Exploring Teachers' Capacity to Reflect on their Practice[1]

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Running Head: Exploring Teachers' Capacity

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Abstract

Although the idea of *pedagogical content knowledge* (PCK) has been elaborated in numerous studies, there has been little clarification of what constitutes it or research into its development. Furthermore, studies that have investigated PCK, or *mathematical knowledge for teaching* as first introduced by Thompson and Thompson (1996), have historically focused on pre-service teachers at the elementary level. This study contributes to filling these voids by investigating in-service secondary school teachers' ways of thinking that supported or constrained their capacity to reflect on their practice as they engaged in activities designed to promote powerful mathematical knowledge for teaching as proposed by Silverman and Thompson (2008). Findings indicate that teachers' whose personal mathematics focused on facts and skills found reflection most difficult; their mathematical knowledge constrained their capacity to reflect on the reasoning that they engaged in through instruction, impeding the level of coordination of meanings required to sustain propitious reflection.

Key Words: Mathematics Knowledge for Teaching; Pedagogical Content Knowledge; Mathematics Teacher Education

Introduction

Stigler and Hiebert (1999), drawing on the conclusions of the Third International Mathematics and Science Study (TIMSS), highlighted the necessity for reform of mathematics teaching in the United States. In the years since *The Teaching Gap* there has been one generally agreed upon theme – that students are not developing a satisfactory level of mathematical proficiency (Baldi, Jin, Skemer, Green, & Herget, 2007; Gonzales, et al., 2000; Gonzales, et al., 2004).

Although elementary and secondary students' mathematics performance has shown some improvement over the past decade, this improvement has not been in all grades assessed and is not equal for all groups of students (Baldi, et al., 2007; Hall & Kennedy, 2006; Planty, et al., 2007). Several documents have indicated the important role that teachers play, not only in what students learn, but also in any mathematics reform effort (Board, 2006; Borasi & Fonzi, 2002; Lampert, 1991; Mathematics, 2000; Panel, 2008; Stigler & Hiebert, 1999; Wenglinsky, 2002). Therefore, in order to most propitiously influence student learning, it is of paramount importance to identify characteristics of effective teaching and determine how best to develop these characteristics in the minds of teachers.

In the past two decades a number of researchers have attempted to identify and explore the characteristics of mathematics teaching and teachers. Several researchers have investigated teacher beliefs (Cooney, 1985; Ernest, 1989; Leder, Pehkonen, & Torner, 2002; A. G. Thompson, 1984, 1992); others have placed a focus on identifying the structure of teacher knowledge in an attempt to identify a knowledge base for teaching (Ball, 1988; Ball & Bass, 2000; Ball, Hill, & Bass, 2005; Rowland, Huckstep, & Thwaites, 2005; Shulman, 1986, 1987). In addition, various researchers have commented on the difficulty inherent in attempting to distinguish between knowledge and beliefs (Fennema & Franke, 1992; Grossman, Wilson, & Shulman, 1989; A. G. Thompson, 1992).

According to Ball (2003), "The quality of mathematics teaching and learning depends on what teachers do with their students, and what teachers can do depends on their knowledge of mathematics" (p. xv-xvi). Although several studies have shown that a teacher's content knowledge influences their teaching practices (Fernández, 1997; Sowder, Philipp, Armstrong, & Schappelle, 1998; Swafford, Jones, & Thornton, 1997), attempts to quantify a link between teacher subject matter knowledge and student achievement have been largely inconclusive (Begle, 1972, 1979; Eisenberg, 1977; Monk, 1994; Rowan, Chiang, & Miller, 1997).

Beginning with Shulman's (1986, 1987) seminal work regarding *pedagogical content knowledge* (PCK), several researchers have focused on the form, nature, organization, and content of teachers' mathematical knowledge (Ball, 1990; Lampert, 1991; Leinhardt & Smith, 1985; Marks, 1987; Steinberg, Marks, & Haymore, 1985; A. G. Thompson, 1984; A. G. Thompson, Philipp, P. W. Thompson, & Boyd, 1994; Wilson, Shulman, & Richert, 1987). In the view of Thompson and Thompson (1996), "This work has highlighted the critical influence of teachers' mathematical understanding on their pedagogical orientations and decisions — on their capacity to pose questions, select tasks, assess students' understanding, and make curricular choices" (p. 2). According to Schoenfeld (2000), "Such studies indicate ways in which teachers' knowledge shapes what the teachers are able to do in the classroom - at times constraining their options, at times providing the support-structure for a wide range of activities" (p. 247).

Mathematical Knowledge for Teaching

In 1996, Thompson and Thompson coined the term *mathematical knowledge for teaching* (MKT) to refer to understandings that "cut across the types of knowledge typically embraced by phrases such as 'content knowledge' or 'pedagogical content knowledge'" (p. 19). Silverman and Thompson's (2008) model for mathematical knowledge for teaching, which expands MKT to the high school level, builds from Thompson and Thompson's (1996) conception, and on Simon's (2006) construct of *key developmental understanding* (KDU), Silverman's (2005) construct of *key pedagogical understanding* (KPU), and Piaget's (2001) construct of *reflective abstraction*.

For Silverman and Thompson (2008), MKT is grounded in the idea that "to know" means to have a scheme of meanings (i.e., mathematical ways of thinking and mathematical ways of understanding) that express themselves in action. Accordingly, for Silverman and Thompson (2008), "to understand" means to assimilate to a scheme.

In Silverman and Thompson's (2008) view, powerful mathematical knowledge for teaching involves developing significant personal understandings of a particular mathematical topic (i.e., a KDU) and transforming these personal understandings to understandings and ways of thinking that are pedagogically powerful (i.e., a KPU). An individual (e.g., a mathematics teacher or mathematics student) has developed a KDU when they have constructed a scheme of meanings that proves foundational for understanding a broad array of mathematical ideas and methods. In order for this KDU to be transformed into a KPU requires the teacher to: become reflectively aware of the KDU; realize the benefit that such an understanding could provide for her students' future learning (i.e., the work that the KDU could do for her students); and build a way of thinking about how to support the development of such an understanding in the minds of her students. According to Silverman and Thompson (2008), both the personally powerful understandings and mathematical knowledge for teaching develop via a process that Piaget called reflective abstraction.

Although, the idea of pedagogical content knowledge has been elaborated in numerous studies (Ball & Bass, 2003; Ball, et al., 2005; Ball, Thames, & Phelps, 2008; Carpenter, Fennema, Peterson, & Carey, 1988; Grossman, 1990; Ma, 1999), there has been little clarification of what constitutes it or research into its development. In addition, the majority of studies that have investigated mathematical knowledge for teaching (or PCK) have been with pre-service teachers at the elementary level (Ball & Bass, 2003; Carpenter, et al., 1988; Hill, Ball, & Schilling, 2008; Marks, 1987). The current study contributes to filling these voids by exploring in-service secondary school teachers' cognitions as they engage in activities designed to promote powerful mathematical knowledge for teaching as proposed by Silverman and Thompson (2008).

The Project and Study

The current study is part of a five-year NSF-sponsored research project, the Teachers Promoting Change Collaboratively (TPC^2) Project, conducted by Professor Patrick Thompson and his research team at Arizona State University. The larger project goal was to help teachers move from a very teacher-centered orientation to a very student-centered orientation. The project participants consisted of local high school mathematics teachers from a large district in the southwestern United States.

A major component of the project were three graduate-level mathematics courses, taken over three years, referred to as Functions 1, 2, and 3. The Functions courses involved activities that were intended to transform the participating teachers personal and pedagogical understandings of the mathematics they teach. The project attempted to affect the teachers' mathematical content knowledge in Functions 1 and 2. The Functions 3 course was designed to both promote and investigate the development of mathematical knowledge for teaching in the Silverman and Thompson (2008) framework.

The main goal of the Functions 3 course was for the teachers to transform their ways of operating, by replacing their current schemes with cognitive structures that are more conceptually oriented. As such, Functions 3 course activities were designed explicitly to create contexts for which the teachers would reflect on their own activity in a way that might translate into them doing things differently in their own classrooms. The Functions 3 course met for a total of 14 class sessions (7 summer meetings, 7 meetings during the fall semester).

