Point/Counterpoint: Should We Teach Calculus Using Infinitesimals?

Rob Ely
University of Idaho
ely@uidaho.edu

Tim Boester
Wright State University
timothy.boester@wright.edu

Abstract

During the first 150 years of its life, calculus was developed and widely applied by mathematicians who conceptualized and notated integrals and derivatives using infinitesimals. Although the rigorous notion of limit took the place of infinitesimals, it has been since shown that infinitesimals can be used to define calculus with equal rigor. So why should calculus not be taught today using infinitesimals? This paper presents a point/counterpoint debate about the merits and drawbacks of the infinitesimal approach to calculus, appealing to educational research findings, issues of notational affordance, formal abstraction, and the various student conceptions of limits as dynamic and static entities.

Background

Mathematicians of the 17th Century such as Cavalieri, Torricelli, and Fermat, routinely appealed to indivisible and infinitesimal quantities in their explanations, particularly when finding curved areas and volumes (Mancosu, 1996). Dividing a figure into indivisible quantities provided a powerful method for determining certain geometric relationships. While Galileo and Cavalieri showed that this technique gave rise to some apparent contradictions, dividing a figure into infinitesimal quantities proved to be a far more promising and coherent method, namely because infinitesimal entities retain the same dimensionality as the original figure. This is why the powerful and general differential and integral calculus was developed, described, and notated entirely as a method of manipulating infinitesimal quantities by its independent inventors, I. Newton and G. W. Leibniz.

On the continent, Leibniz' infinitesimal calculus flourished in the 18th Century, providing a flurry of powerful and deep results from mathematicians such as the Bernoullis, L. Euler, and A. Legendre. Although the infinitesimal foundations of the calculus were ultimately replaced in the 19th Century by the more rigorous δ-ε formulation of limit, we still use Leibniz' infinitesimal notations today: "*dx*" for an infinitesimal difference and "∫" for a sum of infinitely many infinitesimal quantities. In 1961, A. Robinson's creation of the hyperreal numbers firmly established that calculus could in fact be founded upon infinitesimals, instead of the formal definition of the limit, with equal rigor and power.

Virtually every modern-day calculus book teaches the subject using limits rather than infinitesimals. But if calculus was historically born and raised with infinitesimals, if we still use these notations today, and if the infinitesimal approach is provably no less rigorous than the limits approach, why should we not teach calculus using infinitesimals? We investigate this question in a point/counterpoint debate format.

Point: Why We Should Teach Calculus Through Infinitesimals

Before enumerating several points in favor of the infinitesimal approach, it is best to clarify a bit what is meant by the proposal that we teach calculus using infinitesimals. The method of using infinitesimals in calculus classes that I am espousing here is an informal approach, not a formal one. Just as most university calculus classes employ limits informally rather than using the formal definition, the informal approach to infinitesimals would not include the construction of the hyperreal numbers. Therefore this approach is not necessarily in keeping with the approaches found in infinitesimal calculus books such as Keisler (1986) and Henle & Kleinberg (1979).

To approach calculus using infinitesimals, we must start with Leibniz' injunction to treat a curve as a polygon with infinitely small sides. Then a derivative indicates a slope of one of these straight sides, and a definite integral indicates a sum of infinitely many trapezoids. To teach calculus this way would involve first modeling situations that would involve derivatives and (definite) integrals with small finite differences, and then move to infinitesimal differences. For derivatives, this involves generalizing several rules for working with infinitesimal and finite quantities, such as "the product of an infinitesimal quantity and a finite quantity is infinitesimal." Then the fundamental theorem of calculus can be discussed in two ways: first, as a generalization of the relationship between sums and differences of sequences, and second, as a property of the infinitesimal accumulation of area under a curve over an infinitesimal distance. There are several points in favor of this approach:

1. History suggests infinitesimals are the correct way to build notational intuition

The historical story of calculus strongly suggests that infinitesimals are a natural and intuitive way to think about the complex ideas of calculus. The primary reason for this is that this approach makes natural use of Leibniz' notation. This allows the notation to function as a cognitive tool (rather than an obstacle) that: (a) affords relatively transparent representation of graphical and physical phenomena and (b) suggests powerful and correct generalizations.

There are many ways in which calculus notation affords transparent representation of graphical and physical phenomena when using an infinitesimal approach. Here are a few examples which hopefully illustrate this point, but are by no means exhaustive:

• The integral notation $\int_a^b f(x)dx$ evokes a trustworthy image of a sum of rectangular areas with heights given by the function and uniform infinitesimal width *dx*. Compare this with the limits-based approach, which requires students to treat "*dx*" merely as a placeholder, an imminently forgettable vestige that serves only to denote that the integral is "with respect to *x*."

• Using infinitesimals, *dy*/*dx* really *is* a ratio of two infinitesimal quantities, not code language for $\lim_{h\to 0}\frac{f(x+h)-f(x)}{h}$.

• A differential really *is* what it looks like: an infinitesimal change in one quantity given an infinitesimal change in the other: $dy = 3x^2dx$.

