How Do Undergraduate Students Navigate their Example Spaces?

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Abstract

In this paper we report on a study where first year undergraduates were asked to generate examples of real sequences satisfying certain combinations of properties. Following earlier work by Antonini (2006), who classified the example generation strategies of expert mathematicians into three types—*trial and error*, *transformation* and *analysis*—we use both our data and a graphical representation of Watson and Mason's (2005) construct of example spaces to explicate Antonini's classification. We discuss a further dimension of variation within the classification that has not been previously reported, namely instances of *transformation* and *analysis* where at least some of the mathematical assumptions and deductions were invalid.

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Introduction

I can't understand anything in general unless I'm carrying along in my mind a specific example and watching it go. (Feynman, Leighton, & Hutchings, 1997, p.244)

Examples play a central role in within mathematics and in mathematical thinking, forming 'an important component of expert knowledge' (Zazkis & Chernoff, 2008). As acknowledged by Feynman in the above quotation, mathematicians often use examples to help develop their understanding of new topics. In part of a case study Alcock and Inglis (2008) observed two graduate students of mathematics answering questions in number theory. One of these graduate students used examples extensively, including instances of inductively testing conjectures using crucial experiments/generic examples, to check whether generalized arguments are correct by relating them to to specific examples and to generate counter-examples.

Authors in mathematics education are increasingly writing about using example generation by learners as a pedagogical tool; Dahlberg and Housman (1997) found that students who spontaneously generated examples of mathematical concepts were more effective in attaining an initial understanding of those concepts, and Watson and Mason (2005) gave a wide-ranging account of possible uses of example generation in the course of teaching. The use of 'gappy' or partially-populated lecture notes (Burn & Wood, 1995; Tonkes, Isaac, & Scharaschkin, 2009) is conducive to giving example generation exercises in the lecture theatre environment.

Mathematics education researchers have studied not only the use of examples by teachers and students (Chick, 2009; Alcock & Inglis, 2008), but also the process of example generation itself (Zazkis & Leikin, 2007; Antonini, 2006). Here, we follow on from such work and examine the strategies that are used by students when they are asked to generate examples of mathematical objects. In this paper we shall argue that there are several dimensions of variation that should be considered when researching different example generation strategies, including the mathematical assumptions and deductions that a student makes, together with the strategy used (Antonini, 2006) and whether the resulting example is correct.

Example spaces

Theoretical constructs called *example spaces* have been used by different authors for some time without a clearly defined common meaning between the different studies. Michener (1978, p.362) used example space as a noun to refer to a set of mathematical objects which served as illustrative material for a particular mathematical theorem or definition. By constructional derivation each example is built upon another logically (in a positivist sense), extending into a growing example space connected to the object. More recent expositions by Watson and Mason (2005) and Goldenberg and Mason (2008) treat the example space as analogous to Tall and Vinner's (1981) construction of *concept image*, i.e.

> Concept image describes the total cognitive structure that is associated with the concept, which includes all the mental pictures and associated properties and processes (Tall & Vinner, 1981, p.152)

Like a concept image, an individual's example space typically has an idiosyncratic internal structure (Watson & Mason, 2005, p.51); a particular example may be seen as representing a wider class of objects, for instance. Watson and Mason liken an example space to a mathematical toolshed:

> Another way to think of example spaces is as a toolshed containing a variety of tools—examples that can be used to illustrate or describe or as raw material. Some tools are familiar and come to hand whenever the shed is opened, whereas others are more specialised and come only when specifically sought (Watson & Mason, 2005, p.91)

In this paper we shall be using the construct of example space in a similar way to Watson and Mason, but include the structure as 'part' of the space. In other words an *example space* is a set of all examples of a particular mathematical object or concept that an individual is consciously or implicitly aware, together with any associated properties the individual believes the examples to possess, and any links the individual has drawn between examples. The *conventional example space* is the set of all possible examples of a mathematical object or concept where we define the properties and links formally via mathematical axioms and definitions.

