Testing Conceptual Frameworks of Limit: A Classroom-Based Case Study

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Abstract

The purpose of this study is to test three proposed conceptual frameworks of how students come to understand informal (dynamic) and formal (static) definitions of limit: one based on embodied cognition (Lakoff & Núñez, 2001), one based on APOS theory (Cottrill et al., 1996), and one created by the researcher. Predictions are made of how students would respond, based on each conceptual framework, to instructional tasks posed in a first-semester college calculus discussion section. These predictions are then compared to the responses of eight students during a sequence of interviews spanning the course of the semester. The results support the conceptual framework proposed by the researcher, with specific responses suggesting a further refinement, explaining how students come to bridge dynamic and static conceptions of limit.

Background on Limits: Conceptions and Research Trajectory

The mathematical definition of limit has gone through various refinements throughout history (Kleiner, 2001), but its current form, for two-dimensional limits, requires the coordination of two small intervals. One interval's radius, signified by the Greek letter delta (δ), surrounds the input or x-value, while the second interval's radius, represented by the Greek letter epsilon (ε), surrounds the output or the result of the function applied to the x-value. The coordination of the two intervals is accomplished through the logical quantification portion of the definition: for any epsilon, there exists a delta, such that if the input value is within the delta interval, then the output value must be within the epsilon interval. These intervals are symbolized through the distance interpretation of the absolute value, while the logical quantification uses common logical symbols for "for any," "there exists," and the "if …, then …" structure.

Limits can be mathematically conceptualized in two different ways, regardless of how a student could come to understand them: either in a dynamic way or in a static way (Cornu, 1992). The dynamic conception is a motion-based idea of limit, frequently expressed using the word "approaching" (or similar words describing movement), or graphically demonstrated by showing a point moving along a graph towards another point. This was also the first way in which limits were historically conceived by Newton and Leibniz (Kleiner, 2001). Students thinking in terms of a dynamic conception would describe, through words and gestures, a limit as a point traveling along a curve.

The static conception of limit is a function-based idea of limit which coordinates static intervals. This conception matches the formal definition devised by Cauchy and Weierstrass (Kleiner, 2001), and can also be thought of as a way to precisely describe closeness. Students thinking in terms of a static conception would describe, again through words and gestures, limits as coordinated intervals. In summary, the dynamic conception emphasizes motion in describing limits, while the static conception is devoid of motion and instead emphasizes closeness.

The initial wave of research involving students' conceptions of limit focused on misconceptions (Bezuidenhout, 2001; Davis & Vinner, 1986; Tall & Vinner, 1981). This work has evolved into research, or simply proposals, of how students might build appropriate conceptions of limit (Lakoff & Núñez, 2001; Williams, 1991, 2001). However, there has only been one study that attempts to link a conceptual framework with classroom instruction tailored to that framework (Cottrill et al., 1996), and that study was unable to successfully capture evidence of students conceptualizing the formal definition of limit.

Research Questions

This corpus of research presents a chicken-and-egg type problem. Building a representative conceptual framework of limit depends upon witnessing students successfully conceptualizing limits. Watching students learn about limits, particularly the formal definition, rests upon appropriate instructional strategies and curriculum. But these pedagogical strategies rest upon an accurate conceptual framework of limit. Thus, this study investigates three interdependent research questions concerning a concept of limit that includes both informal (dynamic) and formal (static) components:

1) What concept(s) of limit do the students in the case study have, and how can this be determined through their oral and written communication collected during the course?
2) What conceptual framework successfully describes how students, in general, construct a concept of limit?
3) What are the characteristics of an instructional sequence that successfully enables students to build a concept of limit?

Three Conceptual Frameworks of Limit

This study compared three conceptual frameworks: the one proposed by Lakoff & Núñez (2001) based on embodied cognition, the one proposed by Cottrill et al. (1996) based on APOS theory, and one created by the researcher.

