave her explanation, *Instructor* provided specific pieces of data when he asked the whole class what would happen if *s* changed from 10 to 11 and then from 987 to 988. Teachers explained that the values would increase. So at this point in the discussion, *Instructor* and the teachers developed the claim and several pieces of data that supported the claim for Argument 3.

Immediately following this interchange, Ms. Satterfield, one of the case study participants, made a statement that might be interpreted as a challenge to the conclusion for Argument 3. She stated that she thought that *r + s + 1/s* actually decreased, that is, for larger values of s, r + s + 1/s approached r + s. We enter the discussion as Ms. Satterfield continued to explain her ideas. (Note: Ms. Satterfield is Ms. S.)

Ms. S: Well maybe I'm confused because I looked at it like as s increases, even though you are adding more with s you know that $1/s$ is going to approach 0 so that the expression itself is going to approach $r + s$. So it's not going to continue to increase; I put it is going to decrease—approach $r + s$. You know I guess I was thinking about it differently than just the value increases or decreases. Does that make sense?

At this point, Ms. Satterfield provided a different explanation for what happened to the values of $r + s + 1/s$. She in fact actually stated a different conclusion-data pairing—as $r + s + 1/s$ increased it approached $r + s$ *because* $1/s$ *approached 0*. How might *Instructor* address Ms. Satterfield's ideas? Would he interpret her ideas as a challenge to Argument 3, or as a new argument, and so on? As the discussion continued, notice how he incorporated her ideas into the ensuing argument.

Instructor: Why don't you draw it?

Ms. S: I don't think I can draw it, is the problem.

Teacher: Sure you can.

Ms. S: I don't know what I would draw.

Instructor: Well, how about we let r be 2?

Ms. S: Okay.

Instructor: Why did I say 2? I just picked a number, that's all…You can't draw it if you don't know what the numbers are so let's let r be some…They said it had to be positive, so let it be 2.

Ms. S: Okay.

As we see, *Instructor* asked Ms. Satterfield to make a drawing to support her ideas. When she stated that she was not sure how to make a drawing, *Instructor* assured her that she could do so and suggested how she might begin—by letting r be 2. His suggestion about fixing r was important; it made it possible for Ms. Satterfield to generate supports in this case, a table of values for $r + s + 1/s$.

As *Instructor* asked Ms. Satterfield to provide supports in the form of possible warrants, it is not clear at this juncture whether her table of values supported her new claim or the current claim for Argument 3. As we continue with our analysis, it will be important to determine how Instructor draws on Ms. Satterfield's ideas to support either or both claims.

We return to the discussion after Ms. Satterfield had derived these two values. We re-enter the discussion as she commented that the values decreased for smaller and smaller positive values of *s*.

> Ms. S: See I'm saying that once you get to these really large numbers that $1/s$ isn't …I mean I guess maybe I'm saying it's increasing but to me, I guess, okay, I see what I'm saying. We are kind of saying the same thing now. Where I said it decreases to approach $r + s$ (covers $1/s$ with her hand), rather than saying it increases…

Notice that once Ms. Satterfield provided two different values for *s*, she realized that the table of values supported the claim that $r + s + 1/s$ increases for larger and larger values of *s*. At the same time, she continued to argue that the values could be used to show that $r + s + 1/s$ approached $r + s$. So, in a sense, Ms. Satterfield provided an explanation that might serve as support for Argument 3 and her new claim.

It remains to be seen if Ms. Satterfield's new claim, what we will tentatively call, Argument 4, will become a mathematical argument for to $r + s + 1/s$. To better understand if Argument 4 is established as yet another argument for $r + s + 1/s$, we return to the ensuing discussion. As the discussion proceeded, Ms. Satterfield inserted a few more values and then referred to the table of values as she continued to explain her ideas. We reenter the discussion as she and another teacher, Teacher E refer to the table as they restate the claim for Argument 4.

> Ms. S: (Calculates .01 and .001 for *s*.) So to me, I guess…I guess I didn't look at it like now it's decreasing and then it's going to increase (moves her hand over the values in the chart). I looked at it like it's going to decrease until it gets to the point where essentially the answer is going to be $r + s$ (draws a box around 2 + 1,000 and 1,002 and $r + s$ on the chart paper) and then if you keep going the more and more. Even though it may be increasing, it's really just approaching whatever the value of *r* plus *s* is.
>
> Teacher E: But it doesn't really get to $r + s$ because of the extra small piece, $1/s$.
>
> Ms. S: Yes.

