Counting Two Ways: The Art of Combinatorial Proof

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Abstract

Combinatorial proofs are used to show that many interesting identities hold. Typically, after examining an identity, one poses a counting question and proceeds to answer it in two different ways. This poses a challenge for students as it requires a way of thinking other than they have traditionally encountered. The newly introduced proof technique requires students to either create new strategies or adapt their old strategies to write such proofs. We will discuss the results of a preliminary study on the combinatorial proofs written by students in an upper-division combinatorics course and a graduate-level discrete mathematics course. In particular, we will identify some common difficulties that students have and suggest ways to overcome them.

Counting Two Ways: The Art of Combinatorial Proof

When studying the thought processes used to students to construct proofs, the subcategory of combinatorial proofs is particularly interesting. Combinatorial proofs are almost always proofs of identities, and prove the identity by counting a set twice. One of these ways of counting the set is represented by the left-hand side of the identity, and the other is represented by the right hand side. Since these two ways of counting count the same set, they must be equal to each other, thus proving the identity.

The logic of a combinatorial proof is not particularly challenging. Students' written combinatorial proofs are interesting for two reasons. First, students typically do not encounter combinatorial proofs until late in their educational careers, and therefore already have a great amount of experience writing proofs. However, combinatorial proofs are usually structured differently than other proofs, and require students to revise their proof strategies or create new strategies to write this new form of proof. Secondly, students' combinatorial reasoning skills are critical to these proofs, and students may or may not realize how that combinatorial reasoning affects their proof.

This article describes the results of a preliminary study examining students' written combinatorial proofs. In particular, we will examine some of the difficulties students have in writing combinatorial proofs through case studies of several students.

Methods

This study examined the written combinatorial proofs of students in two classes: a graduate level discrete mathematics course and an undergraduate/graduate level combinatorics class at a large public university. Because of the structure of the graduate program at this university, there were some graduate students who were simultaneously enrolled in both of these classes. The discrete mathematics course had 27 students, and the combinatorics class had 38 students, for a total of 55 distinct students. The students' written work was analyzed using discourse analysis (Cameron, 2001; Cruse, 2004; Gee, 2005).

We examined the combinatorial proofs written by students in response to questions on a midterm exam and a final exam. Both classes asked students to prove the following identity on the midterm exam:

$$\sum_{k\geq 0}\binom{n}{k}\binom{k}{m} = \binom{n}{m}2^{n-m}.$$

The identity above was presented on the combinatorics class midterm exam; the discrete mathematics exam used the slightly different form

$$\sum_{k=m}^{n}\binom{n}{k}\binom{k}{m} = \binom{n}{m}2^{n-m}.$$

On the final exam, both classes asked students to prove the identity

$$\binom{3n}{3} = 3\binom{n}{3} + 6n\binom{n}{2} + n^3.$$

By examining the students' written responses to these tasks, we identified students' successes and difficulties with writing such proofs. In both classes, the instructors encouraged students to write their combinatorial proofs by asking a "how many?" question that could be answered by both the left and right hand sides of the identity. This advice was not explicitly followed by all of the students; however, some students may have implicitly used this advice to think of a question that could be answered by both sides of the identity, but did not include this question as part of their written proof.

Quantitative Results

In order to analyze the overall success of the students in writing combinatorial proof, we gave each proof a score from 1 (for very unsuccessful proofs) to 4 (for very successful proofs). We also recorded whether or not each proof explicitly included a question that could be answered by both sides of the identity. The results of this quantitative analysis is given in Table . Students were able to successfully write proofs whether or not they explicitly wrote a questions, but were more likely to be successful if they did write a question, especially on the midterm exam.

Case Studies

Our main analysis of students' written proofs takes the form of case studies. Five students' written proofs were chosen for study: Daisy, Arlene, Russell, Ishmael, and Kathy.

Daisy

Daisy was enrolled in both the combinatorics and the discrete mathematics course, and was very successful in both courses. She was very successful in writing combinatorial proofs.