Conceptual Framework of Limits Based on Embodied Cognition

Lakoff and Núñez introduce limits by first discussing one-dimensional limits: infinite sequences along a number line that approach a particular number (the limit). To demonstrate the process of "approaching" a limit, and how this utilizes the Basic Metaphor of Infinity (BMI), they give an example sequence $\{x_n\} = n/(n+1)$. As *n* increases, the value x_n gets closer and closer to 1 (Lakoff and Núñez , 2001, p. 187). The authors choose to express this notion of "approaching" by utilizing the notation of the formal definition of limits in terms of sequences: the sequence $\{x_n\}$ has *L* as a limit if, for each positive number ε, there is a positive integer n_0 with the property that $|x_n - L| < \varepsilon$ for all $n \geq n_0$ (pp. 189-90). Unfortunately, the formal definition cannot use the BMI directly, because there is nothing being iterated. In order to bridge the gap, they decide to express "approaching" in terms of nested sets, whose iterative quality can be directly used by the BMI: $0 < r < |x_n - L|$, where R_n is the set of all values *r* bounded between zero and $|x_n - L|$. As $|x_n - L|$ gets smaller, the range of values in R_n gets smaller.

This explanation of how we supposedly understand limits feels like an overly complex mathematization of the concept. Why do they do this? Since limits utilize the concept of infinity, Lakoff and Núñez are required to use their Basic Metaphor of Infinity to explain the concept of limits. Seeing as the standard formal definition of sequences doesn't use an indefinite iterate process, they need to formulate a new definition which does, hence nested sets. Thus nested sets can be plugged into the BMI as an indefinite iterative process, which in turn can be used to explain our conception of limit, regardless of how complicated or unnatural this conception might seem.

In their attempt to mathematize the concept of "approaching", they are completely ignoring the natural, physical-ness of approaching. What Lakoff and Núñez are trying to do is characterize the way in which the distance between you and the limit becomes smaller as you approach the limit. However, they use a complicated mathematical concept (nested sets) encoded in a complicated mathematical notation (absolute value), when a description of "approaching" as simply going towards something would suffice. This notion of "approaching" more closely resembles one of the four grounding metaphors (Lakoff & Núñez, 2001), arithmetic as motion along a path. Instead, they use the distance interpretation of absolute value as a symbolic bridge between limits and the grounding metaphor "arithmetic as motion along a path" for the purposes of measuring distances and the linking metaphor "numbers are points on a line".

In fact, thinking of "approaching" in this simple, physical way may even help students understand one very important piece of the formal limit concept. The same linguistic distinction between "jumping" and "swimming" can be made here: "approaching" is an imperfective aspect, because it does not inherently mean that you arrive at what you are approaching. This is actually helpful because it matches the definition of limit (in that a function may not exist at the limit). You may never reach the limit, because it may not exist, even though you are approaching it. Reaching a limit is a consequence of the BMI – in order to have a limit, you must "reach" it at the infinite step (at least potentially, if not actually). While it may be important sometimes to know what happens at the limit (for example, when considering continuity), this is not required when finding the limit itself.

While Lakoff and Núñez overcomplicate matters by using nested sets, they completely oversimplify the ways in which limits can be approached. Recall the example sequence $\{x_n\} = n/(n+1)$. Notice that this example is strictly monotonic, in that it creeps up on the limit in one direction, always getting closer and closer to 1, never farther away. This matches the smooth, motion based "approaching" conception of limit. Taking smaller and smaller steps towards a wall would be a physical example of this.

However, sequences (and functions) can approach limits in far more complicated ways than simply monotonically. The authors make an attempt to fix this by introducing sequences which converge indirectly but still have a limit. For example, they give a "teaser sequence" 3/6, 4/6, 5/6, 9/12, 10/12, 11/12, 15/18, 16/18, 17/18, 21/24 … which bobs up and down while generally trending towards 1 (Lakoff & Núñez, 2001, pp. 192-3). This improves their coverage of possible types of limit convergence, while stretching the metaphorical interpretation of limits as "approaching". This would correspond to a physical example where, for each step taken towards a wall, a step a fraction of that size is taken going away from the wall (remembering that the steps towards the wall are getting smaller and smaller).

But Lakoff and Núñez ignore functions (even ones as simple as $\{x_n\} = (-1)^n/n$) which approach a number from both sides, rather than just one side. Remember that the nested sets they describe are defined as the sets of the values *r* can take when $0 < r < |x_n - L|$. But in all the examples presented, simply writing $0 < r < L - x_n$ would suffice, since all instances of x_n are less than *L*. It seems that they use $0 < r < |x_n - L|$ because using the absolute value is necessary in the formal definition. If their definition did not cover limits which approach from both sides, they would be defining “approaching” in a lopsided way which was fundamentally different from the formal definition.