Here Ms. Satterfield referred to the table of values to explain how she had thought about $r + s + 1/s$. She not only argued that this expression increased for larger values of *s*, but also she argued what value it approximately increased to, $r + s$. The fact that she highlighted this idea when she drew a box around specific values provides further support for this interpretation. Teacher E also contributed in part to this discussion. So at this juncture, Ms. Satterfield's table of values also served as data for Argument 4.

As the discussion continued, note how *Instructor* addressed the teachers' new ideas.

Instructor: So the expression does keep increasing right? As *s* gets bigger, the whole expression, what happens to it?

Ms. S: Yeah, well yeah…

Teacher F: It increases if *s* is greater than 1.

Instructor: Yeah, we're talking about bigger than 1 still, right? When *s* is bigger than 1, okay as *s* gets bigger the whole expression keeps getting bigger. But you are saying something about the way it is getting bigger, is changing.

As the discussion proceeded *Instructor* continued to refer to the table as providing support for Argument 3. His question, "So the expression does keep increasing, right?" is evidence of this fact. After making this statement, he too indicated how the table of values also related to her new claim. He did so by explaining to the class that Ms. Satterfield also provided supportive evidence for the "way it is getting bigger is changing." So at this point in the discussion, *Instructor* and several teachers provided data for Argument 4. Following this line of analysis, then Teacher E's early comment, "But it doesn't really get to $r + s$ because of the extra small piece, *1/s*." provided additional support for that further clarifies why the function can only approach $r + s$. As a consequence we interpret her comment to serve as a warrant for this new data-conclusion pairing. At the same time, *Instructor* and Ms. Satterfield also explained how the table of values supported Argument 3.

So what was *Instructor's* role during this part of the discussion? How did he facilitate the discussion about both claims? From above we note that he was able to highlight ideas that might serve as data or warrants for both arguments by pointing out how Ms. Satterfield's ideas were related to the different mathematical ideas. He also supported her as she constructed the table of values. First he suggested a way to begin to construct the table of values. Secondly, as the table of values was constructed, he asked specific questions about values in the table that might highlight ideas that could be used to support both claims. By doing so, he could support Ms. Satterfield's argument about convergence and at the same time continue to facilitate a discussion about Argument 3.

To summarize briefly what has happened up to this point, we refer you to Figures 1 and 2. As Argument 3 emerged, Ms. Satterfield offered yet a fourth argument. Instructor's role during this part of the discussion was that of asking teachers to provide examples that further supported the claim about what happens to $r + s + 1/s$ as s increases. As he made suggestions for what value to assign r, he helped to launch a discussion in which teachers offered support for Argument 3 as well as Argument 4. So his role was that of making explicit teachers' ideas so that these ideas might serve as evidential support for the claims that they made. Note in the diagram that we have also inserted Argument 2 as data for Argument 3. Elsewhere we have explained that Argument 3 is a complex argument, that is Argument 1 is nested inside of Argument 2 and is data for Argument 3 (Whitenack, Ellington & Cavey, 2009).

As *Instructor* continued to facilitate this part of the whole class discussion, Teacher E constructed a graph that provided additional information to support her and Ms. Satterfield's ideas for both arguments. By doing so, Teacher E and *Instructor* provided explanations that might function as argumentative supports, in this case warrants and backings, for both Argument 3 and Argument 4. Interestingly, as this discussion proceeded, Teacher E's backings in Argument 3 were *augmented* to that of warrants in Argument 4. So, her graph of the expression supported Ms. Satterfield's table of values but in different ways in both arguments. In Figure 1 and Figure 2, we insert these contributions as warrants or backings to further build the scheme for both arguments. Whereas the initial claims and data are different, notice that Ms. Satterfield's and Teacher E's explanations supported both arguments.

Figure 1 Schematic Representation for Argument 3

I substituted values for r & s and made a table of them. *Teacher C's Data*

Instructor: For the part where s is bigger than 1, if s goes from 10 to 11 what happens to r + s + 1/s? How much does it increase by?

Teacher B: One?