The current study investigated Functions 3 participating teachers' ways of thinking that supported or constrained their capacity to reflect on practice. However, reflecting on one's practice entails thinking about more than the mathematics you will teach. It also entails thinking about the mathematical realities of students who will learn that mathematics and the tasks you will use in your teaching. As such, the study attempted to answer the following research questions:

1) In what ways do teachers' mathematical understandings and ways of thinking support or constrain their capacity to reflect on their practice?
2) In what ways do teachers' images of their students' mathematics support or constrain their capacity to reflect on their practice?

The Didactic Triad

A major focus of the Functions 3 course involved discussions and activities centered around the *Didactic Triad* (Thompson, 2009). The *Didactic Triad*, is a means of support that re-orients teachers productively as they design instruction, and emphasizes the interrelationships amongst three aspects of instruction: learning goals – understandings or ways of thinking that the teacher intends for their students to develop; tasks/materials – tasks, activities, and materials that the teacher chooses or creates to support the development of the intended ideas; and, teaching – choreographing activities and conversations about those activities in a manner that enhances the likelihood that students develop the intended understandings or ways of thinking.

The TPC^2 project has found that teachers who successfully changed their instruction to emphasize students' mathematical thinking tended to use the *Didactic Triad* in a very special way. Rather than looking at each component in isolation, these teachers always focused on all three aspects of instruction simultaneously. Although at any given moment a teacher may place an emphasis on one area of the *Triad*, this focus is always in the context of the other two.

Mathematical Realities

From a radical constructivist perspective, individuals construct their own reality on the basis of their experiences; and, an observer (i.e., teacher or researcher) has no direct access to those constructions (Glasersfeld, 1995). Therefore, an observer (i.e., teacher or researcher) must attribute understandings and mathematical realities (Steffe & Thompson, 2000) to their students that are independent of their own understandings and mathematical realities. Steffe and Thompson (Steffe & Thompson, 2000) introduced the phrase "students' mathematics" to refer "to whatever might constitute students' mathematical realities (p. 268); and the phrase "mathematics of students" to refer to an observer's (i.e., a researcher's or teacher's) interpretations of students' mathematics. According to Steffe and Thompson (Steffe & Thompson, 2000) "Students' mathematics is something we attribute to students independently of our interactions with them…[and] is indicated by what they say and do as they engage in mathematical activity" (p. 268); and "is not considered as … knowable independently of the [observer's] ways and means of perceiving and conceiving" (Steffe, 2007).

Analytical Methodology

Data were derived from videotapes, work artifacts, field notes, and teacher interviews generated as the teachers engaged with instruction designed to promote conceptual approaches to teaching major ideas in secondary mathematics. The analytical methodology employed in the study is consistent with Cobb and Whitenack's (Cobb & Whitenack, 1996) method for retrospective analysis of qualitative video data and grounded theory's (Strauss & Corbin, 1998) iterative process of continual review, constant comparison, and regeneration. Data analysis happened at two levels:

1) Review the entire video collection with the intent of identifying episodes that were explicitly designed to provoke teacher reflection;

2a) Provide descriptions of and the rationale for each reflective episode. This gave context to the analysis and provided models of intended instructional outcomes.

2b) Use conceptual analysis to construct models of the teachers' ways of thinking as they engaged with instruction.

Although individual teacher models were developed and refined through conceptual analysis, these results will not be discussed in this paper. Rather, the current report will focus on those epistemic ways of operating that were developed via conceptual analysis of the larger data corpus, and discuss how these ways of operating demonstrated themselves during lessons designed explicitly to provide the teachers with opportunities to reflect on their practice.

Analysis has identified two main epistemic ways of operating that exhibited themselves throughout the Functions 3 course. These ways of operating involved the teachers' meanings, specifically those involving the mathematics that they teach, and those associated with "reform" or "project-specific terminology," and the teachers' mindfulness of students' mathematics.

In the remaining sections of the paper, I will discuss the general (epistemic) ways of operating indicated through analysis of the data corpus, identify how these ways of operating demonstrated themselves during lesson explicitly designed for the teachers to reflect on their practice, and discuss the implications that I envision these findings having for mathematics teacher education.

Meanings

There is a lot of evidence to suggest that the teachers were generally not inclined to make their meanings explicit; specifically, that the teachers were generally disinclined to articulate their meanings and ways of thinking with regards to mathematical concepts. This disinclination exhibited itself at various points throughout the Functions 3 course, especially at times when the course activities were designed to provide teachers with the opportunity to make their meanings explicit or when the course instructor (Pat Thompson) pushed the teachers to articulate and reflect on their meanings.

One of the initial class activities required the teachers (in groups) to examine an Algebra 1 quiz entitled "Interpreting/Creating Graphs" (Appendix A). The quiz consisted of four contextual situations each involving variable quantities. The quiz was designed to provide Algebra 1 students with the opportunity to reason about ideas involving rate of change (e.g., rate, proportion, constant rate of change), covariation (e.g., a focus on varying two quantities simultaneously, a focus on covariation over small intervals), graphs, and the students' coordinations of their meanings regarding these ideas. Furthermore, the quiz was designed to initiate instruction (e.g., in a subsequent lesson) on ideas related to average rate of change, non-constant rate of change (i.e., increasing and decreasing rate of change), and linear functions.

The activity required the teachers to articulate what they believed to be the learning goals of the quiz and to describe instruction that the teachers' believed would prepare their students to be successful on the quiz. The activity served as an introduction to the *Didactic Triad*, which had been briefly described prior to the activity. The teachers worked together in groups of 3-4 and were required to white-board and present their work to the class.

The most commonly indicated learning goals included rate of change (5 of the 6 groups), creating and/or interpreting graphs (all 6 groups), and covariation (4 of the 6 groups). Throughout the group discussions and presentations, the teachers frequently used terms, such as rate of change or interpreting graphs, without articulating the meanings that they had in mind and intended for students to develop. For example, one of the groups listed "understand rate of change" as one of their learning goals, but did not explicate a meaning for rate of change that they intended for students to "understand," or how they envisioned promoting the development of whatever image of rate of change they had in mind in their students. I take this general disinclination in the teachers to make their meanings explicit as an expression of the teachers being insufficiently mindful of their own meanings so as to operationalize them.

Although findings suggest that the participating teachers did not have an operationalized image of their meanings, there is evidence to suggest how the teachers had conceptualized the mathematics that they teach. During the group presentations of the learning goals, one of the teachers (Marcy) commented on the conciseness of another group's learning goals (the group had listed only rate of change and graphical interpretation as learning goals). Marcy stated that, "I'm not saying that ours are learning objectives either…but they should really be things that kids are able to do…and we know what rate of change means, but what's the objective?"

Although Marcy asserted that, "We know what rate of change means," none of the groups had articulated a meaning for rate of change they intended for students to develop. Marcy's comment also provided an indication of how she conceptualized the mathematics that she intends for students to learn. Specifically, that mathematics involves "things that kids are able to do."

Several of the groups identified either objectives or sub-objectives that involved students "doing" things, that is, performing procedures or skills or re-calling facts. Marcy and her partner identified several sub-objectives under the larger objective "rate of change": create graphs, interpret from the graph a real-life application, difference between distance from start and total distance travelled. These sub-objectives involved demonstrable actions on the part of the students, not ways of reasoning or ways of thinking. Other groups identified learning goals, such as: "know the relationship between distance, rate, and time" (i.e., the ability to recall and employ the formula r = d/t), and "understand when you read a graph input is on the horizontal, output is on the vertical" (i.e., the ability to recall facts).

Furthermore, when confronted with situations in which the teachers' conceptions broke down, or did not work, the teachers were disinclined to reflect on how modifications that they were required to make to their own ways of thinking would translate to their instruction; that is how these modified ways of thinking could be promoted in their students, or how certain conceptions could be problematic to their students' future learning.

During the group presentations of instruction that the teachers' believed would prepare their students to be successful on the quiz, the majority of the groups listed "reform" or "TPC^2-specific" terminology, such as "big ideas," "speaking with meaning," and "discovery learning." Although the teachers did not often articulate their meanings regarding these terms, they did frequently employ them, and at times challenged one another to make their meanings explicit. Analysis indicated that the teachings frequently expressed images different from the instructor, or even from other teachers. I take this as an expression of the teachers not having a clear image of these ideas, or at least ones that can be operationalized coherently.

