The Construction of Mathematical Communities in Undergraduate Real Analysis: the Case of Cyan

Paul Christian Dawkins, Northern Illinois University

Abstract: Much university learning occurs outside of the classroom, but little is known about how students engage mathematical learning beyond class meetings. This study describes the manner in which one undergraduate real analysis student, Cyan, built mathematics communities around himself to facilitate his learning and the learning of his classmates. Vygotsky (1978) and Cobb, Wood, & Yackel (1993) have especially informed the research community about the role social interaction plays in individual learning and how mathematical meaning arises in intellectual communities. Rasmussen, Zandieh, and Wawro (2009) identified some of the mechanisms by which brokers transfer and negotiate meanings between mathematical communities. Cyan's membership in multiple mathematics communities allowed him to act as a broker between them both inside and outside of the classroom. I identify several departmental and instructional factors that facilitated Cyan's adoption of this mode of learning and the roles he played in the classroom mathematical community.

Key words: real analysis, emergent framework, case study, undergraduate, broker

The transition from grade school to university transfers much of the onus of learning from the classroom to students' personal time. Where many school children spend at least 30 hours per week in classes of some sort, college students spend half that much, 12 to 15 hours, and graduate students even less. The convention that university students spend twice as much time studying outside of class as they do in class reveals how engrained this transition of responsibility is in the assumptions underlying university practice. However, a large body of evidence indicates that social interaction is often powerful for if not integral to cognitive development and the creation of mathematical meaning (Cobb, Wood, & Yackel, 1993; Fullilove & Treisman, 1990; Treisman, 1992; Vygotsky, 1978) especially because learning mathematics entails a component of enculturation into the practices of the broader mathematical community (Cobb, 1989).

Much recent work in undergraduate mathematics education investigates the nature and mechanism of communal learning inside the inquiry-based classroom environment (Marongelle & Rasmussen, 2008; Nickerson & Bowers, 2008; Rasmussen, Zandieh, & Wawro, 2009; Yackel, 2001). However, comparatively little work has been able to document how student interactions and study outside of the classroom contributes to the development of mathematical meaning. Part of this no doubt stems from methodological issues, but these difficulties must be overcome because so much learning takes place beyond the classroom. Triesman's (Fullilove & Treisman, 1990; Treisman, 1992) work with students learning calculus at Berkeley represents the most prominent account of organic mathematical communities developing connected to the university classroom.

The present study documents one student's (Cyan's) construction of mathematical community around a proof-based undergraduate mathematics course and 1) identifies some of the various roles this student played in these mathematical interchanges, 2) considers aspects of the classroom and departmental context that facilitated the construction of these mathematical relationships, and 3) suggests how this case informs the current literature regarding the scope and nature of social mathematical learning in the university setting.

Theoretical Framework

The constructivist outlook developed by Vygotsky (1978) emphasizes the social dimension of learning. Vygotsky argued that for children, cognitive capabilities appear first in the social realm before they are internalized and become available within the individual mind. Cobb et al. (1993) built the emergent framework as a synthesis of Vygotsky's work in social learning with individual psychological constructivism (von Glasersfeld, 1984, as cited in Cobb et al., 1993) via Blumer's (1969, as cited in Cobb et al., 1993) development of the theory of symbolic interactionism. Cobb et al. (1993) argue that mathematical meaning emerges in a classroom via the interactions between teachers and students regarding mathematical concepts and objects. According to symbolic interactionism, meaning is not inherent within objects or symbols, but arises from the dynamic interaction of object or symbol and interpreter. Thus, in the interactive classroom setting, students and teachers negotiate their understanding of mathematical concepts and objects until these conceptions converge (to some degree) to constitute shared meaning.

Cobb (1989) pointed out that within the broader professional mathematics community the mutual establishment of understanding and the coherence of shared meaning among its members constitutes accepted truth. These phenomena of "institutionalizing" meaning apply at all levels and thus the mathematics classroom can be thought of as its own institutionalizing body of mathematical thinkers (Cobb, 1989). Similarly, smaller bodies of students within a classroom who negotiate understanding of mathematical concepts and objects together constitute such a community. For this paper, I refer to any such institutionalizing body as a "mathematics community" which is marked by the negotiation and establishment of meaning around mathematical concepts and objects.