Instructor: About 1? Why not exactly 1?

Teacher B: Because of the fraction.

Instructor: Because you have that 1/10 thing.

Instructor: If s went from 987 to 988 what would happen to the expression?

Teacher B: About 1 again.

Instructor: Increase about 1 except that tiny little part.

Instructor & Teacher B's Data

Teachers: The values are going down. [Referring to the 1st three rows of Ms. Satterfield's table to the right.]

Teachers' Data

Data

Because

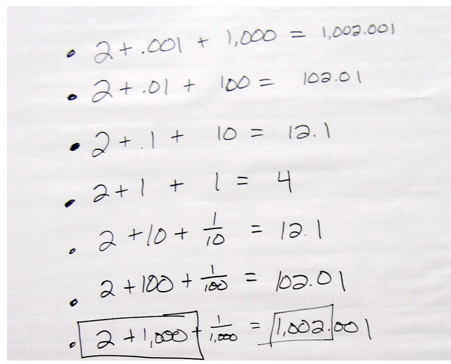
So

As s increases starting at one whole then the value [r + s + 1/s] increases but as s decreases from 1 the value goes up, also increases.

Conclusion

Since

Instructor: How about we let r be 2?



Now it's decreasing and then it's going to increase.

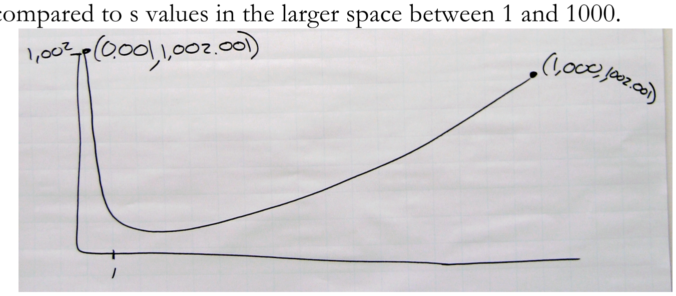
Ms. Satterfield's Warrant

Whether you add a large value or a small value, the expression is increasing.

Teacher D's Warrant

On account of

[Motioning down the table of values listed above] The same change occurs just over a smaller space for s values between 0 and 1 compared to s values in the larger space between 1 and 1000.



Look at a tenth [points to (0.1, 12.01)] and ten [points to (10, 12.1)].

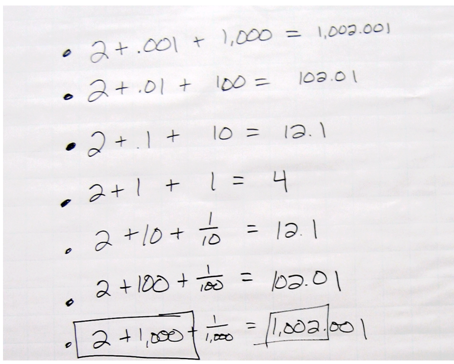
Teacher E's Backings

Figure 2. Schematic Representation for Argument 4

Argument 2

Ms. Satterfield: Because 1/s approaches 0

Instructor: How about we let r be 2?



Ms. Satterfield's Data

It never gets [to the value of r + s] because there is always a little extra. *Teacher F's Data*

Data

Because →

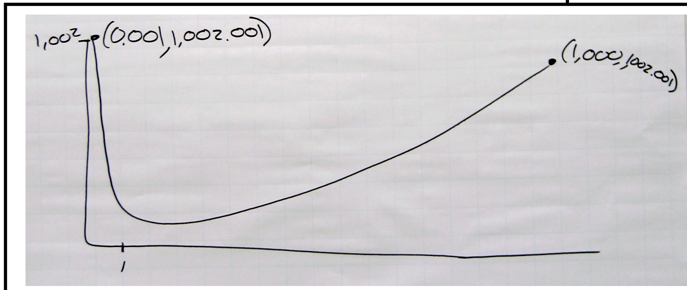
The expression itself [r + s + 1/s] is going to decrease to approach r + s.

Conclusion

So

Since

It doesn't just increase, it decreases until it approaches this [points to 2 + 1000 in the table presented above]. *Ms. Satterfield's Warrant*



It starts getting flatter because it involves 1/s and those values start to get really small.