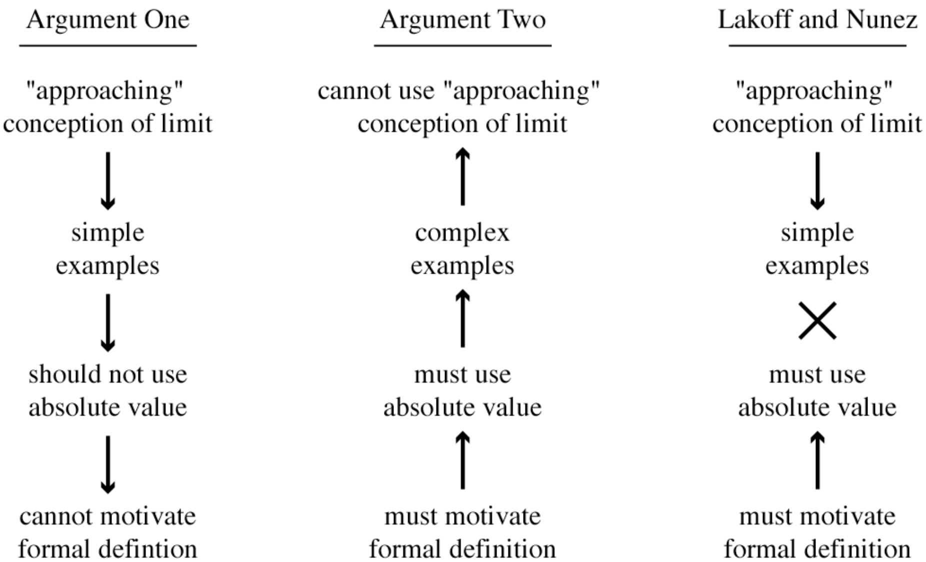

Figure 1. *Linking the informal, motion-based conception of limit with the formal definition*

Lakoff and Núñez are in a bit of a jam (Figure 1). They have chosen to pick a middle ground of one-dimensional limits, hoping to connect it to both the simplicity of the “approaching”, motion-based grounding metaphor, and to the complexity of the formal definition. However, their examples are poorly connected to either extreme: a simple, motion-based conception of limit would not need to be formalized through nested sets, while their examples are not complex enough to warrant the use of the absolute value necessary in the formal definition. And in all of this, by mixing a motion-based definition with an static-based definition, Lakoff and Núñez cloud the real issue: how do people move from the intuitive, grounded, dynamic conception of limit to the formal, static definition? It seems as though they are implying that the dynamic, informal conception of limit is the same as the static, formal definition of limit, since their examples are trying to connect the two.

Conceptual Framework of Limits Based on APOS

Unlike the embodied cognition approach, whose proponents have thus far glossed over the connection between the dynamic and static conceptions, APOS may be able to clarify why limits are so conceptually difficult for students. A process view of a limit (“approaching”) has been found to precede a static view of a limit (the formal definition) both historically (Grabiner, 1992a, 1992b) and in research of student understanding. If students move along this trajectory, which view is more problematic for students? Some believe that the process view is more natural for students to understand and the transition from process to the static view is difficult. Some go so far as to suggest that the process view should be avoided as to not lock students into a conceptual hurdle that they otherwise would not have to leap (Tall & Vinner, 1981; Williams, 1991). On the other hand, some say that the process view is not easy for students to understand, primarily because of unavoidable misconceptions stages of how limits work (Davis & Vinner, 1986). Instead of avoiding the process view, students need to engage it and reconcile certain epistemological obstacles (Brousseau, 1997; Sierpinska, 1987) before they can move on to the formal definition.

Cottrill et al. (1996) agree with the latter view and argue that the process view is challenging for students because of the complexity of the process, which is not simply one process, but a coordination of two processes (which makes it a schema). They propose a genetic decomposition of how students can first learn the dynamic conception of limit, which then can be transformed into a static conception of limit. Students initially evaluate a function at a few points, successively closer to limit, to see what the function "approaches". (In the second pass of building the genetic decomposition, a step before this one was added, where students only evaluate a function at a single point, close to or perhaps equal to *a*. Here, the dynamic conception of "approaching" is replaced by a static conception of "being in the vicinity of". Note that this does not match the advanced static conception of the formal definition.) Next, students construct a schema which interiorizes and coordinates two actions: the action of selecting *x* coordinates closer and closer to *a*, and the corresponding action of selecting *y* coordinates closer and closer to *L*. Coordination of these two actions is facilitated by the function of which the student is taking the limit. This two-process schema is then encapsulated into an object when students start performing actions on the limit concept, such as when rules governing the limits of combinations of functions are considered.

