The Role of Skepticism in the Emergence of the Practice of Proving[i]

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The purpose of this paper is to explore skepticism in terms of its role in the emergence of the practice of proving. In particular, drawing on data from a series of teaching experiments and clinical interviews, three distinct paths to the practice of skepticism in mathematics are described: (1) cultural, non-experiential; (2) experiential; and (3) quasi-experiential. Analyses of students' responses are used to illustrate how two pathways, the experiential and quasi-experiential pathway, support the emergence of the practice of proving.

Introduction

Looking back at the National Council of Teachers of Mathematics (NCTM, 2000) standards for reasoning and proof, one finds the following statements:

- the habit of asking why is essential for students to develop sound mathematical reasoning;
- through the classroom environments they create, mathematics teachers should convey the importance of knowing the reasons for mathematical patterns and truths;
- reasoning and proof are not special activities reserved for specials times or special topics in the curriculum but should be a natural, ongoing part of classroom discussions, no matter what topic is being studied.

These statements are illustrative of a perspective in which rich mathematical understandings are viewed as resulting from student inquiry and justification. While few would argue with the idea that such understandings can arise from inquiry and justification – from knowing not only *what* but *why* – the implications of this perspective for classroom activity are far from clear. For instance, in her discussion of the emergence of proof in the classroom, Mariotti (2006) has argued, "studies exploring the potential of particular contexts are needed in order to shed light on the general characteristics required" (p.192). Similarly, Hoyles, in her discussion of the complexities of learning to prove deductively, argues, "We know now even more about potential obstacles to 'learning the mathematical game'; but need more systematic work on progress over time" (Ball, Hoyles, Jahnke, & Movshovitz-Hadar, 2002, pg. 914).[ii] These comments illustrate that there is a need within the research community to understand the milieu within which proof emerges and the characteristics of classrooms where 'learning the mathematical game' takes the form of a genuine mathematical endeavor as opposed to a mathematical farce.[iii] The purpose of this contributed research report is to discuss three pathways to the development of skepticism: (1) *cultural, non-experiential*; (2) *experiential*; and (3) *quasi-experiential*. In so doing, I will illustrate how two pathways, the experiential and quasi-experiential pathway, support the emergence of the practice of proving.

Background

Many have noted that students and teachers experience difficulties with mathematical proof (cf. Hoyles, 1997; Moore, 1994; Knuth, 2002; Harel & Sowder, 1998). Among other reasons, it can be argued that these difficulties arise from non-negotiated shifts in what constitutes a valid solution to a mathematical task. Prior to students' introduction to proof, computed results are often considered sufficient – it is the ends not the means that are valued. When asked for a mathematical proof, however, students are often provided with the very statement to be proved. Given the end, they are now asked to construct the means. The rules that determine what constitute valid means are often implicit and difficult for students to identify. Moreover, it can be argued that to adopt the new rules one must engage in a shift in one's epistemological stance towards mathematics.

Brousseau (1997), building on the work of Chevallard, argues that knowledge undergoes a didactical transposition. This transposition takes results from scientific activity and translates these results into 'teachable' knowledge. Each transposition poses the risk of distancing the knowledge to be taught from the very situations that supported its manifestation and meaning – "(decontextualization) has as a price the loss of meaning and performance at the time of teaching" (Brousseau, 1997, p. 262). Given the substantial research indicating students' difficulties with proof (cf. Harel & Sowder, 1998), we must be concerned with:

- Students' understandings of the meaning and purpose of the practice of proving that may result from the transpositions that have occurred and, therefore, the distance between commonly used didactical situations and the situations that fostered the manifestation of the practice of proving;
- Researchers' understandings of the complexities involved with shifting one's epistemological stance towards mathematics and the potential lack of appropriate instructional materials and pedagogical approaches.

Defining Skepticism

Zaslavsky (2005) has argued that uncertainty occurs when a group of learners "contemplates over a certain conjecture, without a sense of certitude whether it is valid or not and why it is or is not." Uncertainty then is doubt, without a *basis of belief* – without that which conveys an intuitive acceptance, a feeling 'it must be so' (Fischbein, 1982). Skepticism, on the other hand occurs when a conjecture is viewed as uncertain or unknown, despite the existence of a basis of belief. Skepticism can occur against intuition, that is, against a feeling "that it must be so." Skepticism can be thought of as a state of being; that is, a collective or individual can, at a particular point in time, both obtain evidence for a conjecture and view the conjecture as of unknown truth value. Skepticism, however, can also be thought of as a practice --"classroom mathematical practices are the taken-as-shared ways of reasoning, arguing, and symbolizing established by the classroom community while discussing particular mathematical ideas" (Cobb, 2000, p. 71). In everyday settings, proof of validity often takes the form of an empirical argument, an approximation or a summation of one's experiences. When proving in mathematics, however, the act of suspending judgment against a backdrop of empirical or experiential evidence is the taken-as-shared way of reasoning; that is, it is a mathematical practice. It follows that uncertainty and skepticism differ in form: both occurring as a state of being but only the latter has the potential to evolve into a practice; and, with uncertainty occurring without a basis of belief while skepticism occurs against a basis of belief.