• Using infinitesimals, the arc length of a curve is the sum of hypotenuses of triangles with infinitesimal sides *dx* and *dy*: $\int_a^b \sqrt{dx^2 + dy^2}$. In the standard approach to calculus, this integral becomes the opaque $\int_a^b \sqrt{1 + f'(x)^2}\,dx$.

• To set up an integral for the volume of rotation of a solid, who does not imagine a thin cylinder with "base" πr^2 and thickness *dx*?

The infinitesimal approach also allows the calculus notation to function as a cognitive tool that suggests powerful and correct generalizations. For instance, in the infinitesimal approach the chain rule, $\frac{dy}{dt} \cdot \frac{dt}{dx} = \frac{dy}{dx}$, makes use of prior knowledge of fractions—it *is* cancelation of fractions of quantities. The limits approach asks students to ignore the suggestiveness of the notation, viewing the fractions as code language for $f'(x)$.

2. Some students think using infinitesimals, regardless of their non-existence

What about Berkeley's famous objection (1734) that infinitesimal quantities do not really exist? While students, like Berkeley, may have an ontological anxiety to objects that do not exist, such a concern may be addressed head-on in a calculus class. After all, students by this time are comfortable working with "useful fictions," as Leibniz called his infinitesimals (Ariew & Garber, 1989); "2", "π", "$\sqrt{-4}$"—all are objects that we have imagined and constructed to do our bidding. Besides, there is reason to believe that students often do believe in infinitesimal entities. In a recent study, 31% of calculus students have a conception of the real numbers that consistently includes infinitesimal distances (Ely, in review), and nearly 83% of students believe that "it is possible to find two different real numbers that are infinitely close to one another." In another study, a student explicitly describes the real numbers in terms of infinitesimal numbers and distances, and consistently explores the implications of her conceptions (Ely, 2010), clearly not just envisioning that these entities are merely very small. These student conceptions indicate that Robinson's formulation of the nonstandard real numbers is not just a clever trick, but involves entities that may be far more intuitive for students than limits.

3. Limits are conceptually difficult for students

Research shows that limits are perennially difficult for students to understand. This is not just true for the notoriously difficult formal definition, but also for the informal notion. The idea of a limit is often unmotivated at the beginning of calculus classes, and is often viewed as something that cannot be reached, is only an approximation, is always gotten by "plugging in the numbers," and so forth. (e.g. Schwarzenberger & Tall, 1978; Sierpinska, 1987; Davis & Vinner, 1986; Williams 1991). When the notion of limit is so artificial and problematic, and the notion of infinitesimal so natural, we should make the switch.

Counterpoint: Why We Should Not Teach Calculus Using Infinitesimals

In this section, the above claims, namely that infinitesimals are more intuitive and their notation is less cumbersome, will be questioned. Other problematic consequences of using infinitesimals as a basis for the calculus, that my colleague failed to mention, will also be raised:

1. Infinitesimals in classrooms are entirely unproven

The evidence that limits are conceptually difficult for students is irrefutable. However, limits tend to be poorly supported in calculus classes, mostly because the pedagogical tools drawn from current conceptual frameworks for limits are underdeveloped. Thus we cannot simply point to students' difficulties with the subject and conclude that the currently dominant foundation for the subject must be rejected. Let us first resolve some of the issues surrounding the teaching and learning of limit, and then we can decide if a switch is warranted. Put another way, just because the only study thus far (Cottrill et al., 1996) that has attempted to link a theoretical framework of limit with instruction failed to produce students who could understand the formal definition, doesn't mean it is not possible.

The criticism that limits are not properly motivated in a calculus classroom could just as easily be cast to infinitesimals if not properly motivated at the beginning of a calculus course. The same framework for instruction using limits would be required in using infinitesimals to teach calculus. The brief outline above would necessarily need to be fleshed out, and while the work for limits is progressing, the same work for infinitesimals is in its infancy. Thus, the call to abandon limits for infinitesimals seems premature. I am, however, anxiously awaiting some brave professors to try and then report their findings.

2. The concept of limit is more intuitive for students to understand than infinitesimals

Informally, limits have a simple grounding metaphor, that of motion-based approaching (Lakoff & Núñez, 2001). Even though this metaphor breaks down when stress-tested against more formidable sequences or functions, it serves as a good place to start when discussing limits. Students can rely on their natural experiences with approaching and motion to motivate an informal conception of limit. This conception even works for many (perhaps all) of the examples and problems that first-year calculus students are typically exposed to.

There is no similarly grounded conception for infinitesimals. While it doesn't matter mathematically whether or not infinitesimals actually exist, it may matter a great deal to students. Some students may readily accept infinitesimals as useful objects, but regardless, their creation will require a metaphorical feat similar to the Basic Metaphor of Infinity (Lakoff & Núñez, 2001). This is conceptually more demanding than a simple, grounding metaphor such as motion. This taxing metaphorical leap occurs regularly when applying infinitesimals to other concepts in calculus. For example, it is not at all obvious that treating a curve as a polygon with infinitely small sides is more intuitive than treating a curve as a simple, smooth object.