This is not where the genetic decomposition ends, however, because thus far a student's conception of limit is grounded in the "approaching" metaphor, and thus needs to be reformulated into a "static" form resembling the formal definition. Thus, students must go back and reconstruct the interiorization and coordination step in terms of intervals and inequalities. After this, they must apply a quantification schema which matches the internal logic of the formal definition. Finally, students then apply this new conception to specific instances, either limit proofs which require the formal definition or other concepts which need to be built directly from the formal definition, which in turn encapsulates the limit concept into a new object.

To summarize, here is a list of the steps of their genetic decomposition:

1) Students evaluate a function at a single point to find the limit.
2) Students evaluate a function at several points which are successively closer to *a*.
3) Interiorization of two processes (*x* approaching *a* and *y* approaching *L*) forms a coordinated schema.
4) Schema is encapsulated into an object.
5) Reconstruction of processes in terms of ranges (inequalities and absolute values).
6) Application of quantification schema results in formal definition.
7) Successful usage of formal definition.

This genetic decomposition continues to advance the idea that a dynamic, process-oriented conception of limit must precede a static, formal conception of limit. Since the primary focus of the genetic decomposition proposed by Cottrill and his colleagues is the formation of the dynamic conception, many of the classroom activities of their study, based on the genetic decomposition, are at the dynamic view level. As a result, much of the evidence from their study is also at the dynamic view level. Of course, this means that less emphasis is put in the genetic decomposition, and as a result, the classroom activities, on the reconceptualization stage. The authors then conclude that students have trouble thinking about the static view, when it is most likely their own research priorities and genetic decomposition (which influenced the pedagogy, which in turn influenced student thinking) are responsible.

Let us also think about the trajectory of this genetic decomposition from the students point of view. It is no small undertaking for a student to form a limit problem-solving heuristic while at the same time abstracting what limits are in general from that heuristic. So when this entire conceptual structure is finally formed and is robust enough to solve problems and understand what they mean, it should not be surprising that a student may be reluctant to give up that schema and reconceptualize it in terms of a new idea, "ranges" instead of "approaching". The schema works, why mess with it? The problem here is that the schema doesn't actually work, but most calculus students are never presented with sufficient evidence in the form of counterexamples to show that, under certain circumstances, their definition of limit is not sufficient.

The reconceptualization of the limit concept is necessary under APOS theory (moving from a dynamic to a static conception) in order to build more complex mathematical concepts, just as it is necessary under embodied cognition and mathematics in general (moving from an "intuitive" definition of limit to a formal one). However, I would argue that, to the student, this transition would appear to be unwarranted. The epsilon-delta windows activity in the Cottrill study, designed to enable students to complete the reconceptualization stage, does not seem up to the task of convincing students that there is something wrong with their schema and that it needs to undergo an accommodation step in order to explain this new exercise.

A New Conceptual Framework of Limits

In analyzing both of the preceding conceptual frameworks for limits, a new theory emerged of how students should and should not be taught limits, in order to match their expected learning trajectory. First, it seems clear that the dynamic conception of limit needs to precede the static conception. The dynamic conception is the more natural, grounded conception, and should be introduced as such, by using the "approaching" metaphor. Once students are familiar with the dynamic conception, they should then be presented the static conception in order to motivate the formal definition. This should be accompanied by some sort of motivation to move beyond the dynamic conception. This could take the form of a transition from dynamic to static language surrounding limit, or complex convergence problems that cannot be solved with the simple motion-based "approaching" conception. In any case, a new metaphor underpinning the static conception of "closeness" needs to be emphasized, separate from the original "approaching" metaphor. The non-intuitive static conception needs to be as strongly supported as the intuitive dynamic conception is for students. Ideally, through some way of grounding the static conception to experience (and not just syntax), both conceptions would become equally intuitive. At this point, students would be ready to bridge the two conceptions to form a robust conception of limit that encompasses both the dynamic and static conceptions.