For example, although several of the teachers employed the term "idea" in discourse throughout the Functions 3 course, analysis indicated that their conceptualization of an "idea" was different from that of the project. The TPC^2 project defined an idea (e.g., a factor, a graph) as "something that has context and meaning, is generalizable, and has the potential of being foundational for other ideas and ways of thinking" (Thompson, 2009). Throughout their involvement with the Functions 3 course and the project itself, the teachers engaged in a variety of activities that attempted to build meaning to this definition. Analysis indicated that many of the teachers conceptualized an "idea" as a mathematical topic (e.g., rate of change, quadratic), but one whose meaning has not been operationalized; and, understanding an idea as becoming proficient with the "skill set" (procedures, skills, and facts) associated with that topic. As such, several of the participating teacher's identified instructional goals consisted of getting students to become proficient with a given topic's "skill set"; where the focus was on doing (or demonstrating) something, not on developing ideas and building meanings.

An example of a "skill set" emerged as the teachers' attempted to justify or refute "solving quadratic equations" as an idea during another discussion of the *Didactic Triad*, which occurred at the end of the summer portion of the Functions 3 class.

Pat: Suppose somebody said, okay the learning goal is that…they know how to solve a quadratic equation…that's the learning goal…
Bernardo: Okay
Quiet for six seconds
…
Janine: It could be one of them, but I think my goal would be…for students to understand quadratic, that's the big goal.
…
Gwen: When I first heard you say that I thought, well which way, because there are so many different ways and I think the main goal for me when I teach quadratics is that I want kids to understand, you know, when you're solving…okay, so you find and x and an x, sometimes you find one x, sometimes you don't find any x's, but what does that mean, and that's my main goal is maybe teaching that, I mean because we teach it lot's of different times without actually ever saying solve the quadratic or you're finding the two places on the x-axis where it crosses, you know, I mean, we use all sorts of different language that doesn't have any meaning to them anyway, but…

For Gwen, solving quadratic equations appeared to be a legitimate learning goal (an idea, a mathematical topic), with a variety of methods (members of the "skills set") for solving for x (e.g., quadratic formula, factoring, solving graphically). In addition, Gwen expressed that what gives "meaning" to the "solution," is the idea that the root(s) are the place(s) where the graph crosses the x-axis. This could indicate that for Gwen the focus is on "understanding" the solution, that is, what it "means" (graphically, with the focus solely on one variable, x) to be a solution to a quadratic. In addition, although Gwen asserts that, "We use all sorts of different language that doesn't have any meaning to [the students] anyway," she does not indicate that she views this as problematic, only that it is the norm. In the following (subsequent) exchange, Pat attempted to push the teachers to give meaning to quadratic.

Pat: Suppose that you're learning goal is "I want them to learn how to solve quadratic equations"…versus "I want them to learn…what…to learn what, turn that one into learning an idea.
Story: Well Janine said understand quadratic equations.
Pat: But that's not saying what they're supposed to understand yet…what are you supposed to understand when you understand quadratic equations?
Janine: A whole lot of things…recognize them, use them to solve problems…
Pat: But the hallmark of an idea is how it's related, is it's meaning and how it's related to other ideas…my question was…what does it…mean to understand quadratic equations?
…
Pat: So…if they're going to understand quadratic equations…don't they have to have a meaning for quadratic equations…what meaning do you want them to have? And we can't just say "the meaning", because that doesn't answer, that doesn't make anything…so what meaning do you want them to have?
…
Janine: Well, you have a particular relationship between variables where…
Story: It could be two of them…that work.
Janine: There's a particular…

Story: That relate with one…
Janine: There's a particular relationship between those two variables where squaring is happening…

Although Janine initially expressed part of her "skill set" for understanding the topic of quadratic equations, she was unable to, as were the other teachers, come up with a meaning for quadratic equation. This illustration was indicative of the teachers' inability to conceptualize the mathematics that they teach as being composed of ideas that entail meanings, rather than topics that entail associated procedures, skills, and facts.

As stated earlier, I take the teachers disinclination to make their meanings explicit as an expression of the teachers being insufficiently mindful of their own meanings so as to operationalize them. This disinclination to take their meanings, their reasoning, as objects of thought appears to be a consequence of how they understand the mathematics they teach.

Mindful of Students' Mathematics

Concurrent with a disinclination to make their own meanings explicit, the teachers neither challenged one another to make their meanings explicit (other than the rare occasions where they challenged one another's use of "reform" or "project-specific" terminology), nor attempted to ascertain how others were conceptualizing what they were trying to communicate (i.e., whether the other teachers were working with understandings aligned to their own).

The teachers' disinclination to consider the interpretations of others expressed itself in a variety of situations throughout the Functions 3 course. The following excerpt will serve to illustrate what I mean by a teacher not being mindful of their students' mathematics. During the fall portion of the Functions 3 course, the teachers were requested to share a video of their teaching, in which the teachers attempted to hold a conversation with their students that had the attributes of being conceptual. A conceptual conversation is one that has "a diminished emphasis on technique and procedure, and an increased emphasis on images, ideas, reasons, goals, and relationships" (Thompson, 2009).

The following excerpt is from a video that one of the participating Functions 3 teachers (Johnny) made of a lesson that he taught, in which the class was to employ information that they had previously been taught regarding central angles, intercepted arcs, inscribed angles, and regular polygons to answer what Johnny described as novel problems. In the lesson, the teacher (Johnny), stated his intent to use information that the class "knows" involving the area of a circle to find the area of a polygon.

Johnny draws a circle on the white-board
Johnny: We can already figure out how to find the area of a circle, right? How do I find the area of a circle, what are the things I have to know?
Samantha: The radius or the diameter…
Johnny: Okay, either the radius or the diameter, so I at least have to know that…and if I know the diameter, how does that help me, Nancy, to find the area of a circle?
…
Nancy: Because knowing the diameter of the circle, then you can find the area of the circle, or find the circumference?
Johnny: I want to find the area of the circle…so how does knowing the diameter help me?
Nancy: I don't know…

Johnny: If I know the diameter, then what do I know?
Eva: Where the circle is halved.
Johnny: I know the radius, don't I…If I know the diameter, I automatically know the radius…because the radius is always what?
Several: Half…
Johnny: Half the diameter…
…
Johnny: So Tim, what is the area of a circle?
Tim: It's just $2\pi r$…
Johnny: Ah, ah…area, area…
Fred: Length times width…
Johnny puts head down
Johnny: Length times width is what guys?
James: Area of a rectangle.
Johnny: That's the area of a rectangle…the area [of circle] is what?
Jenny: $\pi \cdot r^2$
Johnny: $\pi \cdot r^2$

In the above excerpt, Nancy and Eva's comments suggest that they were interpreting the information differently from the images that Johnny had intended. Johnny appeared to desire that the students not only recall the formula $A = \pi \cdot r^2$, but that the students understand that the problem required that they make use of the formula. Nancy and Eva appeared to be attempting to answer the question of how the diameter relates to the circle. Rather than attempting to get at the understandings that Nancy, Eva, and other students might have been developing (i.e., how they might have been interpreting the situation), Johnny moved the conversation forward to get to the desired formula. Johnny focused the students' attention on Johnny's conception of the product of "proficient" reasoning (i.e., the formula), rather than getting students understandings and reasoning on the table, as objects of discussion.

In addition, analysis indicated that several of the teachers were disinclined to consider how students might understand and interpret instruction (to attempt to de-center) when designing instruction. The following excerpt illustrates how one of the participating teachers, Janine described her instructional design methodology. Janine stated, "I have this really strong desire…to present…to make sure that I really understand the material, and that I can present the most clear, concise lesson, but for me it's always been about me…."

Janine's response is very indicative of an image of instruction that is based on transmission; one that is grounded in the belief that students learn by being shown. Janine asserted that she has a "really strong desire…to present" and to "present the most clear, concise lesson". Janine's response indicated a lack of de-centration, both in her image of lesson design and in her teaching. Janine asserted that she had a strong desire "to make sure that I really understand the material, and that I can present the most clear, concise lesson, but for me it's always been about me." This comment suggests that Janine's image of lesson design and teaching is such that she makes the material and the presentation clear to her – not necessarily to her students.