The curve eventually goes straight or gets close to the line r + s. *Teacher E's Warrants*

On account of

[Adds the coordinates (0.001, 1002.01) and (1000, 1002.01) to the graph above.]

The same change happens this way [motioning up and down in the area between x = 0 & x = 1] but it takes larger space in this direction to get there [motioning from the point (1, 4) to the right along the x-axis]. *Teacher E's Backings*

It [r + s + 1/s] stops getting bigger in a big way. *Ms. Smith's Backing*

The Rational Number Lesson

For our second example, we use part of a whole class discussion in which teachers shared their methods for finding rational numbers between 1/11 and 1/10. Here we briefly outline strategies that two small groups shared. (See Whitenack, Cavey, & Ellington (2009) for a full description.)

Group 1's method. Previously Teacher G from Group 1 explained that 95/1000 and 99/1000 (or 0.095 and 0.099) were two rational numbers between 1/11 and 1/10. She and her group members explained that 0.095 (or 19/220) made sense because they had used a similar method (converting the fractions decimal form) to show that 21/220 was a rational number between 1/10 and 1/9—one of their homework problems. After Teacher G explained, *Instructor* made a more general statement about their ideas:

> There's lots of rational numbers between 1/11 and 1/10, and the one you found was 95/1000, and I didn't say whether or not I would care if it was in lowest form, but you felt better about making it be that way. I think [rewriting the fraction as] two-hundredths gives you a little more sense of it, yeah?

As he redescribed their ideas, he also restated a claim that might serve as a claim for this potential argument, what we will call Argument 5.

As the discussion continued, several groups offered explanations that also supported this claim. As they did so, they provided several different *sets of warrants* for data that supported this more general claim—*there are an infinite number of rational numbers between 1/11 and 1/10*. In the rest of this section, as we briefly outline the ensuing argument, we illustrate how this substantial argument for the density property was collectively established.

After Teacher G explained her group's ideas, Ms. Sneider, without prompting, stated that she could find other decimals (rational numbers) between these two numbers. As she did so, she too provided supports for Argument 5. We enter the discussion as she explains her ideas:

> Ms. Sneider: Now I am just looking at the 0 in the 1000th place in .0909? And now so the decimal between is .095 so it went from 0 to 5… You could also just have a 1 there, or a 2 or 3 or 4.

Group 1:	Yes, there are an infinite number!
Ms. Sneider:	…91/1000, 92/1000, 93/1000, 96/1000, 97/1000
Instructor:	Yes.
Ms. Sneider:	But it also helps me to think of whole numbers. Say if I'm looking at the 9 and the 0 in the 100ths and 1000^{th} s place, I think of 90, and another 90, then 91, 92, 93…
Group 1:	Yes.
Ms. Sneider:	I think of that just for a split second to get…to solidify my thinking.

Here Ms. Sneider built on Group 1's claim by providing further support, in this case in the form of warrants, for Argument 5. It is as if she indicated that not only did their explanation make sense, but also she could find other rational numbers that satisfied this claim by simply manipulating the 1000ths digit. In fact, as her comment suggests, she could make a similar argument for finding other three-digit decimals, in this case .091, .092, .093 and .094, that were between .0909 repeating and 0.10.

Applying Krummuheur's (1995) scheme, we can state this new data-conclusion pairing as follows: *There are an infinite number of rational numbers between 1/11 and 1/10 because.0909 repeating < .091 < .095 and .0909 repeating < .092 < .095*. Ms. Sneider's comments related to the fraction equivalents for 3-digit decimals between 1/11 and 1/10 are also data that support the conclusion. In addition, we can place her comment about whole numbers inside the scheme. Recall that she explained how she thought of these decimals in much of the same way that she thought about whole numbers. These comments seem to support why she could find the next larger or smaller rational number. For this reason, her second comment served as a warrant for the data-conclusion pairing.

We now return to the ensuing discussion as another group, Group 2 explained their method for finding numbers between two rational numbers. As they presented their strategy, they used an open numberline to illustrate how they simplified a complex rational number. Instructor, too, played an important role during this discussion as he highlighted aspects of their thinking by referring to how one could use fraction strips to illustrate their strategy.