At various junctures during the paper we shall be making reference to a Venn-diagram representation of a conventional example space, which we will use to illustrate the various example generation strategies. Figure 1 is an example of such a diagram illustrating the conventional example space for the sequence properties of increasing and decreasing. Some of the instances of example generation we discuss later on in the paper will involve sequence properties other than increasing and decreasing, but we shall still use the shape of the diagram in Figure 1 each time. This is because we are using the diagram to illustrate the different ways strategies navigate the example space, and such illustrations are dependent on the strategy used rather than the properties (or type of mathematical object) considered in a specific example generation question.

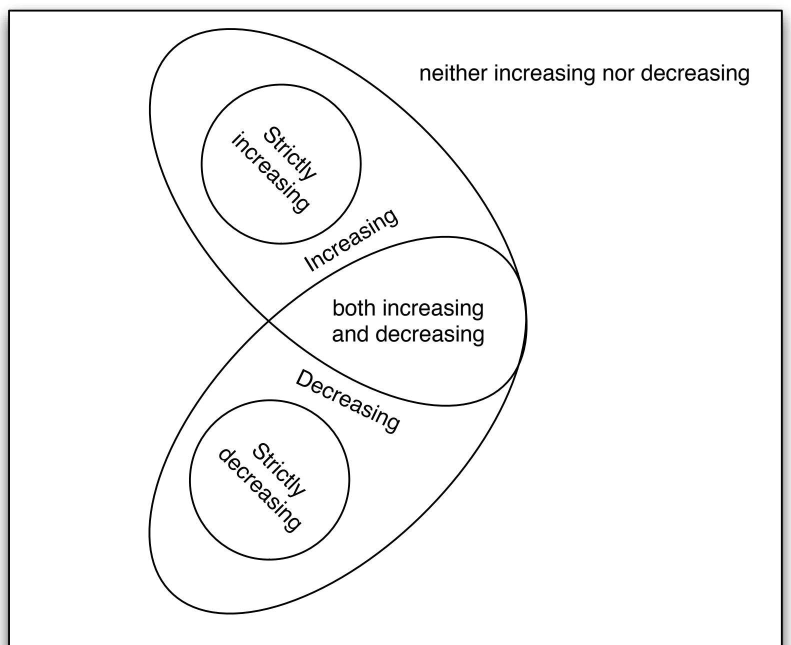

Figure 1. The conventional example space for the sequence properties of increasing and decreasing. We note that each region is non-empty, and each sequence may belong to one region only.

Research instrument and data analysis

Participants

The data that forms the basis of this study was collected during semi-structured clinical interviews with 15 undergraduate students at a central UK university. All the students were in the first year of a three or four-year mathematics degree and, as a consequence of the university's admission criteria, all had achieved a grade A in their A-level examinations (or equivalent), which is the highest grade available for 16-18 year olds in the UK. They were midway through a course on introductory real analysis, and had covered the topics of inequalities, sequences and series.

The task given to students during the interview

The 15 one-to-one interviews began with a general discussion about the students' experiences at university to date. Then, after around 15 to 25 minutes' discussion, the students were given a list of formal definitions of sequence properties that they had met previously in their course, as presented in Appendix A. Each student confirmed that they had seen the definitions before, but was not asked if they understood them. They were then presented with a list of example generation questions, as presented in Appendix B. This sheet consists of eleven questions, each which asks the student to give an example of a sequence with certain properties, or to state that the combination of properties requested is impossible. Of the 11 questions, only Question 10 involves an impossible combination of properties. The student was then asked to complete the questions, "thinking aloud" where possible.

During the task, students were prompted to explain what they were doing or thinking if they were silent for a significant amount of time (this was not measured, but in practice was after around a minutes' silence). When the student volunteered they were finished with the task, the interviewer (first author) informally discussed the student's answers, suggesting correct examples where appropriate.

Interviews were videotaped, and the audio transcribed. The transcript data was then open-coded as part of a Phenomenographical analysis of the data (Marton & Booth, 1997). Concepts which emerged from similar codes were grouped under more general headings and major categories were formed across headings. The variation within one such category (*Students' comments on how they produce answers*) is reported in this paper.