Further analysis of the data corpus indicated that Janine's conception of the mathematics that she intended for her students to learn is one that focused on procedures, skills, and facts. This way of thinking about the mathematics curriculum is compatible with a conception of lesson design and instruction that focuses on presenting clearly and concisely – as exhibited by Janine. Janine made certain that she had a clear understanding of the mathematics (i.e., the procedures, skills, and facts) and attempted to present (i.e., demonstrate) the material to the students in as clear and concise a manner as possible.

When confronted with idiosyncratic understandings, the teachers appeared to assimilate this to students not understanding or possessing different, commonly incorrect understandings. In these instances, which the teachers indicated were common, it was up to the teacher to try to re-teach, to attempt to successfully transmit the mathematics. In the following excerpt, the teachers had just watched a video of a geometry lesson, given by one of the participating teachers (Bernardo). Annie brings up her observation that several of the students in the video had different meanings for perpendicular bisector than the teacher, and from one another.

Annie: That's one of the things that was brought up in the conversation, what is a perpendicular bisector…and the kid's had a lot of different meanings for what a perpendicular bisector was…they weren't consistent on what they thought…

Bernardo: No, they weren't.

Annie: Some of them even had completely different meanings…

Bernardo: I thought they knew what it was, I thought they knew what one was, but they didn't.

Johnny: Maybe that's just important to find out.

Bernardo: Oh, yeah.

Marcy: How can you have a conversation about it, if the vocabulary you're talking about doesn't have the same…

Bernardo: I just gave quiz, and it's so interesting, because we were using the word bisector completely by itself…sketch line AB bisecting segment CD, and quite a few of the students were equating bisector and intersect…they thought they were synonyms.

Alyce: I heard a girl, later in the discussion [in the video] say that they were perpendicular…and you said, "What's perpendicular," and she goes, "They just have to cross."

Marcy: You see and I had that happen in my class the other day, they didn't get what perpendicular meant.

Bernardo: What's the solution by the way…what's the solution to the problem?

Story: The point of concurrency of the perpendicular bisectors.

In this excerpt the teachers were confronted with a situation in which students clearly had idiosyncratic understandings. One of the teachers (Johnny) asserted that it might be "important to find [that] out." Rather than discussing the ramifications of idiosyncratic understandings to instruction (design, implementation, and assessment) the teachers reacted as if idiosyncratic understandings (i.e., interpretations) are simply a matter of students having "correct" understandings or misconceptions. In addition, Bernardo moved the discussion to a focus on the solution to the problem, the product of their reasoning, rather than on the reasoning itself.

These findings suggest that an emphasis on "doing" (i.e., applying procedures and skills, or recalling facts) hindered the teachers' capacity to focus on their students' meanings, understandings, and ways of thinking. Furthermore, the emphasis on procedures, skills, and facts placed the instructional focus on the products of reasoning, not on promoting students to reflect on their reasoning.

Managing the Conversation

Analysis suggests that the teachers' focus on facts, procedures, and skills also impacted their images of the role of tasks and activities – those that they engaged in as students of mathematics, and those that they created or chose to employ in their own classrooms. Specifically, analysis indicated that the teachers' focus on the products of their (and their students') reasoning, and their disinclination to consider others' possible interpretations, impacted the teachers' capacity to conceive of the idea of managing a conversation.

Throughout the Functions courses, the instructor (Pat Thompson) served as a model of what it means to manage conversations so that the ideas are developed as the students (i.e., the participating teachers) engaged with instruction; whereby the students (i.e., the participating teachers) were encouraged to take their mental operations, their reasoning, as objects of thought.

Analysis suggests that the teachers were constrained in their capacity to not only manage a conversation, but were unaware that they were frequent participants of managed conversations or that several of the case study videos that they reviewed exhibited managed conversations. At the initial fall semester meeting of the Functions 3 course, the teachers reviewed and discussed a video from the project's reformed Algebra 1 case study. In the video, which involved instruction pertaining to a unit on quadratics, the teacher (Augusta) used ideas of function, covariation, and rate of change (ideas for which the Algebra 1 students had been building meanings throughout the semester), to push her students to think about how to find the vertices of a quadratic function. The following excerpt illustrates the teachers' difficulties with conceiving of a managed conversation.

Pat: So…what strikes you about this video?
Tami: She's not teaching the formula, the $-\frac{b}{2a}$.
Pat: Okay
…
Tami : She's approaching it from a very different perspective.
Sheila: She's teaching calculus to freshman
…
Pat: Actually, is she teaching this to freshman?
Alyce: No, they discovered it on their own.
Pat: Did they discover it on their own?
Alyce: Well, I'm sure there was plenty of guiding, but…they came up with that.

Alyce indicated that the students were "discovering" the ideas, whereas, Tami attributed the relating, the reasoning, to the teacher (Augusta). The teachers appeared to be unaware of the role that Augusta played in moving the students' to reason in ways that she intended for them to reason – the idea that Augusta had managed the conversation so that the discussion moved towards the ideas and meanings that Augusta intended. Pat continued to push the teachers to reflect on the role that the teacher (Augusta) played in the conversation.

Pat: Right, but why are they even talking about it?
Pat: Augusta chose to talk about it, right?
Tami: She's relating the increase and decrease and covariation to vertex, so she's pulling it one level

deeper…
Pat: Okay, but she's not…see that's why, where Alyce's right…she is not doing that relating.
Tami: The students are
Pat: The students are…but now contrary to what Alyce said about them discovering it, they're not discovering this, so…Augusta's not teaching it and they're not discovering it, so what's happening?
Quiet for eight seconds
Bernardo: Well, they're having a conversation…Augusta's talking with them and they're talking with Augusta to figure it out together.
Pat: Augusta posed the problem right?
…
Bernardo: Right…Augusta posed the problem, and basically threw up her hands and said what do I do, and then they came up with…whoever it was, Ed I think, came up with rate of change function, and then Augusta started asking why.
Alyce: She probed.
Pat: Yeah, he suggested find the rate of change function.
Bernardo: And then Augusta said well why, why would I want to do that? What will that get me?
Pat: Now, was that just a random question?
Several: No.
Pat: What might have been Augusta's motive for asking that?
Quiet for 4 seconds
Annie: Well one to kind of check for understanding, that he's not just throwing something out there.
Pat: All right
Annie: That he had an idea of what he was talking about.
Alyce: And to protect against procedural thinking…

The teachers did not indicate that they had conceptualized Augusta's actions as being purposeful – they had not considered the possibility that Augusta managed the conversation so that there was an increased likelihood that the students would reason in the ways that Augusta had intended. The teachers' comments suggest that they had not considered the possibility that Augusta anticipated, designed, and pushed the conversation so that it went in a particular direction, that Augusta had purposefully managed the conversation. The teachers appeared constrained to conceive of the idea that Augusta had managed the conversation so that the ideas unfolded (in the minds of her students) from the conversation.

In addition, throughout the Functions 3 course, if the teachers did employ the term "managing a conversation," they did so by in a manner that suggests that they understood "managed" to mean "to check for understanding" – that is, a conversation is "managed" so that the teacher can determine whether the students understands (i.e., has developed a proficient "skill set"). From this point of view, a managed conversation is important for the teacher (to see who "understands" and who does not). This point of view differs markedly from that promoted by the project and course instructor – that the teacher manages the conversation in order to get interpretations and meanings out on the table, to move students to reflect on their reasoning, and for the students to see that their misconceptions do not work for them – all for the benefit of the students.

Reflective Episodes

Having discussed the epistemic ways of operating that were identified through analysis of the data corpus, the next level of analysis involved identifying how these ways of operating demonstrated themselves during the reflective episodes, those lessons or parts of a lesson that were explicitly designed to provide the teachers with opportunities to reflect on their practice.

The first such reflective episode that I will discuss (which occurred near the end of the course) was designed to have teachers act, over time, with two hats. The first was with the hat of a student of a lesson involving data analysis (phase 1). The second was with the hat of an instructional designer creating the lesson that they just experienced (phase 2).

Data Analysis – Experiencing the Lesson as Students

Pat initiated phase 1 by stating that a recent study had been conducted at a high school where students were given an anxiety assessment just before taking a mathematics test. Pat then distributed (to the teachers) a two-sided sheet containing unorganized data, explaining that it came from this study. The sheet contained four columns listing test score on one side and four columns listing anxiety level score on the reverse side. Pat asked the teachers what information they could make from the data. This action enabled Pat to highlight the issue that little could be made of unorganized data.