Group 2's method. Group 2 first explained that they converted 1/11 and 1/10 to the equivalent fractions, 10/110 and 11/110. They then commented that they were not sure how to proceed once they had these equivalences. We rejoin the discussion as *Instructor*, in response to Group 2's comment, directed a question to the whole class about using fraction strips to represent 10/110 and 11/110:

> Instructor: (To Group 2) Before you go any further there…(To the whole class) if you have one of these fraction strips, how many pieces would fold it up into now?
>
> Participants: 110.
>
> Instructor: 110 pieces. Can you go from actually folding 8 or folding 12, to actually thinking in your mind [about making] 110 folds? I'm not that good. But I kind of think it's as if I had folded 12 times. It's the same idea [here]. So it's folded into 110 little pieces.

As the discussion continued, Group 2 provided additional information about these equivalences.

> Group 2: 10 ½ /110 was the rational number we found, but this expression does not make sense in its current form.

As the discussion continued, Group 2 then explained how they resolved this issue of working with complex rational numbers as they spoke with one of the visiting instructors, Instructor 2:

> Group 2: Instructor 2 visited our group and discussed this issue with us. He asked us to think about how we could represent 10 ½ /110 if we folded fraction strips. (Draws a number line and marked 10/110 and 11/110 on the number line and marks off the distance halfway between these two numbers to show where 10½ /110 was located on this number line.) So we folded the paper again and found another name for 10 ½ /110—21/220! (Records that they multiplied both fractional parts by 2 to derive 21/220.)

In the above discussion, as Group 2 explained how they found another rational number between 1/11 and 1/10 using a common denominator strategy, they provided new pieces of data that supported the conclusion, 1/11 = 10/110, 1/10 = 11/110, and 10 ½ /110 = 21/220. To convince the class that 10 ½ /110 was between the two given numbers, they used an open number line to draw the interval between 1/11 and 1/10 and marked the halfway point in this interval where 10 ½ /110 was located. As they did so, they provided additional information about 10 ½ /110, precisely where it was located between these two rational numbers. For this reason, their drawing served as a warrant because it provided information about the position of 10 1/2 /110—it was halfway between the two given numbers. Their explanation for how they derived 21/220, by imagining folding the fraction strip, was also a warrant because it further explained why 21/220 (or 10 ½ /110) was halfway between 1/11 and 1/10.

As the discussion continued, Instructor asked Group 2 to explain their answer, 21/220.

Instructor: Wait one second. You folded it in half…

Group 2: (Paraphrased.) We have 220 parts instead of 110 parts.