Research Literature

Given that each mathematical community develops (often) overlapping but distinct mathematical meanings, the question arises as to how these different notions interface and become more globally established and compatible. According to Cobb (1994) engagement in the local classroom should enculturate students into the broader mathematical community. The responsibility thus falls upon teachers to guide the meanings emerging in a particular classroom mathematics community toward forms compatible with those more-or-less agreed upon by the broader mathematics community. To describe specific mechanisms by which this enculturation happens, Rasmussen et al. (2009) borrow Wenger's (1998, as cited in Rasmussen et al., 2009) description of "brokers" as mediators and connectors between independent communities. Brokers are defined as individuals who have membership in multiple communities and thus can act as links between them. "Brokers facilitate the translation, coordination, and alignment of perspectives between communities" (Rasmussen et al., 2009).

The most common form of broker in the classroom is the teacher because they are both peripheral members of the broader mathematics community and full members of the classroom mathematics community. As previously mentioned, the teacher influences the classroom community's development of shared understanding with strong deference to the accepted meanings of the broader community. Rasmussen et al. (2009) also observed students acting as brokers between the overall classroom community and their respective small group communities with whom the students engaged in classroom discussion where new ideas often developed. Rasmussen et al. identified instances of brokering actions within the classroom.

As mentioned before, one of the only research accounts of organic mathematical communities forming outside of an undergraduate classroom comes from Triesman (1992). To understand what made certain ethnic minorities perform poorly in college calculus at Berkeley while other minorities thrived, he investigated the various groups to identify factors that distinguished their social and mathematical backgrounds and their respective modes of engagement in undergraduate calculus. He found that the less successful groups of students tended to study in isolation and forego aid from instructors while the successful students organized themselves into groups for study and look to outside resources for help (Triesman, 1992). Working on the hypothesis that social engagement powerfully determined these students' success in calculus, he developed a program for the less successful body of students that emulated the social learning environment he observed organically arising among the successful students (communities arise organically when they are not guided or formed by educational structures). This model of calculus instruction, often called the Emerging Scholars Program, has found remarkable success over a range of institutions over time (Fullilove & Treisman, 1990) providing strong evidence for the salience of social interaction in undergraduate mathematics learning. Artificially enacting the communal learning environment that some ethnic minorities automatically construct appears to be able to yield strong learning benefits.

Methods

This case regarding the student Cyan was observed during a study of classroom communication in undergraduate real analysis. The study followed a grounded theory approach to data collection and analysis (Strauss & Corbin, 1998). I made written records of all class meetings, interviewed a set of five volunteers weekly throughout the semester, interviewed the professor biweekly, and collected copies of student exams. Student interviews focused on identifying student conceptions of key concepts, recording their perception and recall of classroom interactions and explanations, observing student proving and problem solving, and exploring their study habits. Teacher interviews identified the professor's expectations and intentions for class activities, her perceptions and insights into student thinking and learning, and her accounts and interpretations of particular course events after they occurred. In addition, I spent time regularly interacting with students in the mathematics club's lounge where many real analysis students often gathered.

Context

Cyan was a strong student in his final year of an undergraduate mathematics degree. He had previously taken an introduction to proofs course upon which he worked consistently at one table in the lobby of the mathematics building. He invited others to join him such that a small cadre of students from that course began to regularly meet there to work; a large portion of this group were reunited in the analysis course which was the setting for the present study. In the analysis course, the professor integrated small group and whole class discussions during class meetings and students turned in assignments in groups of three or four. The class adopted social norms of proposing, explaining, arguing, and justifying. The class also directly discussed issues of mathematical language, notation, and communication. During the semester of study, the mathematics department set aside the afore-mentioned classroom as a lounge for the student mathematics organization in which students gathered regularly to study and socialize. The professor's office was one floor above, and student attendance often matched her regular presence there.

Results

Cyan and his classmates

Over the course of the interviews with Cyan, it became clear that he placed strong value upon interaction with his classmates and perceived these interactions to play a significant role in his learning. He perceived his identity in the class as a leader, if not a teacher, and took personal responsibility for the learning of his classmates. Cyan indicated that helping other students, which he did often in the mathematics club lounge, provided both deeper understanding and personal satisfaction in his own learning. He said:

> I have to try to explain it to somebody, and trying to explain it takes an additional level of understanding.... I mean it could be late in the evening, and I am explaining something and I find that whoever I was explaining to stops listening and I am looking at it myself and I am like, "Ok, I really like this now. I really like what, how I understand this."