The Study

The data discussed in this paper is drawn from a multi-year project focused on students' understandings of mathematical induction. Studies conducted during the project included: observations of introduction to proof courses and clinical interviews with the students; teaching case studies involving novice proof writers; and a series of five teaching experiments involving undergraduate mathematics and science students enrolled in the second semester of a year-long calculus course, with no prior exposure to proof at the university level as determined through a curricular history questionnaire and assessment. The teaching experiments, which ranged in length from 3 to 6 weeks, involved 3 to 8 students depending on the university. Students met with the author for a minimum of two 75-minute sessions per week. The courses are best described as focused on advanced problem solving and proof, with students engaging in small group problem solving and whole class analysis of solutions. The tasks used during the sessions were developed primarily by the author, with some begin drawn from the work of Harel and Sowder (1998). The tasks represent modifications of tasks commonly used in introduction to proof textbooks. Data for the study included videotaped classroom sessions, transcripts of all sessions, instructor field notes, and students' written work.

Findings

Three pathways to skepticism were identified in data from the clinical interviews and classroom teaching experiments: (1) cultural, non-experiential; (2) experiential; and, (3) quasi-experiential. The *cultural, non-experiential* pathway is a didactical pathway that arises when an authority within the collective dismisses a basis of belief but there is no experiential history that the collective has shared. In this case, a normative way of reasoning is established through the power of an authority rather than negotiated. The authority's warrant for introducing the practice is that it is a practice of the larger community and, therefore, must be adopted by those who wish to legitimately participate within the community.

Consider the following example, drawn from the clinical interviews. Evan, an introduction to proof course student, was asked to evaluate four proofs of the claim "The sum of the angles of a triangle is 180°." When examining the standard Euclidean proof, which included both a written argument and a diagram (see Figure 1), Evan rejected the argument on the grounds that it included a "picture."

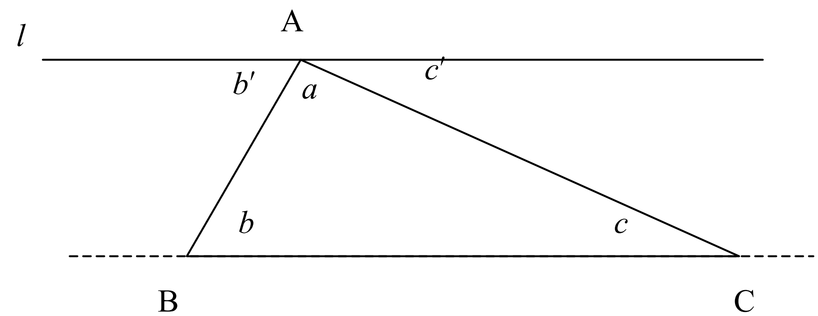

Figure 1.

SB: Why does the picture matter?
Evan: Pictures aren't proofs.
SB: Why aren't pictures proofs?
Evan: (pauses) Dr. Black told us that.

After a brief discussion about the lesson[iv] during which his professor had made this statement, Evan was asked to explain how the diagram or "picture" was used in the proof. Evan's response, however, provided no indication of his understanding of the role of the diagram, as illustrated in the following excerpt:

SB: In this one, do they use some particular part of the diagram?
Evan: (pauses, looks at the diagram) It doesn't matter, you're not allowed to use pictures.

The root of Evan's rejection of the argument appears to be his instructor's comments rather than a particular aspect of the diagram or his knowledge of instances for which such diagrams could lead to false conclusions. In this case, doubt exists in relation to the validity of a claim and is a result of the actions of an authority of the community, who had dismissed the basis of belief, rather than a result of a shared experience.

The second pathway, the *experiential* pathway, has three distinct phases. First, the collective encounters an unusual or unexpected result. Second, the collective uses the result to debase particular forms of reasoning, in terms of their capacity to function as a means for verification. Third, the collective renegotiates the status of particular forms of evidence; that is, new socio-mathematical norms are established.

Early in the teaching experiments, students were asked to solve a modified version of the Towers of Hanoi task (Figure 2).

Towers of Hanoi

Three pegs are stuck in a board. On one of these pegs is a stack of disks graduated in size, the smallest being on top. The object of this puzzle is to transfer the stack to one of the other two pegs by moving the disks one at a time from one peg to another in such a way that a disk is never placed on top of a smaller disk. How many moves will it take to transfer a stack of 1275 disks to another peg?