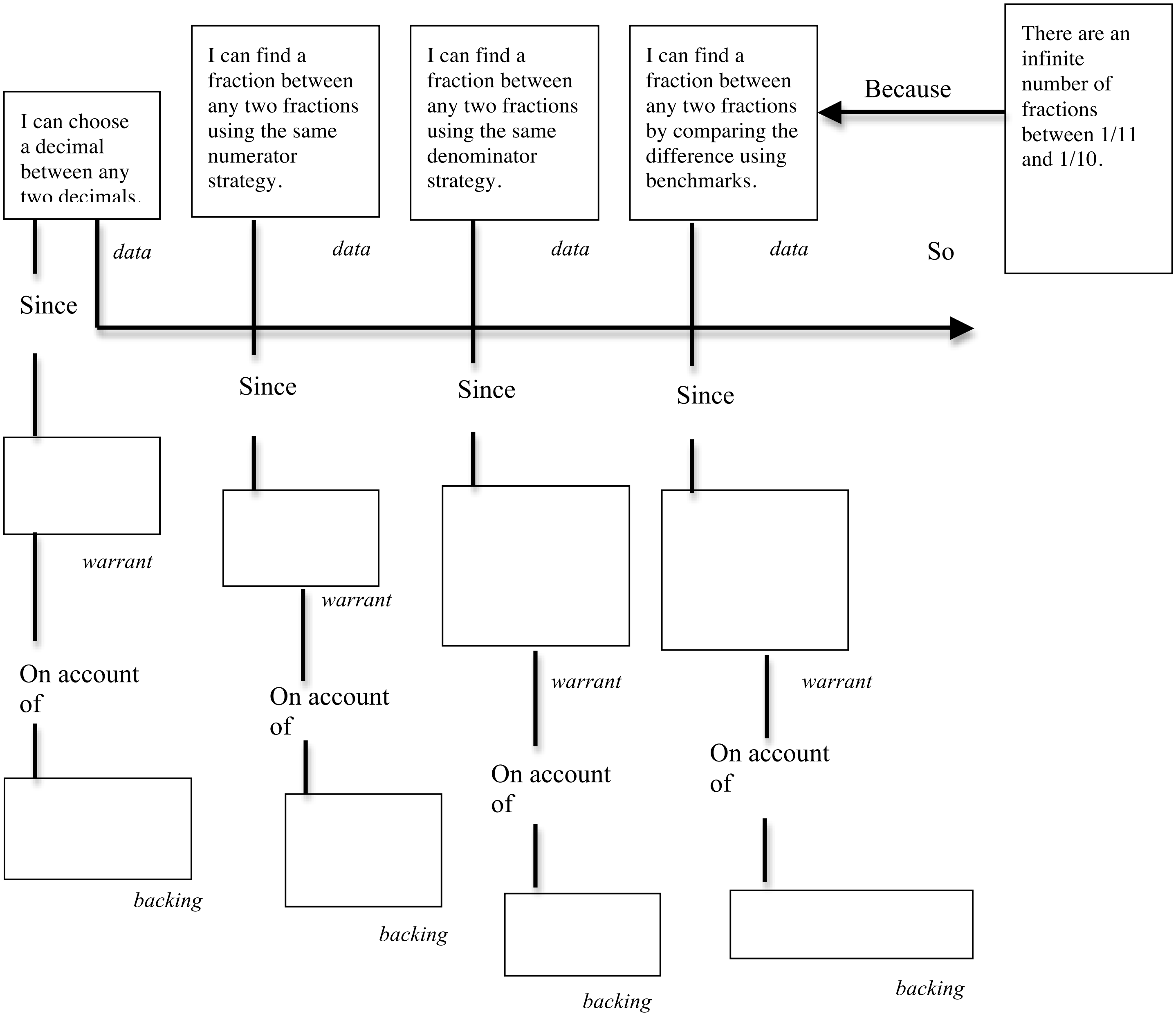
Instructor: 220 pieces, right? So you folded it one more time, every one of those 110 pieces is gonna be in half. And that would be how many?

Here as *Instructor* prompted Group 2 to further explain their strategy, they gave additional information about the 220—by repeatedly halving (or making half-folds), they could partition the strips into 220 instead of 110 equal parts. *Instructor*, too, offered additional information as he referred to folding (or imagining folding) the fraction strip. As he did so, he explained what happened when one folded an interval that was already divided into 110 parts. If each of these pieces were folded again, one would partition the unit interval into 220 pieces. So at this point, he, too, offered additional information that supported the warrant that Group 2 had provided. As such, both Group 2 and *Instructor* provided backings for the warrants.

In retrospect, we can see how the argumentation established by Group 2 and *Instructor* provides additional data for the claim—there are an infinite number of rational numbers between 1/11 and 1/10. In this case, it one repeatedly makes half-folds using a fraction strip, one will find more and more numbers between 1/11 and 1/10. Although neither *Instructor* nor the teachers explicitly indicated such, at this point in the discussion, the teachers along with *Instructor* have give several explanations for finding rational numbers between 1/11 and 1/10. Put another way, they have provided additional data and warrants (Ms. Sneider) that provide support for the general claim.

The discussion continued as other groups offered different methods for how they found rational numbers between 1/11 and 1/10. (See Figure 3, on the next page, for a general outline of the scheme for this argument.) Here we have illustrated only the first two groups' explanations to highlight the types of strategies that teachers developed. At the same time we have provided an analysis using constructs associated with argumentation to illustrate how these arguments were collectively accomplished. As we developed the argument, we did not imply that either the participants or *Instructor* intended to develop this more general argument for a specific case of the Real Numbers. Instead, using our interpretive lens, we have illustrated how *Instructor* and the teachers collectively engaged in a discussion that we then recast as an ensuing argument. As we have done so, we have highlighted the important but different roles that *Instructor* and the participants played as they engaged in this argumentation.