The environment within the classroom fostered interactions between students, and Cyan often contributed to the classroom interchange. During the meeting in which the class discussed the Bolzano-Weierstrass Theorem, the professor began the discussion by letting the class give examples of divergent sequences that do and do not have convergent subsequences. The professor then wrote on the board, "A bounded divergent sequence (always, sometimes, never) has a convergent subsequence." Locke, one of Cyan's classmates, quickly claimed that "always" should complete the statement, but the rest of the class took some time trying to find a counterexample. Locke supported his claim by arguing that any bounded, divergent sequence must "converge to two different things." Cyan extended Locke's claim to say that a sequence is bounded and divergent if and only if it has two different limits of subsequences.

The professor recorded this claim as a true/false question for the class to consider, but redirected the discussion by reminding the class of the previously proven monotone convergence theorem. Almost immediately, Locke pointed out that the class needed to prove that every bounded sequence has a monotone subsequence. The professor affirmed Locke's suggestion writing on the board: "To prove B-W using MCT, we will prove [Locke's] lemma: Any bounded sequence has a monotone subsequence."

The professor guided the class toward a proof of Locke's Lemma using "peaks," which are sequence terms with value greater than or equal to that of any term with a greater index. The proof's first case considers sequences with infinitely many peaks. The professor represented the sequence by a zigzagging line mimicking a mountain range, which is reproduced in Figure 1. The points marked by a heavy dot are peaks of the sequence.

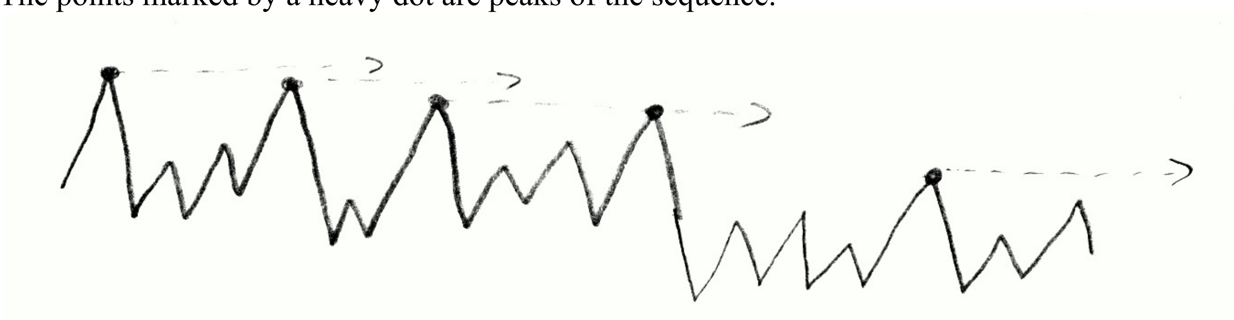

Figure 1: The professor's diagram showing the peaks of a divergent sequence.

Once she had argued that a sequence with infinitely many peaks has a monotone subsequence made up of peaks, the professor asked the class to consider what a sequence with finitely many peaks would look like. Cyan told the teacher that he was unable to imagine such an example.

The professor invited Cyan to the board and asked him to draw a sequence with exactly three peaks. He was able to produce a similar zigzagging diagram with three descending peaks matching her earlier diagram, but then wanted to make the rest of the sequence increase without bound to ensure that none of the later sequence terms would be peaks. He recognized that this would mean that the first three high points would no longer be peaks. Locke then volunteered to finish the diagram for Cyan and produced the diagram presented in Figure 2. The professor translated this diagram into the one presented in Figure 3 in order to complete the proof of Locke's Lemma.

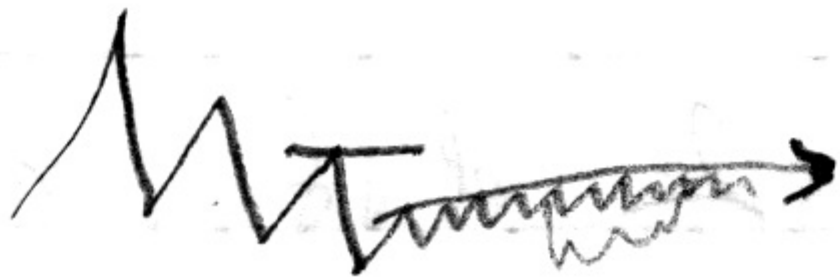

Figure 2: Locke's diagram of a sequence with exactly three peaks.