Each theory also had a few suggestions for instruction. APOS suggests using holistic spray, foreshadowing upcoming topics, giving proper motivation to ideas, and generally giving adequate time for reflective abstraction. Here, this would mean asking difficult questions which prompt students to think about each conception of limit. Embodied cognition implies that we need to call specific attention to metaphorical mappings when used. Since each conception of limit is motivated by a different conceptual metaphor, these metaphorical mappings should be made explicit for students though class discussion.

Method

Materials and Setting

This study was conducted during the 2002 Fall Semester at a large, Midwestern public university. An instructional sequence was devised to first introduce students in a discussion section (attached to a large lecture) to the informal, dynamic conception of limit. Then necessary background concepts pertaining to the formal definition (such as the distance conception of absolute value) were introduced, along with a conscious change of language from dynamic (words like “approaching”) to static (words like “closeness”). The final activity in the sequence was a special problem (the bolt manufacturing problem) designed to introduce the logical structure of the formal definition:

> Suppose we run a bolt manufacturing company. We have lots and lots of contracts, with lots and lots of different companies. As you might expect, everybody’s needs are a little different. Bolts that we provide for home construction have to be of good quality, whereas bolts that we provide for NASA to be used on the space shuttle have to be of exceptional quality. For the sake of simplicity, let’s look at only one variable that goes into the quality of our bolts: length. Bolts for home construction that are supposed to be, say, four inches long can be a little more or a little less than four inches. But bolts for the space shuttle that are supposed to be four inches long have to be within a much smaller target range in order to be acceptable. The length of the bolt depends directly on how much raw material we put into the bolt making machine (assuming that the diameter of the bolts for the home and for NASA are the same), according to some function.
>
> How do we create bolts that we know will be of a length that falls within our target range?

The answer to this question is that, for any bolt length tolerance (or output range), there is a raw materials tolerance (or input range), and if the amount of raw materials we put into the machine falls within the raw materials tolerance, then we will get a bolt with a length that falls within our bolt length tolerance. This explanation matches the quantification schema of the formal definition of limit (Table 1).

This story problem was meant to be a way to *ground*, in the sense of embodied cognition, the formal definition, before the students were introduced to the actual notation of the formal definition. However, this was not supposed to be a grounding metaphor for the formal definition (in terms of actual physical things, like factories or bolts), but rather was to serve as a linking metaphor to a particular piece of the function concept, specifically the input-output relationship of functions.

Participants and Protocol

Eight students from the discussion section, taught by the researcher, participated in the experiment. There were five male students and three female students; one African-American female and one Indian male, all others were Caucasian. Seven of the eight were first year students, the eighth was a second year student who was required to take college algebra before calculus, based on placement test scores. By the end of the fall semester, three students had declared their intent to major in an engineering program, two were majoring in the sciences, and three were majoring in the arts. Two students received A's at the end of the semester, one received a B, four received BC's, and one received an F. The average grade of these eight students was a BC, which roughly matched the averages of the discussion section as a whole and the lecture.

Source Domain Bolt Problem		Target Domain Formal Definition of Limit
For any bolt length tolerance	$\rightarrow$	$\forall \varepsilon > 0$
There exists a raw materials tolerance	$\rightarrow$	$\exists \delta > 0$
Such that	$\rightarrow$	Such that
If …, then …	$\rightarrow$	… $\rightarrow$ …
the amount of raw materials put into the machine is within the raw materials tolerance	$\rightarrow$	$0 < \lvert x - a \rvert < \delta$
the length of the bolt the machine produces will be within the bolt length tolerance	$\rightarrow$	$\lvert f(x) - L \rvert < \varepsilon$

Table 1. *The bolt problem is the formal definition of limit*

The experiment consisted of a sequence of interviews throughout the semester (one interview before limit instruction, one just after limit instruction, and two more interviews after derivative and integral instruction respectively). In these interviews, the students were asked to explain their conception of limit, and then asked to explain how the bolt manufacturing problem related to the formal definition of limit. These interviews were transcribed and coded for instances of dynamic and static language in order to determine the presence and strength of their dynamic and static conceptions of limit. Interviews for the same student were examined for changes over time in their conceptual structure of limit, paying particular attention to the presence of the static conception and its relationship to the dynamic conception.