Much of the variation within this category can be captured via Antonini's (2006) classification, which is described in the next section. We note at this time that a number of codes describing students' comments on how they produce answers are not fully captured by Antonini's classification. These are statements and actions that a student goes through regardless of the actual question, and so outside the scope of Antonini's study but still captured within our open coding. In our data this consists of instances such as copying down the relevant definition(s), underlining parts of the question and sketching a pair of axes in anticipation for a sketch of the sequence. Such codes will not be considered further in this paper.

Antonini's classification of example generation strategies

A general comment on example generation strategies

Throughout this paper the word strategy is used to describe what an individual does when asked to generate an example of a mathematical object subject to certain constraints or properties imposed on the object. In this respect we follow the conventions of the literature on example generation, but we should like to make clear that neither we, nor other authors, claim that the use of a particular "strategy" is an explicit or conscious choice by the individual generating examples, although for a small subset this may indeed be the case. We are using the word *strategy* to describe the variation in what we, researchers in mathematics education, observe and describe in our data.

Antonini's categorisation

In 2006, Antonini reported on a study conducted to explore how expert mathematicians produce examples of concepts (Antonini, 2006). In this study, he interviewed seven graduate students of mathematics, asking them to produce examples of various mathematical objects with specified properties (for example, a binary operation that is commutative but not associative). He found that there were three major strategies: *trial and error*, *transformation* and *analysis*. A more detailed description of each strategy can be found in the next three subsections.

In a similar study, Iannone, Inglis, Mejia-Ramos, Siemons, and Weber (2009) found that undergraduate students almost solely used the *trial and error* strategy, with few cases of *transformation* and *analysis*, and (possibly as a result of this) almost no switching between strategies. We too found instances of the each strategies with a majority of *trial and error*, although we hesitate to include a measure of the frequencies due to the small sample size. In the remainder of this section we expound each strategy with

1. Antonini's definition of the strategy
2. An incident of this strategy from our data
3. A diagrammatic representation of each strategy

Then, in the remainder of the paper we introduce another dimension of variation which was not commented upon by either Antonini (2006) or Iannone et al. (2009), the mathematical validity of the assumptions and deductions within a particular example generation attempt.

Trial and error

Antonini describes the *trial and error* strategy as follows:

> The example is sought among some recalled objects; for each example the subject only observes whether it has the requested properties or not. (Antonini, 2006, p.58)

An example of a student who used this strategy is Guan[1], who was attempting to answer Question 9: A sequence that tends to infinity that is not increasing:

> I think it can just be this one [points to her answer to Question 6: $(a_n) = (-1)^n n$], because—oh, that doesn't tend to infinity[. . .] this is just a sub-sequence tending to infinity but the whole sequence doesn't.

The student then moved to a different question, and midway through answering it, returned to Question 9, remarking:

> Yes, I think I've found an example for [Question] 9. It's, I remember the lecturer talked about a sequence which is not this [points to Question 9], so it tends to infinity but it's

[1]The names of students used in this paper are pseudonyms. We have not changed their gender or ethnicity.

> not increasing because this term is obviously not greater than the previous term. So, we can write it as $1,0,2,0$.

This student believed this answer to be correct (although we note that her answer does not tend to infinity). Here Guan has used various sources to obtain candidate solutions for her 'trials', including answers given to previous questions on the task and sequences she had seen previously in class. The *trial and error* strategy is illustrated in Figure 2.