Figure 3. Schematic Representation for the Rational Number Argument



From Substantial to Analytic Arguments

Although of the teachers' strategies may not have been explicitly tied to the more general claim, it is certainly possible that some of the teachers made these types of connections. Retrospectively, it seems that it would have been feasible for *Instructor* to connect teachers' explanations to the more general claim, if that had been *his* instructional intent. As a result, we wonder, "How might *Instructor* have used the resulting substantial argumentation to facilitate a discussion so that teachers could verify the more general claim?" In other words, "How might *Instructor* use the substantial argumentation to establish an analytic argument for the general claim?"

Krummheuer (1995) indicated that a substantial argumentation becomes analytic when "the whole information of the conclusion is already included in the backing" (p. 244). However, he also cautioned about using ethnography of argumentation to make comparisons with some analytic ideal. This is not what we are attempting to do. Rather, we are merely observing that the analysis of the rational number and probability class discussion has allowed us to see something bigger than what happened during the lesson. As a result, we wonder what *Instructor* might do if verifying the general claim were actually his instructional goal.

For example, *Instructor* might decide to build off of any one of the groups' explanations (and strategies) to develop activities in which teachers explored proof making. Because of space limitations, we will only highlight one hypothetical scenario. Suppose, for instance, *Instructor* posed a task using Group 2's common denominator strategy during a subsequent lesson. Let's assume that he challenged teachers to use Group 2's strategy to find other rational numbers between 1/11 and 1/10. After they work in small groups to find others, he could reconvene the teachers for a whole class discussion so that they could share the different numbers that they had found. As they did so, he could record their ideas in the following table (see Table 1).

Table 1. Instructor makes a table of rational numbers that the teachers generated with each half-fold.

Number of half-folds	Easy to identify rational numbers between $\frac{1}{11}$ and $\frac{1}{10}$
1	21/220
2	41/440, 42/440, 43/440
3	…

After filling in the table, *Instructor* might ask participants if they saw a pattern in the table. The discussion might proceed as Instructor recorded the pattern in a new table (see Table 2).

Table 2. Instructor makes a record to count the number of rational numbers between each interval.

Number of half-folds	Number of new rational numbers
1	1
2	2
3	4
4	8
5	?

At this point, *Instructor* and the teachers might discuss how they could continue to insert new values into Table 2 indefinitely. As such, Table 2 could serve as a record of infinitely many rational numbers that could be identified between 1/11 and 1/10. So as they engaged in this type of discussion, they would make a proof for the general claim using this specific case. Moreover, they may also have the opportunity to count the number of rational numbers and even make a generalization about the number of rational numbers for the nth row of the table. In Krummheuer's (1995) terms, Table 2 would thus serve as a backing that contained (and supported) warrants and the conclusion for Argument 5, that is, it would become analytic.

Final Comments

The purpose of our discussion has been to better understand one instructor's role in supporting argumentation in a masters' course for K-5 mathematics specialists. As we have done so, we have also identified the kind of mathematical ideas and the possible learning opportunities that might have surfaced during each lesson. As we have illustrated, teachers participated in different but important ways as they engaged in these whole class discussions. *Instructor* was able to support teachers in different ways as they explained their ideas. As he did so, he made it possible for teachers to contribute to the discussion when and however they could. And when teachers offered more sophisticated ideas, he was able to capitalize on their contributions so that others might have opportunities to understand these ideas. So *Instructor's* role during the discussion was particularly important.

From a methodological standpoint, we have illustrated how one might use ethnography of argumentation as an interpretive framework to understand the instructor's role in supporting mathematical arguments in different ways. In the first example, we used our lens to highlight the complex ways in which the instructor might coordinate two (or more) arguments almost seamlessly during the discussion. In our second example, we highlight how substantial arguments might be used to provide opportunities for teachers to engage in formal proof making. As we did so, we illustrated the utility of using ethnography of argumentation to help us better understand how participants can collectively establish ways to reason mathematically that is couched in their understandings of the ideas at hand.

By carefully examining *Instructor's* role, we develop a deeper appreciation for the range of decisions that instructors might make to support unfolding mathematical arguments. *Instructor* listened to and supported the teachers' ideas. At the same time he aligned their ideas with those that fit with the mathematical practices of the mathematical community at large. By carefully balancing these sometimes competing agendas (Ball, 1993), he and the teachers were able to contribute in part to the rich ideas and the possible learning opportunities that may have surfaced during the lesson.

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