In the first interview, it was expected that many students would only know of limits through a dynamic conception. This was coded as "Stage A." Once students had gone through the designed limit curriculum and learned of the static conception, it was expected that they would (at least initially) treat the two conceptions separately and would not be able to describe a relationship between them. This was labeled "Stage B," as it represented a more advanced overall conception of limit, one that included both the dynamic and static conceptions. It was then expected that a few students would exhibit signs of a merged conception of limit. These would be labeled as “Stage C” (and this coding would be further refined as a result of the interview data, as elaborated below).

Results

During the first interview, students described limits exclusively using dynamic language (“approaching”, “gets close to”), if they knew what limits were at all. Jason’s first interview (9/5/02) was coded as Stage A, because he only describes the limit using dynamic language:

4. J: A limit is, um, I ‘spose it depends on how you look at it a little bit, but, it’s when uhh a number approaches another number, or a umm, uh an equation can have a limit as um, um like a series rather…

In Harriet’s first interview (9/6/02), she only mentions limits using dynamic language once:

3. T: Ok, so what’s a limit?
4. H: I know how to write it.
5. T: Ok. Why don’t you, here, why don’t you write it, at the bottom of that sheet.
6. H: It’s like, you do one of these, like limit as x approaches zer…, zero, and the there’s like a function.

Notice that while she uses the word “approaches” (line 6), she is really just vocalizing the limit symbol. This usage of dynamic language was also coded as Stage A.

By the second interview, all of the students could, with moderate success, explain the formal definition in the context of the bolt problem. Because of this, we can investigate the relationship between students’ dynamic and static conceptions. Five students described the bolt problem using static language (“is close to”), while acknowledging that the problem had to do with limits, which they continued to describe separately using dynamic language. Jason uses both dynamic and static language in his second interview (9/16/02). However, his language use switches between the two, using only one conception at a time (Stage B). He starts with dynamic language to describe limits:

4. J: A limit is, um, the point which a function approaches at a given value…

When asked to provide the formal definition, he can’t remember all the details, but he uses static language to describe the logical quantification:

16. J: … it’s when, you have an epsilon, uh, and you can have an equivalent delta, …
17. T: Ok.
18. J: … and they’re, and they have um, a relationship between, uh, how close they are, like, for every epsilon within a certain range, you get a delta within a certain range.

He explains, using static language, how the epsilon determines what the delta will be (“how close they are” and “within a certain range,” line 18). He uses a similar explanation, and similar static language, for the bolt problem:

26. J: Um, when you set a range for the um, length, you get a range for the material.
27. T: Ok.
28. J: And then you'll know how closely you have to measure.

However, when asked what the bolt problem has to do with limits, he switches back to dynamic language:

32. J: Yeah. Um, as you, as the um, hmm, I guess, I don't know exactly what you want, but, as the um, as you get closer and closer to the target material, you get closer and closer to the target length.

Here, Jason has switched back to a dynamic conception, because he is no longer discussing a range, but rather a "target" (line 32), and he is no longer using static language, but rather dynamic language ("closer and closer," line 32). While this may be laying the groundwork for a merged conception in the future, right now his two conceptions are separate, based on his language use.

The other three students in the second interview had already begun connecting their dynamic and static conceptions of limit. The most typical example of a student connecting their dynamic and static conceptions occurred during Lisa's third interview (10/29/02). Lisa is still having difficulty stating the formal definition of limit. But when asked to try to explain the definition, she draws the picture in Figure 2 and says:

30. L: I don't, not really, 'cause I can't really remember. But um, like, so you're like going towards a point on a graph…
31. T: Mmm-hmm.
32. L: … like this is [unintelligible]. And then, you have like here, here, so this is your um, this is your, uhh, delta, and this is your other delta…
33. T: Mmm-hmm.
34. L: … and as you're like getting closer this way …
35. T: Ok.
36. L: … then you're getting closer that way, I think.
37. T: Ok.

Here, she is drawing (Figure 2) the two intervals, which she calls "delta" and "your other delta" (line 32), one on each axis. While this is part of the formal definition, what she does next is not. She adds arrowheads onto the intervals, pointing inward, in order to signify that these intervals will be put into motion. She then explains "as you're like getting closer this way, then you're getting closer that way," (lines 34 and 36), showing how the motion of the interval along the x-direction is coupled with motion of the interval along the y-direction.