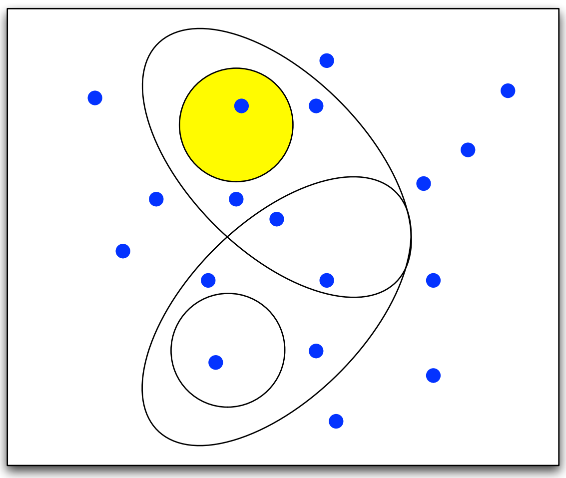

Figure 2. Diagrammatic representation for the *trial and error* strategy. Sequences within the shaded region have all the desirable properties. Sequences are tested one-by-one to see if they have the desirable properties.

Transformation

Antonini describes the *transformation* strategy as follows:

> An object that satisfies part of the requested properties is modified through one or more successive transformations until it is turned into a new object with all the requested characteristics. (Antonini, 2006, p.59)

An example of a student who used this strategy is Madhu, who was attempting to answer Question 6: A sequence that has neither an upper bound nor a lower bound. She eventually gave the answer $(a_n) = (-1)^n n$:

> I thought of $(-1)^n$, and thought that is bounded, so I needed to change it at each n, so I timesed it by that [n], and it'll be different every time.

Here Madhu has taken a sequence she gave to an earlier question (Question 4: A sequence that is neither increasing nor decreasing), which has some features that are desirable for Question 6 also. She then transformed the sequence so that each point's magnitude would increase relative to its predecessor. The *transformation* strategy is illustrated in Figure 3.

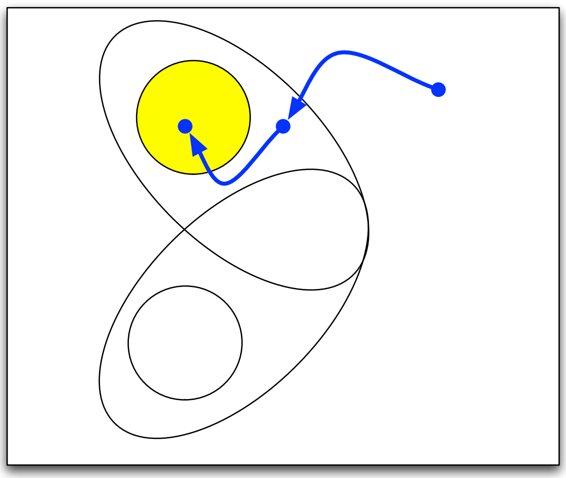

Figure 3. Diagrammatic representation for the *transformation* strategy. Sequences within the shaded region have all the desirable properties. A sequence without all the required properties is modified by transforming it or incorporating additional features until it has all the required properties.

Analysis

Antonini describes the analysis strategy as follows:

> Assuming the constructed object [exists], and possibly assuming that it satisfies other properties added in order to simplify or restrict the search ground, further properties are deduced up to consequences that may evoke either a known object or a procedure to construct the requested one. (Antonini, 2006)

Unlike Iannone et al.'s (2009) study, we found several instances of analysis, although in most students tended to be making deductions and assumptions which were mathematically invalid. This is something that we do not expect experts to do, and perhaps consequently Antonini did not discuss this possibility. The next section is devoted to this added dimension to Antonini's classificaton, but here we report on one student, Ian, who arguably came close to analysis in the sense of Antonini. Ian was justifying why his answer, $(a_n) = (-2)^n$ satisfied the properties given in answer Question 6: A sequence that has neither an upper bound nor a lower bound:

> It's going to increase, a subsequence is going to increase and decrease too. An increasing subsequence, decreasing subsequence, and for each term is going to exceed the upper bound and the lower bound.

Although it is not clear where we find analysis in this somewhat jumbled description, we argue that one possible interpretation of Ian's deductions is as follows:

- If a sequence (a_n) has neither an upper bound nor a lower bound it must contain an increasing subsequence and a decreasing subsequence
- For any candidate upper bound, there must be a term in the increasing subsequence that exceeds it (or we can modify the increasing subsequence accordingly), and a similar property for candidate lower bounds and the decreasing sequence
- $(a_n) = (-2)^n$ has suitable subsequences

The analysis strategy is illustrated in Figure 4.