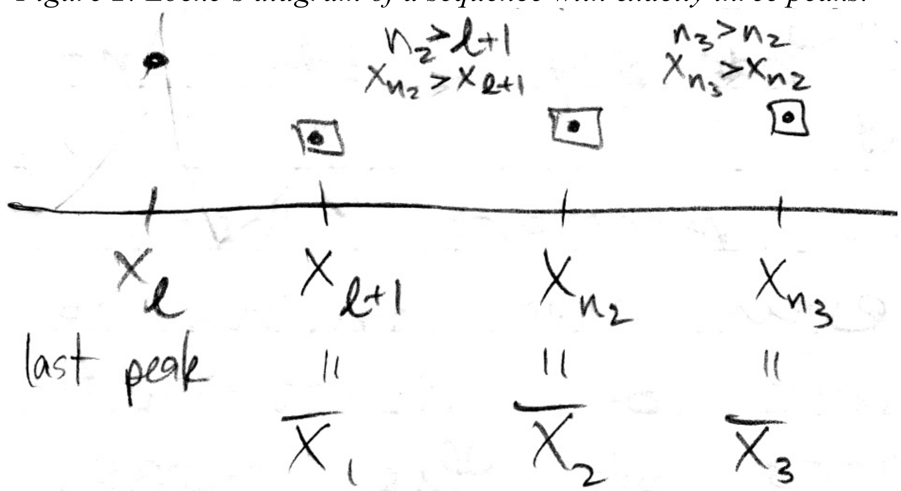

Figure 3: The professor's translation of Locke's diagram of a sequence with finitely many peaks.

Locke quite often provided arguments or proposed ideas before the rest of the class. Cyan expressed particular appreciation for Locke's mathematical input, citing the significance Locke's insights had upon his own reasoning:

> [Locke] always comes up with great arguments and so I love to hear his point of view.... In fact, a couple of times he convinced me quite quickly that I had not understood one of, part of the implication of the definition. So then later on when I was doing my homework I would come across that part and I would say, "No, I understand now this has to be a certain way because the definition implies this."

Several times, Cyan expressed this enjoyment he derived from his mathematical interchanges with Locke in real analysis. He said, "I expected the hardest class of my upper level undergraduate courses not to be so fun."

On several occasions, Cyan expressed appreciation for working with other students who could provide him with formative feedback on his reasoning. He purposefully tried to work with classmates who helped him find weaknesses in his own understanding. In one interview he described an instance in which he had corrected Barrett's proof because he applied a theorem that assumes convergence to a sequence that tends to infinity. Later that week, Cyan made the same mistake and was pleased when Barrett similarly pointed out his oversight. Students from previous semesters also frequented the mathematics club lounge, and current analysis students sometimes consulted with them about particularly difficult problems. Cyan's desire for critical feedback made him devalue this practice though because he said, "Usually I get along better with somebody that's in the class because they are the ones that want to argue their point to you."

Cyan and the Professor

The professor reported that during one conversation early in the semester Cyan revealed a latent misconception regarding infinity. Cyan constructed a bijection between the natural numbers and another denumerable set using a common classroom diagram in which the elements of one set were aligned horizontally above their corresponding elements in the other set with vertical arrows connecting them. To the right of the elements of the two sets that he identified explicitly, Cyan wrote ellipses and then matched the natural number "n" with the element "∞" of the other set. The professor noted that Cyan treated infinity as a number in this way, and directly addressed the fact that this was generally mathematically unacceptable.

On a later exam, she introduced several problems that she suspected might confuse Cyan if he had not overcome this conflation of the arbitrarily large finite with the infinite. One true/ false question stated, "If (a_n) converges to zero and (b_n) converges, then (a_nb_n) converges to zero." Cyan answered this question "false" noting on his test for the teacher to "please see me about this." The other true/ false question related to this idea said, "Suppose the sequence X converges to -1. For $K\in\mathbb{N}$, let X^K denote the sequence obtained by changing the first K terms of X to the value K and leaving the rest of the terms as they are. Then, for all $K\in\mathbb{N}$, the sequence X^K converges to -1." Cyan answered this question "true" before later revisiting his logic and changing his answer to "false."

Cyan was visibly bothered by his conflict over these two questions when he finished taking the test. He shared his reasoning with his classmates many of whom provided correct arguments that did not satisfy him. He reported, "I struggled with them for a couple of hours after the test, and it was bad because I had to go on a flight to [a conference]… I was… in [the professor's] office 20 minutes before I had to leave. I would not rest. I had to get it out."