Figure 2. Modified Towers of Hanoi Task[v]

In each case, that is, in all of the teaching experiments, students developed data tables and recognized both a recursive pattern $2k + 1$ for the relationship between consecutive cases and an n^{th} term pattern $2^n - 1$. Students' claims of validity related to the latter formula were rooted in explanations of the form, "it matches" the data, as illustrated in the following transcript segment.

Susan: I think it ($2^n - 1$) works. Look, 2^6 is 64.
Johan: Minus one is 63.
Paula: 2^5 is 32 minus one is 31.
Johan: 2^4 is 16.
Susan: It works … that's it.
Johan: We got it.
Susan: Now we're done with that one.
Paula: All right, next one! Come on!

Subsequent efforts to perturb the students through instructor posed questions were, at best, unproductive, as illustrated in the following excerpt:

SB: … is there a way of verifying that …that would actually be the case?
Jill: Mathematically? I mean you could just go through and do it every time.
…
Calvin: I don't understand what … what you really want to know.
Jill: She wants us … she wants us to prove it mathematically.

These responses indicate that the questions of validation posed during the lesson were *alien* (Duffin & Simpson, 1993) to the students. Moreover, after identifying the n^{th} term pattern $2^n - 1$ and testing this pattern for $n = 1$ through $n = 6$, the students' inquiry completely stopped and they used the expression $2^n - 1$ to compute the total number of moves for 1275 disks. This suggests that the students had accepted the validity of the expression $2^n - 1$. To address the lack of doubt for empirically-verified claims, the students were asked to work on the *Chords of a Circle* task.[vi]

Question: Suppose you have a circle with n points marked on the circumference. By connecting each pair of points with straight-line segments the circle can be partitioned into a number of regions. Is there a function for calculating the number of regions?

Figure 3. Chords of a Circle Task

Unlike the Towers of Hanoi task, the n^{th} term pattern, 2^{n-1}, which is quickly recognized by students fails to hold for all $n \geq 6$. When $n = 6$ one may obtain either 30 or 31 regions but not the anticipated 32 regions. The production of a pattern that fails to match the 6^{th} case surprises the student, as seen in the segment below.

Boris: How many did you get, Jill?
Jill: Thirty-one but it has to be wrong … because it has to be even.
Calvin: I got thirty-one too … but it's wrong.

After having discussed the Chords of a Circle task and identifying features of the context which lead to multiple solutions for $n = 6$, students in the teaching experiments were asked to revisit their solution to the Towers of Hanoi task. As is illustrated by the segment below, during which the students discussed the validity of their solution, the collective experience of encountering an unusual or unexpected result supported the collective reconsideration of particular forms of evidence (in this case, empirical evidence) in terms of its capacity to function as a means for verification.

Calvin: Our table shows it.
Jill: But how do you know at one point it might not … it might not happen? I understand what you're saying here, if it works for this one it's going to work for that one but it … what if at one point it doesn't? Like the circle thing?

Jill's comments to Calvin prompted the students to reexamine the process by which they had generated their data and to generate an argument[vii] that contained the key ideas of a proof by mathematical induction.

Calvin: Suppose it [$2^n - 1$] works for some stack, then we know the next stack takes [writes "$(2^n - 1) + 1 + (2^n - 1) = 2^{n+1} - 1$"] since we know the two k plus one [$2k + 1$]formula works. So, if it works for five it works for six and we have all of these [reference to data table] so we know it always works. We don't have to worry about the circles.

The experiential pathway to skepticism is, therefore, distinct from the cultural, non-experiential, in that skepticism emerges as a result of the collectives experiences rather than as a result of an authority dismissing a basis of belief.

The third pathway to skepticism, *quasi-experiential*, develops when the collective acquires knowledge of structural and historical aspects of mathematics as a subject-matter domain, which – in the eyes of the collective - function as cultural warrants for the practices of the community. This pathway was observed in a teaching experiment, during a series of sessions focused on the Convex N-gon task (Figure 4).

Let n be a natural number. Is there a formula for the sum of the interior angles of a convex n-gon?

Figure 4. Convex N-gon Task

In response to this task, the cohort of students, Johan, Susan and Paula, constructed the diagram shown in Figure 5.

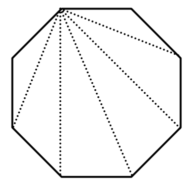

Figure 5.

Using this diagram, the students argued that every convex *n*-gon contains $n - 2$ triangles, therefore, the sum of the interior angles is $(n - 2)180$ degrees.[viii] The instructor then asked the students to justify the claim, "every convex *n*-gon contains $n - 2$ triangles."