Pat next distributed a second data sheet, which showed anxiety level scores together with test scores. Pat asked the teachers whether they felt that this organizational display was better than the first. This action focused the teachers' attention on identifying the benefits that organized data provided and constructing images of more propitious displays. In addition, the conversation anticipated the notion of statistical case, by emphasizing the necessity for the teachers' to assume each row of data went with one and only one student.

Pat encouraged the teachers to generate hypotheses pertaining to the data. One teacher suggested that higher test scores go with lower anxiety scores. When several of the teachers attempted to explain why this hypothesis might be true, Pat insisted that the teachers generate hypotheses based on the numbers, not construct explanations for a hypothesis.

Analysis suggests that a few of teachers impeded their reasoning by focusing their thinking on the development of plausible explanations to relationships within the data, based on the quantities under consideration (e.g., test scores and test anxiety levels), rather than maintaining their focus on developing hypotheses based on the numbers themselves. I believe that this constrained their capacity to build meaning to the role of organization of data, although Pat continued to bring them back to the data and to the role that organization was playing in the ability to develop hypotheses. The following excerpt serves to illustrate this point.

Pat: What are some of the things that you can tell?

…

Rachel: It seems…a lot of them…the ones with a higher [test] scores, a lot of those, have…not all of them… seem to have the score for the anxiety level is a little bit lower…

…

Pat: So…

Annie: I don't think I agree with that…

Rachel: Not all of them, but some of them…your 100's your 90's, the ones where…your scores are in the 100's, in 90's, the anxiety level is quite a bit lower
Pat: Okay, we hear some disagreement…so, what's your general hypothesis?
Tami: The lower the anxiety level, the higher the test score…
Rachel: Or you could go the other way, the higher the test score, the lower the anxiety.
Tami: Well if you think you're prepared for the test, then you don't have anxiety over the test.
Pat: Okay, that's an explanation for the hypothesis [Tami], but not a hypothesis.
…
Rachel: I'm saying not on all of them…
Pat: So why is it hard to make that argument?
Annie: But then you have a couple of kids that don't care that have seven and eights there
Johnny: You could have outliers
Alyce: It could be that they're clueless, and don't know what they are doing
Rachel: And they don't care
Several laugh
Johnny: But see that's the thing is that because you're clueless and you don't really care, that doesn't mean that you're gonna, so you have that as an issue too…
Pat: Johnny, stay to the numbers…what is keeping you from moving forward?

Pat attempted to get the teachers to think about how the organization (or lack of organization) played a role in their ability to generate hypotheses, and how a different organization could possibly help them to make and justify claims (hypotheses).

Pat continued to question the teachers as to why they believed they were experiencing difficulty in generating claims. This action led to the suggestion (by Eve) that they order the test scores from highest to lowest, and look for a relationship between the variables. Pat used Eve's suggestion to move the conversation towards discussing what the teachers' envisioned such a display (ordering test scores from highest to lowest) would provide. This conversation led to the notion that particular types of organization offer particular types of information and again highlighted the importance of assuming that each row represented data from a unique individual.

Pat moved the conversation back to generating hypotheses. After two hypotheses were constructed, Pat pushed the teachers to imagine organizations of the data that would enable them to judge the hypotheses' veracity. This action emphasized the role that data organization plays in generating and testing hypotheses, thus giving meaning to the idea that how displays of data are organized matters greatly in data analysis.

Pat next introduced of a third variable (each student's teacher) to the data set. After each of the teachers had received the third data sheet, Pat asked what hypotheses they could with the addition of the third variable. This instructional action required the teachers to coordinate their notion of statistical case with their images of organizational displays to construct hypotheses based on the data. In addition, this action served to reinforce the idea that particular organizations could provide particular kinds of information.

Pat next sorted the data, as requested by the teachers, using an Excel spreadsheet with the data embedded. The sorted data was displayed on an overhead so that the teachers could see the results of their requested organization and attempt to generate hypotheses. This visual display introduced the idea of statistical case and highlighted a need for a better way to organize the data, thus reinforcing the role that organization plays in analyzing data (i.e., in generating and testing hypotheses).

Some of the teachers hindered their reasoning by focusing on the outcomes of the activities (e.g., developing and testing hypotheses, establishing the statistical case). For these teachers their focus was placed on the products of their reasoning, thus constraining their capacity to reflect on what is was that supported the development of their hypotheses (i.e., organization) or how the conversation was managed so that the idea of statistical case came about.

Pat next introduced *TinkerPlots*, an exploratory data analysis computer program, by constructing a variety of organizational displays of the data provided in the third data set. In addition, Pat focused the conversation on how each of the various displays addressed specific questions regarding the data. These actions served to provide the teachers with a sense of the program's capabilities and anticipated the necessity to construct images that addressed different questions related to the data. After concluding the *TinkerPlots* overview, Pat next partitioned the teachers into four groups and distributed (to the teachers) a sheet with questions that pertained to the data (Appendix B). The questions were designed to reinforce the teachers' images of the role that organization plays in data analysis and to provide a natural way in which the idea of conditional probability could arise in the context of instruction.

Each group then selected a question from the sheet, with no two groups having the same question, and worked to develop a sketch of the organizational display that they believed would allow them to address their question. Each group then described their organizational display to Pat, who generated the requested display using *TinkerPlots*. The displays appeared on a projector screen as Pat generated them. This action required that the teachers imagine organizing the cases in a manner that they believed was most propitious for addressing their chosen question.

During the last part of phase 1, each group presented their designed display and discussed how and why they believed the display addressed their question. This allowed Pat to address specific issues related to both data analysis and instructional design, including: a focus on analyzing distributions, interpreting the attributes of a case, attending to how others (i.e., students) might interpret a question, attending to how others (i.e., students) might interpret a display, and a focus on how certain understandings or ways of thinking might be propitious or deleterious to future learning.

Data Analysis – Reconstructing the Lesson as Instructional Designers

Prior to discussing the re-construction of the lesson, I believe that it is important to discuss Piaget's (1968) *On the Development of Memory and Identity*. In one of his many experiments, Piaget (1968) showed several children, of ages 3-7 years, an ordered configuration of 10 sticks, varying in size from about 9-15 centimeters. The children were asked to look at the configuration so that they could draw or describe verbally what they were shown at a later date (one week later and six months later). After six months, a large majority of the subjects had better recollection than they did after one week. For Piaget (1968), these results indicated that a cognizing subject's memory-images are constituted by the subject's current schemes. This is pertinent to the reflective episode, because the teachers were attempting to re-construct a lesson that they had just participated in as students; they were attempting to re-construct the lesson with the cognitive structures that they engaged the lesson with. Therefore, the activity of re-construction gives insight into how the teachers assimilated the lesson (i.e., it helps to reveal the teachers' understandings and ways of thinking).

The second phase of the lesson provoked reflection on the part of the teachers as they attempted to reconstruct the very lesson in which they had participated as students (e.g., create a

Lesson Logic, the logic of the lesson). Specifically, this phase was designed to provoke reflection on the instructional actions that the teachers might take to support student development of the intended meanings and ideas, and the reasons why those actions might work. These images also include the tasks and classroom discourse that the teacher would employ.

From a global perspective, the teachers were unable to re-construct the lesson that they had just experienced as students. As the teachers began to reconstruct the lesson that they had just experienced as students, they began to disagree about what each conceived of as the products of their reasoning. Rather than reflecting on their reasoning, the teachers attempted to recall the products of their reasoning. As the lesson re-construction progressed the teachers began to disagree about the sequencing of the outcomes (e.g., "Did we first establish a case or a hypothesis?"). Further on in their attempt to re-construct the lesson, the teachers began to introduce actions that had not occurred in the actual lesson, actions pertaining to how they (or would) teach the lesson. By the end of the lesson (as time ran out), the teachers were constructing a lesson that they themselves would teach.