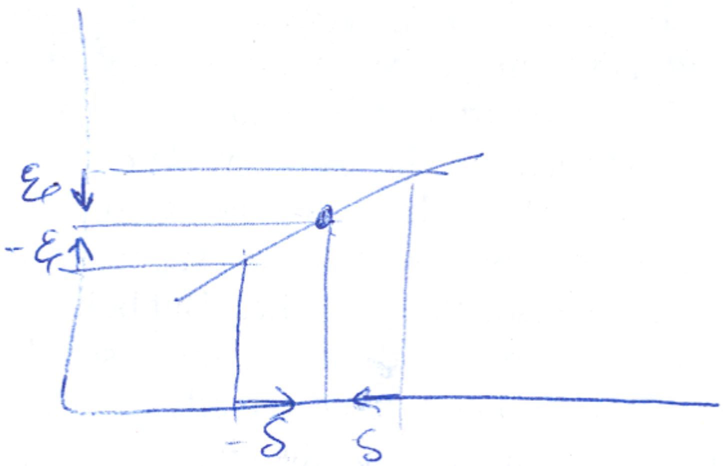

Figure 2. *Lisa's Diagram Showing a Stage C Conception of Formal Definition*

This suggests a refinement of the original conceptual framework proposed by the researcher (Table 2). Instead of simply lumping together the two conceptions, students preferred to connect them in one direction by initially embedding the static conception in the dynamic one (Stage C). In other words, students' dominant conception of limit was the dynamic conception, as they used the language of motion in order to explain why the static conception made sense.

The final stage (Stage D, an "expert" stage) reversed this hierarchy, making the static conception the dominant limit conception. In Steven's fourth interview (12/6/02), when asked about the bolt problem, his explanation uses solely static language, with the exception of the word "approaches" (line 39) when vocalizing the limit symbol:

37. S: To go back to that thing, I mean, what they're saying is that, um like if epsilon is the range, you know, greater than, you know, um, less than or greater than the value of y, then you can always find a delta which is always in the range …
38. T: Mmm-hmm.
39. S: … um, I don't know how to really put that in words but, [pause] like you can, you can get an accur…, as accurate an answer as you want, because within every epsilon you can always find a delta, umm [pause], yeah, so if you wanted, like, you know, a certain function approaches, umm, approaches zero, let's say, and um, uh, you had like epsilon like you know, infinitely small, you can always find a delta which is still within that range, so then zero and then plus or minus that epsilon.

<table>
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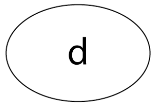
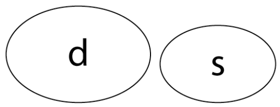
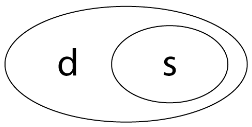
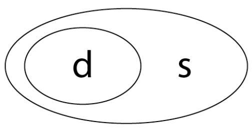
<tr><td>
</td><td>Stage A: dynamic conception only
Dynamic language only</td></tr>
<tr><td>
</td><td>Stage B: dynamic and static conceptions separate
Dynamic language dominant, separate from static language</td></tr>
<tr><td>
</td><td>Stage C: dynamic encompasses static conception
Dynamic language dominant but integrated with static language</td></tr>
<tr><td>
</td><td>Stage D: static encompasses dynamic conception
Static language dominant but integrated with dynamic language</td></tr>
</table>

Table 2. *Proposed Conceptual Framework for Limits*

While he is still a bit ambiguous about the mapping of epsilon and delta (saying that delta "is always in the [epsilon] range," line 37), the logical quantification is correct. When asked how this relates to limits, Steven says:

44. S: Yeah, just using like delta and epsilon …
45. T: Ok.
46. S: … you could use that here, because, um [pause], so, ok, um, epsilon would be, wait, the amount of material would be like the independent variable and the length of the bolt would be the dependent variable.
47. T: Ok.
48. S: So, depending on how, on how, like what you want for the length of the bolt, uh you, like whatever the, whatever error you could have, whatever you're, you know, your tolerance or whatever the error is possible, you could, uhh, find a delta within the range. Suppose epsilon is your, you know, your error …
49. T: Mmm-hmm.
50. S: … your margin of error, then you could find a delta within that [pause], knowing um, knowing the function, you know, whatever…
51. T: Ok.
52. S: … that you have to make the bolts from the material …

Thus, Steven has successfully identified how the static conception can be applied to the bolt problem. He identified which value belongs with which range symbol (line 46), and explained how the “error” or “tolerance” of each variable relates to the delta and epsilon ranges (line 48). All of this is accomplished by strictly adhering to static language, clearly the dominant conception of limit for him.