Discussion of Antonini's classification

Antonini (2006) found the *trial and error* strategy was typically used in the first instance amongst his (expert) participants. He then went on to consider the different circumstances that may result in an expert switching to one of the other strategies. For instance, an example used unsuccessfully in a *trial and error* strategy may be used as the initial representation in a *transformation* strategy if it is considered sufficiently 'close' to a correct example and a way to transform the object is known. In contrast we observed few instances of undergraduate students switching strategies, which is similar to the observations of Iannone et al. (2009).

In his paper, Antonini only reported on transformations and analyses that were successful both in eventual example and internally, i.e. that had mathematically valid steps, deductions and as-

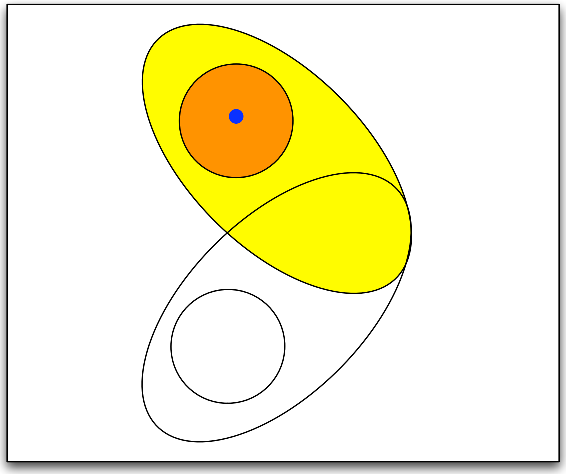

Figure 4. Diagrammatic representation for *analysis*. Sequences within the lighter shaded region have all the desirable properties. Further deductions, assumptions or constraints are deduced or included to narrow the region (to the darker shaded part), and an example is found (or an algorithm to construct an example) within the smaller region.

sumptions. We take this to suggest that when experts use the *transformation* and *analysis* strategies, the deductions and assumptions that are made explicit to the interviewer are typically mathematically valid, or quickly rectified. Iannone et al. (2009) observed far fewer instances of the *transformation* and *analysis* strategies with their undergraduate participants, but the incidents which were reported contained valid mathematics. If our categorisation is constrained by recording only those strategies with mathematically valid deductions and assumptions, we too found few instances of *transformation* and fewer still of *analysis*.

The need for another dimension of description in Antonini's classification

Novice mathematicians are likely to make incorrect assumptions and deductions when they attempt mathematical questions, whether these are example generation questions or some other type. By the very nature of the *trial and error* strategy, errors can only occur when checking to see if a candidate example has the properties required, and so we only remark here on incidents of incorrect assumptions and deductions within a *transformation* or *analysis* strategy.

In this section we shall present two further instances of *transformation* and *analysis*. Unlike those reported in the previous section, these instances contain deductions and assumptions that are invalid relative to formal mathematics. We see the prevalence of such errors as a further dimension of variation within a particular strategy, for there is a clear qualitative difference between two students who provide the same example to a question using the same strategy, one of whom using mathematically incorrect deductions within the strategy.

Transformation strategies with incorrect assumptions

Consider the following incident when Lewis was attempting to answer Question 9: A sequence that tends to infinity that is not increasing.

> Ah, actually, yeah that one is possible. You could have something like a sine curve doing that [see Figure 5 for Lewis's sketch], I need to think of the equation. Yeah, something like $(a_n) = n + sin(n)$.

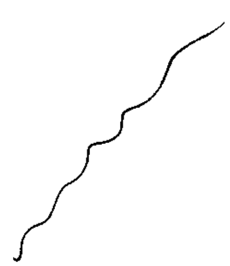

Figure 5. Lewis's sketch of the sequence later labelled as $(a_n) = n + sin(n)$.