When discussing his reasoning on the first question (substituting x for a_n and y for b_n), Cyan explained:

C: So however far out I go, there can be a convergent sequence that far. And so when you multiply the reciprocal times some number that large then you get 1… So you can't show y converges to some number because there is always a greater, my problem was this, I was trying to sit down and look at this times this.

I: If I give you the hypothesis that y is convergent to L, does that mean that L is fixed or can I keep saying make L bigger.

C: See I was taking it to be, take L to be, and then take L to be…. You can't take L to be something in particular cause there is always another one greater than whatever you pick.

Cyan struggled with knowing how to balance the simultaneous variation of both sequences x and y as n increases. He understood that not every product of sequences involving a sequence that converges to zero necessarily converges to zero. He pointed out "when you multiply the reciprocal times some number that large then you get 1." However, he attempted to preserve the fact that L could be arbitrarily large by continually increasing it: "You can't take L to be something in particular cause there is always another one greater than whatever you pick." Thus he tried to multiply ever-increasing numbers by ever-decreasing numbers to no avail, saying, "my problem was this, I was trying to sit down and look at this times this." Cyan conflated sequences that converge to an arbitrarily large finite number with sequences that diverge to infinity. Similarly for the second true-false question for which he changed his answer, he reasoned that if and arbitrarily large number of terms are being changed in a set of sequences (for all $K \in \mathbb{N}$), then an infinite number of terms are being changed bringing the convergence of the resulting sequences X^K into question.

However, through his further explorations Cyan showed evidence of understanding that the definition of sequence convergence provides a way to unambiguously deal with the variation of the sequence values. He reported that this rigor introduced by the definition was highly satisfying to him:

> After discussing it with [the professor] I just realized that there's just no notion of [multiplying by ever-increasing numbers]. I can't, you throw it away. And I was struggling with this all my life trying to figure out what does that imply. But that is why, and now I even understand why we call this undefined in the first place. Why at, the reason it is undefined in the first place, 1/0 or 1/inf. That reason is because we have the definition. And that's why I like this definition so much now.

Cyan connecting the student community with the professor

While Cyan spent significant amounts of time in the mathematics club lounge, he took it upon himself to discuss course material with his classmates. Because of this, Cyan understood the ways in which many of the other students were reasoning and communicated those ideas to the professor. During part of the first half of the semester, the class discussed bounded functions and infimums and supremums. The following questions were posed to the class during homework assignments:

> Let $f: \mathbb{R} \to \mathbb{R}$ and $g: \mathbb{R} \to \mathbb{R}$ be bounded functions. Decide whether each statement below is true or false, and prove your answer.
> (a) If $f(x) \leq g(x)$ for all x, then sup f $\leq$ inf g.
> (b) If $f(x) \leq g(x)$ for all x, then sup f $\leq$ sup g.

and later, "If $f(x) \leq g(y)$ for all x and y, then sup f $\leq$ inf g."

Cyan reported, "So that little distinction [between g(x) and g(y)] changes the behavior of the functions completely, right? And so in that case when you ask the same question, is the sup of f less than the inf of y or of g for all real numbers. That is a huge distinction and we weren't [seeing it], and I say we because I didn't get it at first either. It finally came across where, ok, this is the distinction that needed to be made for the class because [the professor] was asking us all this stuff, but hadn't really addressed it, so I asked her to go ahead a point that out [during class]."

Cyan perceived that his role of leadership caused his misunderstandings to influence other students. He explained, "Some people don't know it at all, and then they come and we discuss it and then they see my point of view, and then whatever flaws I have... some people carry those flaws into the classroom." So as Cyan found answers to his questions from the professor, he made it a point to try and disseminate his clarifications either personally or by asking her to address them in class. In the case of the afore-mentioned problems, he said, "I asked her to go ahead and point [the distinction] out.... because I knew it then, but some of the other people didn't and there were some of my friends that I knew I wanted to make sure caught that before they went on and got the answer wrong and they wouldn't know till a week after the test why it was wrong or something."