Even though Pat had emphasized that the construction of the *Lesson Logic* (see Appendix C for the teachers' constructed *Lesson Logic*) required that the teachers think about and construct images of the understandings that they want their students to develop, the teachers appeared disinclined to think about what it was that they needed to understand in order to participate productively in the lesson or the understandings that they developed as the lesson progressed. Throughout phase 2, the teachers appeared disinclined to reflect on the meanings they held, the meanings they were building, or the ideas that were being developed; they simply attempted to recall the order of the activities (those things that Pat's actions required of them) that they had experienced.

I believe that there are several reasons for the difficulties that the teachers exhibited in their attempts to re-construct the lesson. One reason involved the teachers' own mathematical understandings - the teachers had not developed a KDU for the role that organization plays in data analysis (i.e., they had not transformed their own mathematics understanding). I believe that there are three additional influences that constrained the teachers' capacity to reflect, each involving ways of operating that have been previously discussed: meanings, being mindful of students' mathematics, and managing a conversation.

Throughout the re-construction process the teachers failed to take their meanings as objects of thought, this constrained their capacity to reflect on their reasoning as they attempted to re-construct the lesson. Rather than reflecting on their coordinations of meanings (e.g., the role that organization plays or can play in data analysis, statistical case, hypotheses), the teachers focused on the products of their reasoning (e.g., established statistical case, established hypotheses, organization matters). This lack of operationalized meanings contributed to their difficulties in re-constructing the sequence of these products as they attempted to recall the lesson's progression. The following excerpt illustrates this assertion.

Rachel: Now what?
Story: Then he gave us another set that had more information on it.
Johnny: Well I'm just trying to think though, there was a couple of more things I think that we did with this one.
Rachel: But you could almost put those together, because…
Annie: It's the same thing.
Rachel: They were sort of asking the same question, too.

Story: Yeah.
Annie: You don't need to say that he handed out another set of data, it's the same thing.
Story: No.
Rachel: It's the same thing.
…
Marian: Can we add them to the reason, because I think then if we're gonna include all the data that he gave us, because it wasn't all to organize data but to establish a…
Story: Relationship.
Marian: Case.
Story: Oh.
Annie: Establish a hypothesis.
Rachel: Case, hypothesis?
Faye: Hypothesis, right? Because a case is just one…
Marian: Yeah, well see that was the whole between the first and the second and the third…
Faye: Okay, yeah.
Rachel: So…case?
Marian: Maybe both
Bernardo: A case and then a hypothesis.

In addition, the teachers did not indicate that they were mindful of the instructor's (Pat's) intentions during the lesson re-construction (i.e., the teachers made no attempt to de-center). Specifically, the teachers did not discuss or disagree about meanings, about the possibility that they might have different meanings from one another or from what Pat intended, they simply disagreed about the progression of event. I take this as an expression of the teachers being insufficiently mindful of others' mathematics (e.g., students, instructor).

Throughout the course, the instructor (Pat) served as a model of what it means to manage conversations so that the ideas are developed as the students engage with instruction, whereby students were encouraged to take their mental operations, their reasoning, as objects of thought. During the lesson and lesson re-construction, the teachers appeared to be unaware that they were participants in a managed conversation. In addition, the teachers exhibited a conception of conversation management as a means for checking for "correct" understanding, rather than a tool to provoke reflection on the part of their students – conversations were managed to do work for the teacher, rather than students.

Although analysis suggests that both non-operationalized meanings and not being mindful of students' mathematics contributed to the teachers' inability to re-construct the lesson (i.e., constrained their capacity to reflect), the data corpus allowed for analysis to go one step further.

Speed – Reflective Episode

The teachers engaged in a similar reflective episode, in which they first engaged in an activity as students of mathematics, and were then asked to re-construct the lesson as a teacher intending to teach the lesson. The lesson involved the teachers experiencing and re-constructing a lesson pertaining to ideas related to the concept of speed (i.e., an object whose speed is always increasing). Throughout the earlier (summer) portion of the Functions 3 course, the teachers engaged in several activities designed to transform their conceptions of constant speed, average speed, and increasing/decreasing speed. Therefore, the teachers (in general) had a more developed KDU than in the data analysis lesson; although the data corpus did not provide any indication of the teachers' engagements with concepts involving speed after the summer portion of the Functions 3 course and the reflective episode (roughly five months had passed) – keeping in mind Piaget's *On the Development of Memory and Identity* (1968).

Although the teachers were able to, for the most part, re-construct the lesson involving concepts of speed (or at least more effectively than the data analysis lesson), they did so by being more proficient at re-constructing the sequence of events – where, these events were again the products of their reasoning. In addition, although the teachers were more proficient at identifying how one outcome (i.e., product of their reasoning), led to the next outcome, they were unable to re-construct how ideas were developed or meanings were built.

I claim that the teachers more well-developed mathematical understandings (of constant speed, average speed, increasing/decreasing speed) supported their capacity to reason about the sequence of events, but these understandings were not enough to support propitious reflection. The teachers were unable to identify how the products of their reasoning came about (via Pat's managing of the conversation), they spoke about these outcomes as if they were "discovered" (or came out), not in terms of the reasoning that was involved that provided for the "discovery". In addition, the teachers did not consider Pat's hand in managing the conversation so that the teachers were indeed moved to reason.

Discussion and Implications for Future Research

I believe that results from the two reflective episodes support the Silverman and Thompson framework, which proposes that teachers must first develop a key developmental understanding prior to developing understandings that are pedagogically powerful (i.e., a KPU). Future research should seek to develop an instrument to measure the "level" of KDU-development in participating teachers, and investigate how such "KDU-development" influences the development of key pedagogical understandings.

Findings suggest that how a teacher understands the mathematics that they teach greatly influences the teacher's capacity to reflect on their practice. Teachers whose personal mathematics focused on facts, skills, and procedures were disinclined to take their meanings as objects of thought, thus constraining their capacity to achieve the coordination of meanings that would sustain the kind of reflection that yields thematic, summative images of that reasoning, as they attempted to re-construct the lesson.

I also believe that the study's findings pertaining to the data analysis reflective episode has implications for mathematics teacher education. Mathematics teacher educators often think that if you want to help teachers improve their teaching, then you should teach teachers what you think they should teach, how you think they should teach it, and with the materials that you believe they should teach with. The data analysis reflective episode was an example of such a lesson and the participating teachers couldn't re-construct lesson. Therefore, these teachers could not go off and teach that lesson that they were just taught to their own students.

If the teachers did have a more developed KDU in the reflective episode involving concepts of speed than they did in the data analysis reflective lesson, and this development played a role in supporting reflection, then what impeded their capacity to reflect productively? I believe that the answer lies with exploring the role of managing a conversation. Throughout the lesson re-construction, Pat brought up the idea of "managing the conversation", and he does so at several points throughout the re-construction, such as: manage the conversation so that the idea of speed is brought out, and manage the conversation so that the students conclude that the speed was not constant.

The teachers frequently made the assertion, "We came up with that," when discussing how ideas came about in the lesson – they appeared not to be cognizant of Pat's hand in their development of the ideas. The teachers appeared to be unaware of how Pat managed the conversation, or that they were participants in a managed conversation. Although the teacher's bring up Pat's actions, they do not discuss his possible intentions. In addition, the teachers exhibit a conceptualization of ideas and meanings as things that are established, not constructed; and, do not indicate the possibility of quite varying meanings being developed.

It seems reasonable to suggest that if there were no idiosyncratic understandings (no interpretation occurring), then there would be no need to manage a conversation in the sense employed by the project. Throughout the re-construction the teachers consistently failed to discuss the ideas that were developed or the meanings that were built, as if the issue of knowledge development was not part of their thinking. Therefore, I suggest that that the teachers' lack of an operationalized epistemology appears to be playing a significant role in the teachers' inability to re-construct the lesson (i.e., constrained their capacity to reflect) – the teachers are not cognizant of how understandings (either their own or their students) could develop, or how instruction should be designed and implemented in order to promote the development of intended understandings and ways of thinking.

Although the teachers do not express a cognizance of knowledge development, they do exhibit an epistemology in practice – one of a transmission of procedures, skills, and facts; a transmission of "skill sets" for given mathematical topics, which precludes the possibility of individuals constructing their own idiosyncratic understandings. Teachers whose personal mathematics focused on "skill sets" were also not mindful of students' mathematics; this disinclination to de-center constrained their capacity to reflect on their practice.