Below is her proof written in response to the midterm in the combinatorics class:

$$\sum_{k\geq 0}\binom{n}{k}\binom{k}{m} = \binom{n}{m}2^{n-m}$$

How many ways can we choose a committee of any size from a group of n people where the committee has a subcommittee of m people?

LHS: First we'll choose each committee of size k from n people. In $\binom{n}{k}$ ways. Then from these k people in the committee, we'll choose the subcommittee of m people in $\binom{k}{m}$ ways. Since we'll do this for committees of all sizes, from 0 to n, we first apply the mult. principle for $\binom{n}{k}\binom{k}{m}$ ways to choose the committee of size k, then apply the addition principle to sum the disjoint possibilities for each different size. Thus is $\sum_{k\geq 0}\binom{n}{k}\binom{k}{m}$.

RHS: First we'll choose the m subcommittee members from the entire group of n. This is done in $\binom{n}{m}$ ways. To fill the remaining spots for the committee of any size we survey the remaining $n-m$ people. Since each person will either be on or off the committee, there are 2^{n-m} possibilities. By the mult. princ. there are $\binom{n}{m}2^{n-m}$ ways to create the possible (work cut off).

The final sentence of Daisy's proof was cut off in our copy of her proof, but it is clear that Daisy has written a very successful proof. She has followed the advice of the instructors in writing an explicit question to help her organize her proof. Her written proof on the discrete mathematics midterm is almost identical, as is omitted here for the sake of brevity.

Daisy was similarly successful in her proofs on the final exams. Her response to the combinatorics final is as follows:

$$\binom{3n}{3} = 3\binom{n}{3} + 6n\binom{n}{2} + n^3$$

Q: How many ways can we select 3 students from a student body of $3n$, where n are singers, n are dancers, and n are musicians?

LHS: From the total group of $3n$, choose 3 in $\binom{3n}{3}$ ways.

RHS: Case 1: Choose all 3 students from 1 group. We can choose the group in $\binom{3}{1}$ ways, then the students in $\binom{n}{3}$ ways. By the mult. principle, there are $3\binom{3n}{3}$ ways to do this.

Case 2: Choose 2 groups, then 1 student from the first group and 2 students from the second group. Since order matters, there are 6 ways to choose a ranked pair of groups and $\binom{n}{1}$ and $\binom{n}{2}$ ways to choose the students from those groups. Thus Case 2 is covered in $6(n)\binom{n}{2}$ ways.

Case 3: Choose 1 student from each group. by the mult. priniciple this is done in $\binom{n}{1}\binom{n}{1}\binom{n}{1} = n^3$ ways.

Since the cases are disjoint, by the addition principle there are $3\binom{n}{3} + 6n\binom{n}{2} + n^3$ ways to do this. As the LHS and RHS count the same thing, they are equal.

Her response to the final exam in the discrete mathematics course is, again, very similar. However, she does not write an explicit question as part of her proof, and she uses a different example of a set to make her counting arguments. The argument itself, however, is almost identical to her earlier proof. Again, this proof is omitted for brevity.

Daisy's written proofs provide a good example of highly successful proofs. In the remaining case studies, we will, in part, be comparing written proofs to this standard.

Arlene

Arlene was a graduate student who was only enrolled in the combinatorics class. Arlene struggled with writing combinatorial proofs throughout the semester.

$$\sum_{k\geq 0}\binom{n}{k}\binom{k}{m} = \binom{n}{m}2^{n-m}$$

LHS: On this side we are adding the number of ways we can select people to do k jobs. So of the n people we can select k of them that are eligible to do the job and then m of them that actually complete the job. Each person can do more then one job (that's why they're not taken out of the set).