Discussion

Cyan participated in several mathematics sub-communities within his real analysis class. He discussed and negotiated classroom material with Locke and Barrett considering them assets and resources for his own learning. Cyan's persistent articulation of his reasoning during class allowed him to participate in direct mathematical exchange with other members of the class. With some other students he described taking on a teaching role, acknowledging the responsibility that entails. He also discussed real analysis with the professor at length both inside and outside of the classroom. Because Cyan possessed membership in these various communities, he was able to act as a broker between them.

In the case of the homework questions about infimums and supremums, the teacher and the sub-community of students working with Cyan in the mathematics club lounge interpreted the statements "$f(x) \leq g(x)$" and "$f(x) \leq g(y)$" differently. While the professor produced the two statements to differentiate between two sets of examples, the students failed to distinguish between the cases. The fact that the two problems appeared on the same sheet led Cyan to inquire about the issue with the teacher. Cyan therein informed the professor that some of the class failed to understand her intended meaning for the symbolic expressions, and he requested that she address the issue during a class meeting. Cyan's recognition that the two communities (the professor who represents the broader mathematical community and the group of students with whom he was studying) gave the same symbols different meanings led him to act as a broker by ferrying interpretations back and forth.

The professor paid careful attention to the reasoning Cyan displayed during their mathematical interchanges. When she identified that Cyan held a misconception, as she did regarding infinity, she decided to probe the whole class for that misconception by including the afore-mentioned true/false test questions. This indicates that the professor interpreted Cyan's reasoning as being to some extent representative of a significant portion of real analysis students.

The professor's comments during interviews indicate that she interpreted Cyan's ideas this way according to a belief in the reliability of student misconceptions, meaning that certain misconceptions occur over time among many students. For instance, she would pose other true/false questions on tests in response to students' statements or questions during previous real analysis courses. From the viewpoint of this study, the professor's instructional choice to test the whole class for Cyan's misconception appears quite appropriate for a different reason. Mathematical meaning arises communally, especially in a negotiated learning environment like this classroom or the mathematics club lounge. Cyan's ideas were likely shared by his classmates because of the central role he played in this classroom's various communities' establishment of mathematical meaning. Thus the professor's pedagogical response to Cyan's reasoning was likely suited to a decent portion of this particular classroom mathematics community of which Cyan was an active member.

Cyan's actions as a broker of mathematical meaning between the sub-communities of the classroom were facilitated by a number of personal factors. Cyan's personal value of responsibility to his classmates motivated him to teach them and shepherd their cognitive development. He displayed an avid pursuit of coherence in his own understanding, which revealed itself when he worked for hours after the test to reconcile his confusion, even to the point of almost missing a plane flight. He used very personal if not exaggerative terms to describe his learning: "I really like this now. I really like… how I understand this," "I love to hear his point of view," and "I was struggling with this all my life." Cyan's social connectedness also facilitated his role within the classroom. Cyan became the president of the mathematics club the semester after he took real analysis.

Several contextual factors also influenced Cyan's role in the classroom mathematics community. The contextual factors that contributed to Cyan's activity came from the department and from the professor. The department provided the lounge for mathematics majors to gather and work in, and it became the physical center of social interaction among the real analysis students. The department also hired many of the junior and senior mathematics majors in the math clinic across the hall from the mathematics club lounge, which increased the frequency with which real analysis students came to the mathematics building and spent time in the lounge. The professor promoted mathematics community through the interactive classroom environment, requiring homework assignments to be completed in groups, and holding extra study sessions weekly in which homework groups worked together and interacted with her in smaller groups.

The professor fulfilled her role as a broker between the broader mathematical community and the classroom community when she addressed Cyan's misconception in the form of a test question. She recognized that Cyan used the symbol "∞" in a way generally rejected by the mathematical community. She perceived his action to reveal a deeper discontinuity between Cyan's understanding of infinite processes and standard conceptions thereof. She assessed how Cyan's misconception would influence his reasoning in a different domain; she identified his reasoning about constructing bijections but questioned the class about sequences. She provided the test questions as opportunities for Cyan (and other students with similar ideas) to recognize this discontinuity with standard meaning and possibly alter his understanding. Because of the professor's careful analysis and response to Cyan's communication, his interactions with the professor substantially influenced the class' enculturation to meanings consistent with those of the broader mathematical community.

Conclusions

Much student learning in the university context occurs outside of the classroom, but many students benefit from developing mathematical meaning within a group context. Little is known about how frequently students study in isolation outside of the classroom or how they cope with this learning context. Also, the mathematics education literature explains little about how mathematics students develop community on their own for learning. This study provides one account of semi-organic mathematics community arising in a proof-based undergraduate classroom. These communities are not fully organic because the teacher required students to work in groups, but Cyan built community with many students outside of his homework group.