I believe that future research should explore the role that an operationalized epistemology could play in a teacher's transformation. Such research could involve teacher education in academic or professional development settings, in which teachers would engage in critical epistemological discussions, in concert with analyses of how students interpret instruction (and the teacher's role in that interpretation).

In addition, I believe that these findings suggest that teachers, whose mathematical understandings involve topics and associated "skill sets," in concert with un-operationalized understandings of knowledge development, bestow unwarranted agency to tasks and activities; rather than to conversations about those tasks. I envision future research into teachers' conceptions of the role of tasks, materials, and activities in student learning.

Finally, I believe that the study's findings pertaining the use of "reform" or "project-specific" terminology has implications for mathematics teacher education. Teacher educators often use terms, such as "conceptual understanding" or "big ideas" in both academic and professional development settings. Some teachers may commit to reform ideas and employ the "proper" terminology (or not commit, yet employ "proper" terminology), without having the same image as the teacher educator, or even of other participating teachers. Therefore, if teacher educators or professional developers intend to employ "reform-oriented" or "project-specific" terminology in their instruction, then they must make a point to make both their meanings and those of the participating teachers explicit. This is true even amongst mathematics education researchers – take the case of "mathematical knowledge for teaching."

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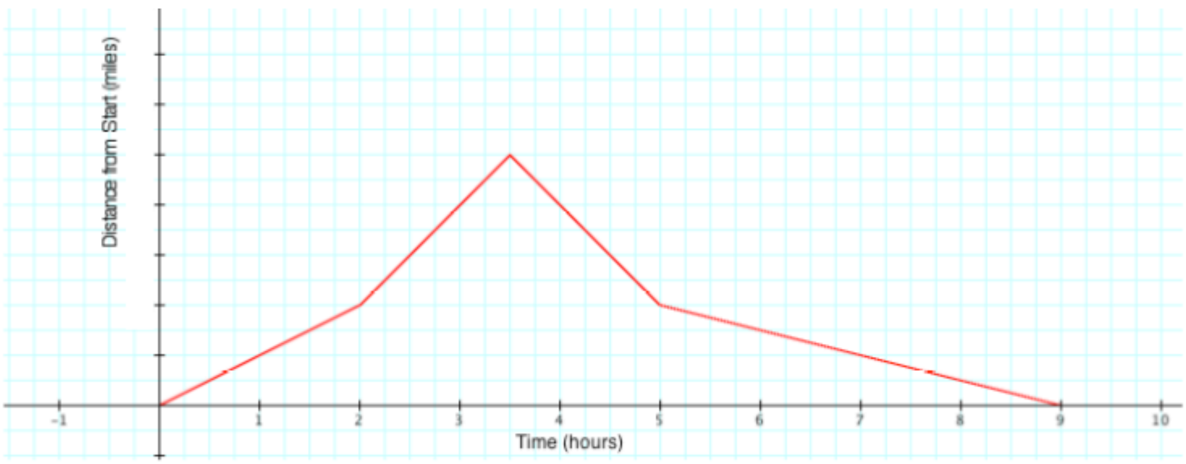
APPENDIX A

Quiz Page 1

Interpreting/Creating Graphs Quiz

1. Clown went on a bike trip. The following graph shows the number of miles that Clown was from the start relative to the number of hours since he started.



a) In the first part of his trip Clown traveled at a speed of 5 mi/hr. Put numbers on the vertical axis so that the graph is accurate.

b) How many miles did clown travel in the third part of his trip? How do you know?

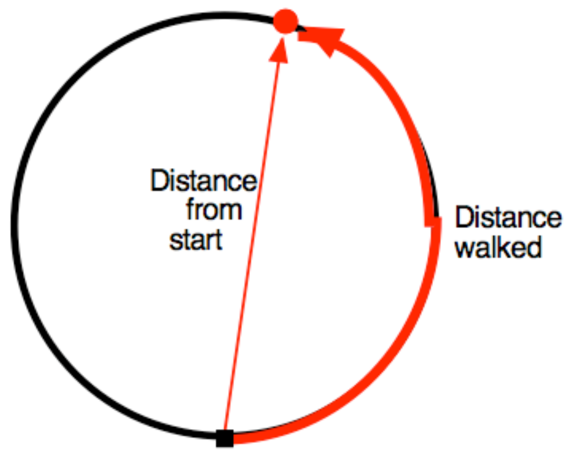
c) How fast did Clown travel in the fourth part of his trip? How do you know?

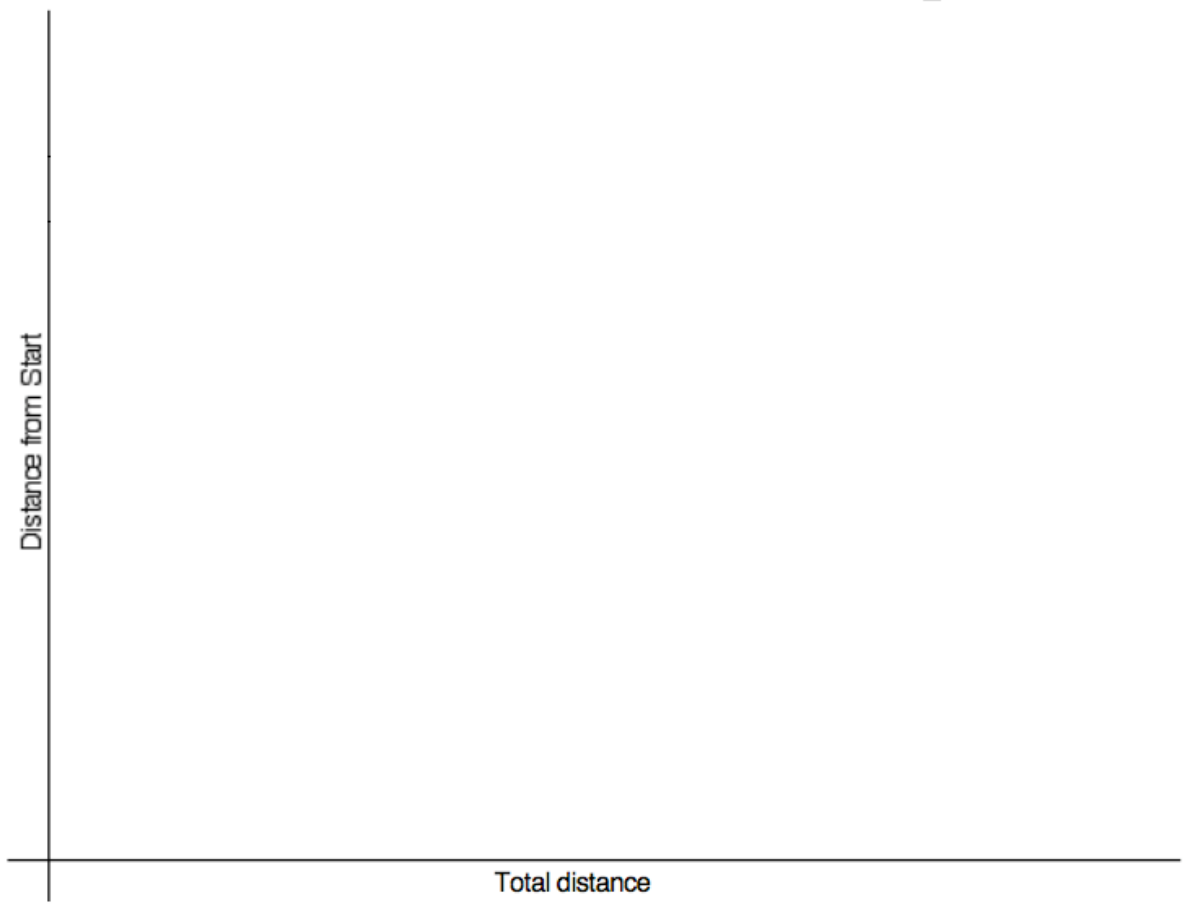
d) In one part of his trip, Clown blew a tire and had to walk. When did this happen? How do you know?

Quiz Page 2

2. Matt walked abound a circular track (see diagram). The black square shows where he began. The arrows show that we are tracking Matt's distance from his current location to the starting point and the distance he had walked to get to his current location.

   a) Imagine Matt walking around the track once. Use your finger tool to sketch a graph of Matt's walk. Keep track of how far he had walked with your *horizontal* finger and his distance from the start with your *vertical* finger. (Hint: Think of playing Matt's walk a frame at a time to keep track of both.)