RHS: On the right hand side we are choosing m people out of n to complete the job and 2^{n-m} is the number of ways we can have them do the job. (2^{n-m} is the number of subsets the empty set is the case where $k = 0$)

Arlene did not write a "how many?" question to be answered on either her midterm or final exam. The choosing people to do jobs context and the idea of having 2 to a power represent the number of ways to have people on or off of a committee (number of subsets of a given set) are common ideas in combinatorial proofs. Arlene does appear to understand the notation $\binom{n}{k}$ as being the action of choosing a group of size k from a group of size n, but she does not appear to recognize that the objects she is choosing are the same as in the original set of n objects. Her last line about the number of subsets is particularly puzzling as she states "the empty set is the case where $k = 0$."

For the final exam, Arlene showed no improvement in her understanding of how to write a combinatorial proof. As we see below, her proof appears to be based on trying to write sentences that look and sound right with the correct numbers/variables in the correct places.

$$\binom{3n}{3} = 3\binom{n}{3} + 6n\binom{n}{2} + n^3$$

LHS: Of the $3n$ jobs we are choosing 3 people to do them.

RHS: This is the number of ways we could have 3 people do the first n jobs in $3\binom{n}{3}$ ways, we could have 2 people do the second n jobs in $6n\binom{n}{2}$ ways, and we can have one person do the 3rd set of n jobs in $n^2\binom{n}{1}$ or n^3 ways.

Observe that Arlene has again chosen the job context to try to write her proof. Her sentences are very short and give little detail about how she is counting. For the left hand side of the identity, she again interprets the choose notation using a mix of jobs and people to do them. On the right hand side of the proof, she appears to be making three statements, one for each set of symbols that has been presented to her, but she has given no indication of why the addition of the three sets of symbols is appropriate. It is our belief that Arlene has a very procedural approach to mathematics and proof writing that inhibits her ability to creatively think up an appropriate context for her combinatorial proofs.

Russell

Russell was enrolled in the combinatorics class only. His written proof on the midterm exam is highly problematic, and although he does show some improvement on the final exam, there is evidence that he never is really able to move beyond trying to make his argument "sound good." Below is his written proof on the midterm exam:

$$\sum_{k\geq 0}\binom{n}{k}\binom{k}{m} = \binom{n}{m}2^{n-m}$$

RHS Of the total of the n population we want to make a committee of m people.

Then from the $n - m$ people a either in the committee or not in the committee.

By multiplication principle, this will tell us the total of m people how many are in m committee.

LHS From the n total population we want to choose a committee of k people. Then from that k group we want to choose who will be in the m committee.

As $k \geq 0$ to n. This will determine the total amount of people that will be in m for each different k.

By the addition principle, the summation of all the k will tells us of the n population how many k people are in the committee m.

Thus, since RHS is the same as LHS, they are the same.

Russell's proof has several problems. He appears to be thinking *procedurally*, rather than *conceptually*. In his combinatorial reasoning, he appears to have grasped the idea that a power of two should be explained by choosing people to be on or off a committee, but he does not have a clear idea of how this should be integrated into his argument. In fact, it is not clear that he attempts to integrate it at all, since he has already chosen a committee of m people. He also invokes the addition principle, for no apparent reason other than that a summation appears in the identity. Finally, Russell does not seem to understand that the two combinatorial arguments should count the same set; one argument counts committees of m people, the other counts committees of k people. Thus, it appears that Russell has a fundamental misunderstanding of the logic of combinatorial proofs.

By the end of the semester, Russell shows some improvement in writing combinatorial proofs. His response to the final exam is below:

$$\binom{3n}{3} = 3\binom{n}{3} + 6n\binom{n}{2} + n^3$$

Left Hand Side: $\binom{3n}{3}$

We have 3 groups of n people. We want to take 3 of them to form a committee. This tells us the way to get 3 people from a population of size of $3n$ people.

Right Hand Side: $3\binom{n}{3} + 6n\binom{n}{2} + n^3$

$3\binom{n}{3}$: Counts the number of ways we can get a committee of 3 people of n population. Multiplying by 3 provides us the ways we can get a committee of 3 from 3 different groups of size n population for $n \geq 3$.