3. Infinitesimals put the burden of understanding in the hands of algebra

The algebra associated with both standard limits and infinitesimals is unavoidable. Even when both methods are treated informally, students will eventually be asked to perform computations using the algebra of limits or infinitesimals. While both types of algebras present difficulties for students (because algebra in general is difficult for students), the main advantage that Robinson (as summarized in Kleiner, 2001) and Sullivan (1976) propose could also be thought of as a disadvantage. They claim the algebra of infinitesimals is better because it is simply an extension of the algebra for real numbers. But this simplicity, while good for mathematicians, may be a poor tool for learning about limits. Students may fail to recognize the distinction between real and infinitesimal algebra, since on the surface, the algebra itself is indistinguishable.

Take, for example, finding the derivative of a function using the definition. Both systems ask students to perform a series of computations, and in a penultimate step, to treat what was just a moment ago considered a regular variable denoting a number as something else. In limits, this step is notated by the $\lim_{h \to 0}$ symbol. One might think this notational reminder is a benefit to students, but anecdotally we see students frequently misuse this symbol, as it requires students to consider functions as objects, that in turn are operated on by the limit. For infinitesimals, students need to know when to stop treating *dx* like a real number and haul out their algebraic rules for dealing with combinations of reals and infinitesimals.

For students who have simply proceduralized all of this under either method, at first glance it is unclear which has a better grasp of the derivative. However, because of the lack of a grounding metaphor, infinitesimals are more closely tied to their algebraic or numeric nature than limits are. Thus, I would claim that if a student does not understand the infinitesimal algebra of computing the derivative, they cannot fully understand the definition of the derivative, because, using this method, the definition of derivative *is* the infinitesimal algebra. A student who has memorized the limit procedure has at least the potential to have a grounded conception of the derivative based on the metaphor of approaching, whereas the infinitesimal student is stuck with the numerical conception of the derivative.

Finally, some may believe that I have made a contradiction in the above statements, that the algebra of infinitesimals is simultaneously too easy and too hard for students. In a way, I have made that claim. The algebra of calculus tends to be difficult for students, and thus the algebra of infinitesimals is also difficult. At the same time, the algebra of infinitesimals is indistinguishable from the algebra of the reals. Students may simply assimilate this new algebra as part of the old, completely missing the subtle point of even having infinitesimals. In this sense, the algebra is too easy. In both cases, the algebra is obfuscating the fundamental concepts of calculus, in either being too dense, or too muted in its differences. This presents, I believe, two hurdles for students' learning: first, to attenuate to the ramifications concerning real versus infinitesimal algebra, and then second, to extract the meaning of the calculus concepts from the algebra.

4. What about courses after an infinitesimal calculus?

There is ultimately a practical consideration that underlies all of this discussion. There is the inescapable argument that limits are a necessity of courses after calculus. There are several possible options to remedy this. We could reshape these courses to include infinitesimals. We could accommodate for them by translating infinitesimals into limits when covering each concept (such as when reinterpreting the derivative for more than two dimensions). We could simply introduce limits at the beginning of these classes and start using them to replace the notion of infinitesimals. Whatever the case, if we decide to replace limits with infinitesimals in first-year calculus courses, something will need to be done to accommodate this decision if and when students proceed to higher-level math courses.

Rebuttal

The counterpoint makes several extremely important points that warrant a bit more discussion. First is the point that students who are going on in mathematics must eventually learn limits. Will the transition to limits be problematic for these students?

It is worth mentioning that the number of students who go on to take advanced calculus is a very small number compared with the number of students who take Calculus I. It may simply be worth doing the tiny minority a disservice in order to better serve the majority. But even for this tiny minority, it is not clear that the transition from infinitesimals to formal limits would be rockier than the transition from informal limits to formal limits. First of all, the transition to the formal definition of limit should take place in advanced calculus, since convergence of series and sequences in Calculus II can and should be covered without epsilonics. When the transition to the formal definition of limit must be made, the best approach might be one that motivates this formalization historically. The goal must be to answer for students the question of why such an intuitive approach must be formalized in such a different manner, by deliberately investigating situations in which the infinitesimal (or informal limit) approach is inadequate or problematic.

Finally, one counterpoint that was made was that infinitesimals do not employ the physically grounded metaphor of dynamic motion in the same natural way that limits do, and that they make too much use of algebraic formalism. I would argue that infinitesimals could be constructed using conceptual metaphors that are more straightforward and more grounded than what was implied. One possible conceptual metaphor is by generalizing the familiar algebra of regular old numbers. While this metaphor may not have the same properties as the approaching metaphor of limits, it has considerable power in bringing forth the correct expectations of how infinitesimals should behave. Infinitesimals might also be constructed using the "approximation" metaphor, which has been shown to be a particularly successful metaphor in learning calculus (Oehrtman, 2009). While further work needs to be done to research which metaphors students are using when thinking in terms of infinitesimals, to say that they are not using any sort of grounding metaphor is presumptuous.

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