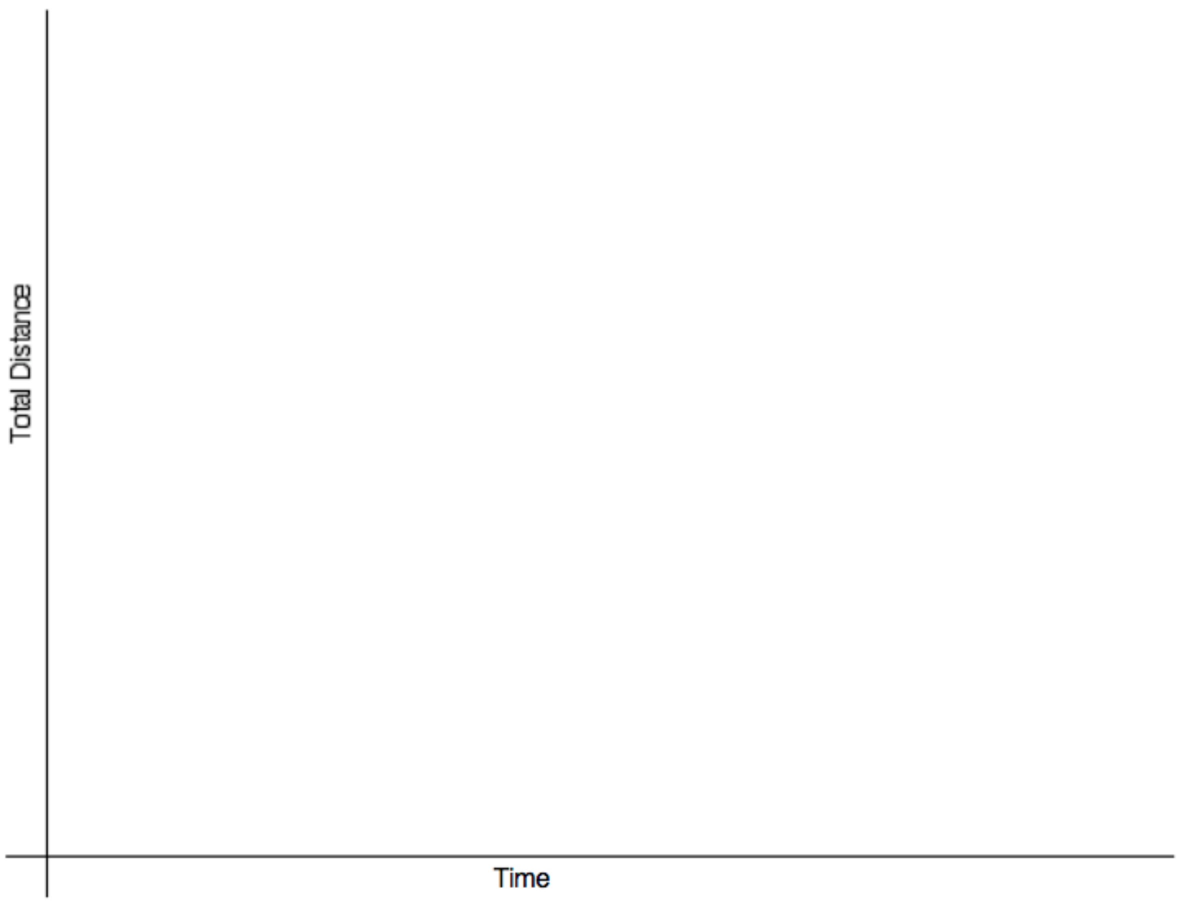




Quiz Page 3

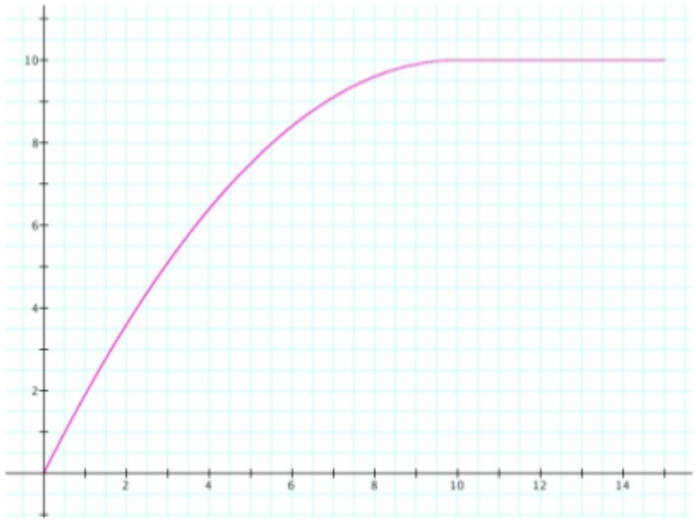
3. Miss Coombs' new years resolution is to run more. Sketch a graph of her TOTAL distance relative to the number of minutes she's been jogging while she was out on the following run:

   a) She began at a steady jog for 5 minutes (warming up, you know).
   b) Then, she picked up the pace to a run, running top speed for 3 minutes.
   c) She got a huge cramp and had to slow way down. She went at this slower speed for 4 minutes.
   d) The cramp would not go away so she stopped for one minute
   e) After that, she decided to toughen up and she sprinted all the way back home



Quiz Page 4

4. The following graph tracks an object's SPEED relative to the number of seconds it had been moving:



a) Describe this object's motion over the 15 seconds shown in the above graph.

b) Sketch a graph (don't worry about total accuracy) of the object's distance from start in relation to the number of seconds it has been moving. Explain your graph (use the back of this sheet if necessary).

APPENDIX B

Anxiety-Test Data (In Groups)

- Consider these questions one at a time, focusing first on the one you chose.
- Think about how the data could be displayed so that you can answer the question just by looking at the display.
- Sketch how you want the display to be organized. Don't try to guess how the data would look, just say how you want it organized so that you can look at it and answer the question.
- Come up to my desk and show me your sketch. I'll make the display you want and then print it out for your group.

1. Is test anxiety a severe problem among Math 1 test takers?

2. Is test anxiety more severe among some classes than others?

3. Did some classes do better than other classes on this test?

4. Suppose you talk to a middle test scorer. How anxious were they likely to be when taking the test?

5. Suppose you talk to a highly anxious student? How likely are they to have a low test score?

6. Suppose you talk to a highly anxious test taker. To which class are they most likely to belong?

APPENDIX C

(Lesson Name): Logic of the Lesson

(Author)

The following is a lesson logic for teaching (the major idea or ideas) in (course, topic, or grade level).

A lesson logic is the outline of how you will develop the lesson's main ideas. It does not pay attention to time, meaning that the "lesson" may transcend several class periods. It does not give the level of detail that a lesson plan gives, meaning it might not say how you will organize the classroom, how you will transition from one activity to another, etc. Instead, it focuses on the ideas you will develop, the way you develop them, and why you take the approach you take.

The following lesson logic provides a structure in which the surrounding conversation unfolds these ideas:[1]

1. Data has cases and each case has several attributes.
2. (Major ideas of the lesson, listed in a way that summarizes the logic)

Meanings students must have before the lesson:[2]

1) What is meant by a hypothesis.
2) (Things students must understand at the outset if they are to participate productively in the lesson. This is not the same as things they must be able to do.

Steps in the Lesson Logic

Step	Action	Reason
1.	Hand out raw data and ask "Can we analyze this data?"	Establish need to organize data
2.	Discuss why organization of data matters.	Particular organization can provide particular kinds of information. (i.e. ordering test scores from high to low)
3.	Create a hypothesis based on the data we have.	Establish a case and see a relationship between the data
4.		

[1] Your description of the major ideas that this lesson addresses should evolve *from* your attempts to create the lesson logic for teaching them. In other words, someone reading your lesson logic will read your description of the big ideas *before* reading your steps for teaching them. But, you will have created your description of big ideas *after* having created the steps for teaching them.

[2] Your list of "meanings students must have" should evolve *while* you write the steps in your lesson logic. Write them as they occur to you, and be alert to when you are plan a step that presumes students have a meaning that is essential for them to participate.