Here, Lewis had taken the sequence based on the local minima and local maxima of the sine curve, and wished to rotate it to resemble the function he sketched (see Figure 5). However, when he attempted to give a formula for this sequence, he gave $(a_n) = n + sin(n)$, a sequence which is increasing[2], and so does not satisfy the properties requested in the question.

If we argue that Lewis is using a transformation strategy, we must conclude that within a transformation strategy attempt it is not necessarily the case that each step (or transformation) will

[2]If we differentiate the associated function over the reals we obtain $f'(x) = 1 + cos(x) \geq 0$, and so the sequence must be increasing

both (a) keep the properties which made the initial example choice salient and (b) add further properties which bring it 'closer' to the combination of properties required. In Lewis's transformation of $(a_n) = sin(n)$ some properties of the initial object which were desirable in the first incarnation of the example were removed. Thus within strategies categorised as *transformation*, we may have steps such as the one illustrated in Figure 6.

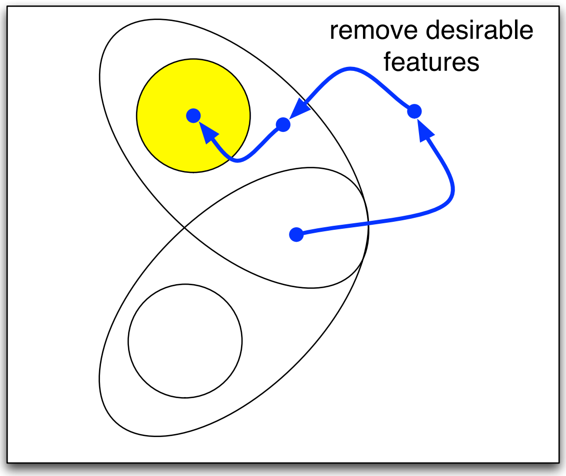

Figure 6. A transformation strategy may contain steps which remove some of the properties which were desirable in the initial object.

Analysis strategies with incorrect assumptions

Within an analysis strategy, there are two main types of mathematical arguments that can be made: mathematical deductions about further properties the example must have (presuming it exists), and further assumptions which can be introduced to restrict the region of the conventional example space under examination (see Figure 4). When non-expert mathematicians use an analysis strategy it is probable that not all of these deductions and assumptions shall be true relative to formal mathematics. Take another of Ian's answers, this time his answer to Question 5: A sequence that has no upper bound.

> I'm not sure you can say that but any strictly increasing sequence will do that. So $(a_n) = (n)$ diverges to infinity so it has no upper bound.

This chain of logical statements contains deductions which are mathematically incorrect. It is not the case that any strictly increasing sequence will have no upper bound, and so within this analysis strategy incorrect deductions have been drawn. The next statement, that a sequence which diverges to infinity will have no upper bound, is correct and the answer given, $(a_n) = (n)$, is also correct. We argue that the mathematically-false deductions within a strategy should not be ignored even if they are followed by true ones, and so when we consider Ian's use of the analysis strategy we conclude that it is qualitatively different to the incidents Antonini reported, with respect to this added dimension of variation.

Discussion and further questions

> *An important part of mathematics is appreciating the scope of a particular theorem or technique.* (Mason, 2002, p.16)

In this paper we have complemented Antonini's (2006) and Iannone et al.'s (2009) research on example generation strategies in mathematics. We have argued that in order to give a rich account of the process undergone during an event of example generation it may be important to consider not just the strategy used, but also the mathematical assumptions and deductions within a particular strategy, and of course the eventual example generated. Such considerations will be rarely necessary when studying expert mathematicians generating examples, but come increasingly to the fore when investigating the strategies used by undergraduate and other novices.

For studies that do investigate the strategies used by novice mathematicians when generating examples, we argue that such considerations should form a dimension of variation within the analysis. It is perfectly possible to generate a correct example via an arguably (mathematically) unsophisticated *trial and error* strategy, or via incorrect assumptions and deductions within a superficially sophisticated *transformation* or *analysis* strategy, for instance.