Cyan constructed a series of mathematical communities to support his and his classmates learning of real analysis. His membership in these communities enabled him to act as a broker of mathematical meaning between sub-communities both in and beyond the classroom. The professor was also empowered in her instruction by the access to student meanings Cyan provided. This case points to some of the ways in which the development of mathematics sub-communities for learning outside of the undergraduate proof based classroom can facilitate the development of mathematical meaning in that classroom mathematics community. The success of these communities might depend upon contextual factors such as physical sites to gather and a professor sensitive to brokers from the classroom community.

Directions for Future Research

The body of research regarding social learning in the proof-based undergraduate context continues to grow, but there is much room for expansion in research upon how students engage their mathematics classes when they are not in those classes. Further research should explore how mathematical meanings developed in communities in the classroom translate into student's private mathematical practices outside the classroom. Does social learning within the classroom facilitate communal learning outside of the classroom? Are there departmental and instructional conditions that can reliably facilitate the development of mathematical communities around undergraduate classrooms?

References

Cobb, P. (1989). Experiential, cognitive, and anthropological perspectives in mathematics education. *For the Learning of Mathematics, 9*(2), 32-43.

Cobb, P., Wood, T., & Yackel, E. (1993). Discourse, Mathematical Thinking, and Classroom Practice. In E.A. Forman, N. Minick, & C.A. Stone (Eds.) *Contexts for Learning: Sociocultural Dynamics in Children's Development* (pp. 91-119). New York, NY: Oxford University Press.

Cobb, P. & Yackel, E. (1996). Sociomathematical Norms, Argumentation, and Autonomy in Mathematics. *Journal for Research in Mathematics Education, 27*(4), 458-477.

Fullilove, R.E. & Treisman, P.U. (1990) Mathematics Achievement Among African American Undergraduates at the University of California, Berkeley: An Evaluation of the Mathematics Workshop Program. *The Journal of Negro Education*, *59*(3), 463-478.

Marongelle, K. & Rasmussen, C. (2008). Meeting New Teaching Challenges: Teaching Strategies that Mediate between All Lecture and All Student Discovery. In Carlson, M. & Rasmussen, C. (Eds.) *Making the Connection: Research and Teaching in Undergraduate Mathematics Education MAA Notes #73* (pp. 167-177). Washington, DC: Mathematics Association of America.

McClain, K. & Cobb, P. (2001). An Analysis of Development of Socio-mathematical Norms in One First-Grade Classroom. *Journal for Research in Mathematics Education, 32*(3), 236-266.

Nickerson, S. & Bowers, J. (2008). Examining Interaction Patterns in College-Level Mathematics Classes: A Case Study. In Carlson, M. & Rasmussen, C. (Eds.) *Making the Connection: Research and Teaching in Undergraduate Mathematics Education MAA Notes #73* (pp. 179-190). Washington, DC: Mathematics Association of America.

Rasmussen, C., Zandieh, M., & Wawro, M. (2009) How Do You Know Which Way the Arrows Go? The Emergence and Brokering of a Classroom Mathematics Practice. In Roth, W.M. (Ed.) *Mathematical Representations at the Interface of Body and Culture*, USA: Information Age Publishing, Inc.

Strauss, A. & Corbin, J. (1998). *Basics of Qualitative Research: Techniques and Procedures for Developing Grounded Theory* (2nd edition). Thousand Oaks, CA: Sage Publications.

Treisman, P.U. (1992). Studying Students Studying Calculus: A Look at the Lives of Minority Mathematics Students in College. *The College Mathematics Journal*, *23*(5), 363-372.

Vygotsky, L.S. (1978). *Mind in Society: The Development of Higher Psychological Processes*. Cambridge, MA: Harvard University Press.

Weber, K. (2004). Traditional instruction in advanced mathematics courses: a case study of one professor's lectures and proofs in an introductory real analysis course. *Journal of Mathematical Behavior, 23,* 115-133.

Yackel, E. (2001). Explanation, justification, and argumentation in mathematics classrooms. In M. van den Heuvel-Panhuizen (Ed.) *Proceedings of the 25th Conference for the International Group of Psychology of Mathematics Education*, *1*, 9–24, Utrecht, the Netherlands.