Student	Interview 1	Interview 2	Interview 3	Interview 4
Steven	A	B	D	D
Dave	A	B	C	D
Lisa	A	B	C	C
Jason	A	C	C	C
Erica	A	C	C	C
Harriet	A	C	C	C
Andrew	A	B	C	C
Daniel	A	B	B	A

Table 2. *Stage coding for each limit interview*

All of the students started at Stage A in their first interview. By the end of the interviews, all but one student had advanced to at least Stage C, with two students ending up at Stage D. Seven out of the eight students moved beyond an extremely limited conception of limit, mostly based on the dynamic conception of a boundary, that they had at the beginning of the interview process. Six of these students advanced to a conception of limit that incorporated both dynamic and static pieces in a way which had not been predicted prior to the study, that of putting into motion the ranges used in the formal definition. In this way, these students were able to connect the intuitive dynamic conception with the formal definition. Two students went even further, indicating through their use of language that they predominantly thought of limits through the static conception.

Discussion

Going into the study, it was expected that treating the dynamic and static conception of limit synonymously (Lakoff & Núñez, 2001) would be shown to be incorrect. During all but the first set of interviews (because students at that point did not have a static conception), students treated the two conceptions differently, and spoke of them separately by segregating their dynamic and static language. Therefore, as students described limits with two sets of terminologies, the dynamic and static conceptions of limit cannot be one and the same.

The genetic decomposition proposed by Cottrill et al. (1996) predicts that the dynamic conception would be frozen into place in order for students to build the static conception from the pieces of the dynamic one. However, the exact opposite was witnessed during the interviews: students took the static conception and put it into motion (by changing the sizes of the delta and epsilon intervals) to connect it to the dynamic conception.

In addition, Cottrill and his colleagues focused much of the genetic decomposition on the dynamic conception and not so much on the static one. They believed that students will have more difficulty constructing the initial, dynamic conception of limit and have less difficulty reconstructing the static conception of limit from the dynamic one. This difference should also be discernable from the interview data collected. If students have difficulty forming the dynamic conception of limit, but are then able to more easily transform that conception into a dynamic one, they are conceptualizing limit as APOS predicts. Instead, we saw that students more easily formed the dynamic conception of limit and had more difficulty understanding the static conception (which, coincidentally, is what also happened in the Cottrill study). While students' base, dynamic conceptions were very weak in their first interviews (Stage A), it was clear in later interviews that they were more likely to express limits by way of the dynamic conception (with the exception of those students who made it to Stage D), using it to help them bridge the gap between the dynamic and static conceptions.

Thus, the proposed model appears to be the best way to explain this data, especially once the new refinements (Stages C and D) are added. One might say that, because the curriculum was based on this model, it was natural for the data to come out this way. However, if the model had been incorrect, and students had thought in terms of the models derived from APOS or embodied cognition, students should have done poorly on the interview questions, as the mismatched curriculum would have worked against their construction of an overall conception of limit.

These results suggest a breakthrough for future limit conception research, not the least of which is having overcome the chicken-and-egg stalemate of previous research by way of an innovative curriculum. The instructional sequence instantiated during the study was successful in providing students the necessary framework to build a static conception of limit. In observing students as they formed this conception, a common trajectory of how students come to negotiate the dynamic and static conceptions of limit into one, overall conception was also observed. Future research will need to confirm these results and explore in more detail this trajectory. Other problems that need to be addressed include how supporting concepts (such as absolute value) enable a conception of limit, whether or not Stage D is truly an expert stage shared by mathematicians, how the norms of the constructivist-based discussion section affected the results (as opposed to a traditional lecture-based class), and how the formal definition of limit should fit into an undergraduate calculus course sequence. Finally, one study (Oehrtman, 2009) has suggested how the dynamic conception can be refined into different metaphors, suggesting future research using the bolt problem to determine whether particular metaphors are more prevalent when linking the dynamic and static conceptions of limit. In Steven's fourth interview, he not only mentions static and dynamic language, but also the notions of "infinitely small" and "accuracy", all of which could be more carefully parsed in order to further refine the base metaphors of the dynamic and static conceptions.

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