Awareness of this dimension is important when using example generation as a pedagogical tool. If we encourage students to generate examples of mathematical objects and concepts, incorrect assumptions and deductions may be hidden from the lecturer, perhaps as part of what may be an otherwise sophisticated strategy and correct answer. We note, however, that the same arguments could be made for practically any other unguided or open-ended task. As mathematics educators we should clearly pay attention to the assumptions and deductions students make during example generation, without ignoring the strategy used or actual example generated.

It would be tough to dispute the quotation from Mason at the top of this section, and examples have a clear role to play in determining the scope of a particular theorem or technique. We believe that by encouraging students to generate examples of mathematical objects they are studying, they will become more competent and fluent using examples, and in time begin to use examples in the way expert mathematicians do. In this paper we have found Venn-diagrams useful for describing the ways various example generation strategies navigate the conventional example space, and we suggest that perhaps presenting conventional example spaces to students in this way, or encouraging students to draw diagrams themselves, could perhaps help reveal the rich and varied structure of examples within a particular mathematical topic.

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Appendix A

The list of definitions given to students

Remember that a sequence is a list of real numbers

$$(a_1, a_2, a_3, a_4, \dots)$$

where $(a_n)_{n=1}^{\infty}$ denotes the whole sequence.

Definition: A sequence $(a_n)_{n=1}^{\infty}$ is *increasing* if and only if $\forall n \in \mathbb{N}$, $a_{n+1} \geq a_n$.

Definition: A sequence $(a_n)_{n=1}^{\infty}$ is *strictly increasing* if and only if $\forall n \in \mathbb{N}$, $a_{n+1} > a_n$.

Definition: A sequence $(a_n)_{n=1}^{\infty}$ is *decreasing* if and only if $\forall n \in \mathbb{N}$, $a_{n+1} \leq a_n$.

Definition: A sequence $(a_n)_{n=1}^{\infty}$ is *strictly decreasing* if and only if $\forall n \in \mathbb{N}$, $a_{n+1} < a_n$.

Definition: A sequence $(a_n)_{n=1}^{\infty}$ is *monotonic* if and only if it is increasing or decreasing.

Definition: A sequence $(a_n)_{n=1}^{\infty}$ is *bounded above* if and only if $\exists U \in \mathbb{R}$ s.t. $\forall n \in \mathbb{N}$, $a_n \leq U$.

Definition: U is an *upper bound* for the sequence $(a_n)_{n=1}^{\infty}$ if and only if $\forall n \in \mathbb{N}$, $a_n \leq U$.

Definition: A sequence $(a_n)_{n=1}^{\infty}$ is *bounded below* if and only if $\exists L \in \mathbb{R}$ s.t. $\forall n \in \mathbb{N}$, $a_n \geq L$.

Definition: L is an *lower bound* for the sequence $(a_n)_{n=1}^{\infty}$ if and only if $\forall n \in \mathbb{N}$, $a_n \leq L$.

Definition: A sequence $(a_n)_{n=1}^{\infty}$ is *bounded* if and only if it is both bounded above and below.

Definition: A sequence $(a_n)_{n=1}^{\infty}$ *diverges* if and only if it does not converge to any finite limit.

Definition: $(a_n)_{n=1}^{\infty}$ tends to infinity if and only if $\forall C > 0$, $\exists N \in \mathbb{N}$ s.t. $n > N \Rightarrow a_n > C$.

Appendix B

The task Sheet given to students

Give an example of each of the following, **or state that this is impossible**.

You may write your sequence in any way you choose (e.g. using a formula or a list of numbers).

1. A strictly increasing sequence
2. An increasing sequence that is not strictly increasing
3. A sequence that is both increasing and decreasing
4. A sequence that is neither increasing nor decreasing
5. A sequence that has no upper bound
6. A sequence that has neither an upper bound nor a lower bound
7. A bounded, monotonic sequence
8. A sequence that tends to infinity
9. A sequence that tends to infinity that is not increasing

10. A sequence that tends to infinity that is not bounded below
11. A strictly increasing sequence that does not tend to infinity