> SB: How would someone know the number of triangles?
> Susan: They just do! They have to do it themselves.
> …
> Paula: They just do!

The students' comments illustrate a sense of conviction -- a basis of belief, potentially derived from perception. This conviction may have been derived from a specific instance of perception ("I can see it works when $n = 8$") or from perceiving in the diagram a structure ("I can see how I could partition any convex *n*-gon").

Instructor posed questions, regarding how one would know for certain that every *n*-gon can be partitioned, prompted discussions within the collective regarding what would cause an *n*-gon to fail to partition in this way.

> Susan: Why couldn't it be? That's what I don't understand.
> Paula: It has to be.

Essentially, the issue the students had arrived at was, "Why, in mathematics, does one rule out that which one cannot imagine?" Instructionally, this raised the questions: what is the root of this cultural practice; and, what experiences might students need in order to understand this practice? Recall that students enrolled in the teaching experiments were recruited from the second semester of calculus. Building on the students' concurrent coursework, the cohort was asked to define continuity and then differentiability. After which, the students were asked, "Is it possible to have a function which is continuous but nowhere differentiable?"

> Johan: It would have to have corners everywhere … like a fractal.
> Susan: That's scary.
> Paula: That's … just wrong.

These reactions, which are similar to Poincaré's, highlight the students' surprise over the existence of such functions -- objects one might fail to imagine, if their existence was determine by perception alone. Upon returning to the *Convex N-gon* task, students' comments moved away from claims of the form "you can just see it" towards attempts to construct a general *n*-gon. In other words, visualization no longer functioned as the primary warrant in the students' efforts to verify their solution to the *Convex N-gon* task.

Thus, doubt against a basis of belief, in the case of the quasi-experiential pathway, is not the result of a collective investigation, as is the case with the experiential pathway, nor is it the result of a mandate from an authority, as is the case with the cultural non-experiential pathway, but rather develops from the collective acquiring knowledge of structural and historical aspects of mathematics as a subject-matter domain, which serve as a warrant for practices within the mathematics community.

Concluding Remarks

While much of the research on proof in the classroom has centered on inquiry (Mariotti, 2006) and the development of a culture of "why" questions (Jahnke, 2005), less has looked into the development of doubt and its role in the emergence of the practice of proving. While doubt may arise from a sense of uncertainty (Zaslavsky, 2005), it is not always possible to create a sense of uncertainty; especially, when a basis of belief develops in conjunction with either empirical or perceptual evidence. In such cases, doubt must arise against a basis of belief; that is, take the form of skepticism. How might doubt arise against or despite a basis of belief? The data presented in this paper, which are derived from a series of teaching experiments and clinical interviews, indicate that skepticism has multiple pathways, not all of which are productive; such as, the cultural, non-experiential pathway. The data also indicate that in the case of the experiential and quasi-experiential pathways, skepticism arises when, in addition to a basis for belief, one also possesses grounds for doubt. This conclusion aligns well with the claims of philosophers who took a pragmatic as opposed to Cartesian view of doubt; such as, Charles Saunders Pierce, who argued that a doubt must have grounds, and Ludwig Wittgenstein who claimed that a doubt is always based on sequestered beliefs, for "a doubt that doubted everything would not be a doubt."

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[i] This is a brief contributed research report presented at the *13th Annual Conference for Research on Undergraduate Mathematics Education*, held in Raleigh, North Carolina in 2010. An extended discussion of this work is in preparation. For more information please contact the author, Stacy Brown, at Stacy_Brown@pitzer.edu.

[ii] It is likely that Hoyles is referring to Brousseau's notion of an *obstacle* and his metaphor of *the game*. See Brousseau (1997) for more details.

[iii] I use the term farce not to ridicule particular approaches or discipline specific instructional practices but to honor the multitude of students who have made comments to me of the form, "proving is ridiculous, why should I waste my time showing something is true when we already know it is true."

[iv] Interview participants often referred back to classroom lessons, in part, because they knew that I had attended the classes with them.

[v] This is a non-standard version of the task. Typically, the student is asked to determine the number of moves for n disks.

[vi] This task has been discussed at length by many researchers (see, for example: Mason, 1985; Orton & Orton, 1999; Brown, 2003; Harel & Brown 2008; Stylianides & Stylianides, 2009). Mason refers to this task as the circle and spots problem. The modification used in the experiment removed the task constraint that no three line segments meet at a point.

[vii] Due to space limitations a full description of the students rationale for the recursive formula $2k+1$ is not provided.

[viii] Certainly, in some classrooms, such an argument would be considered sufficient. Recall, however, that the focus of the teaching experiment was the development of mathematical induction.

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