$6n\binom{n}{2}$: Counts the number of ways we can get a committee of 2 from n population.

We also write this as $\underbrace{2n\binom{n}{2}}_{n_1} + \underbrace{2n\binom{n}{2}}_{n_2} + \underbrace{2n\binom{n}{2}}_{n_3}$.

This subcommittee generates n_1, n_2, n_3 counting the different subcommittee making up from each n population 3 times when choosing a committee of 2 people from a population of n people. Multiplying by the $2n$ generates from the population of $2n$ people. So, $6n\binom{n}{2}$ is the possible combinations of 3 n groups to make a subcommittee of 2 people for $n \geq 2$.

n^3: Counts the population of all total n groups and there permutations they can have.

Thus, by the addition property,

$$3\binom{n}{3} + 6n\binom{n}{2} + n^3 = \binom{3n}{3}$$

Russell does not write an explicit question as part of his proof, but he does appear to be counting the number of possible committees of three people chosen from $3n$ people in two ways. Some of the phrases he uses as part of his proof are non-standard, and it can be difficult when reading his proof to determine exactly what his argument means. Toward the beginning of the "right hand side" section of his proof, he writes, "Multiplying by 3 provides us the ways we can get a committee of 3 from 3 different groups of size n population for $n \geq 3$." We believe that he is correctly invoking the multiplication principle, by first determining the number of ways to choose three people from a population of size n, and then determining the number of ways to choose one of the three groups of size n. However, this is not entirely clear from his written proof.

In the second section of the right hand side, Russell uses a similar phrase: "Multiplying by the $2n$ generates from the population of $2n$ people." His phrasing here is somewhat inscrutable; we believe that Russell may be trying to (correctly) communicate a choice of 2 people from a group of size n, and a choice of the remaining member of the committee from the remaining $2n$ people. However, this is far from clear, and it is certainly possible that Russell is, instead, simply trying to piece together likely sounding words to get his proof to "sound right," similar to Arlene. This interpretation is further borne out by his final case, in which he counts "permutations." In this final case, Russell is clearly lost, and this could be seen as evidence that Russell does not have clear idea of what he is counting in the earlier case, either.

Although Russell is not completely successful in his proof on the final exam, his proof writing does show improvement. In particular, he shows improvement in connecting the two sides of the identity by counting the same set in two ways.

Ishmael

Ishmael was enrolled in both the combinatorics and discrete mathematics courses. Ishmael, like Russell, has difficulty writing combinatorial proofs on the midterm exams. However, by the final exam, Ishmael shows a marked improvement.

Ishmael's combinatorics midterm:

$$\sum_{k\geq 0}\binom{n}{k}\binom{k}{m}=\binom{n}{m}2^{n-m}$$

Let S={0,1,2,...,n}

L.H.S: Choose one person (k) of the group:$\binom{n}{k}$

Then, from the k person we choose sp. person m: $\binom{k}{m}$

By the mult. princ, we have $\binom{n}{k}\cdot\binom{k}{m}$, since we have more than one set. Thus

$\binom{n}{0}\binom{0}{m}+\binom{n}{1}\binom{1}{m}+\binom{n}{2}\binom{2}{m}+\cdots+\binom{n}{k}\binom{k}{m}=\sum_{k\geq 0}\binom{n}{k}\binom{k}{m}$

R.H.S: From the s group we choose a sp. person m $\binom{n}{m}$, and from the committee we ask the sp. person if he in or off committee: 2^{m-n}

By Mult. princ, $2^{m-n}\binom{n}{m}$

Ishmael's discrete mathematics midterm:

$$\sum_{k=m}^{n}\binom{n}{k}\binom{k}{m}=\binom{n}{m}2^{n-m}$$

Let S be a set with n as a committee w k leaders and m special group.

R.H.S: Choose the special group from a committee in $\binom{n}{m}$ and the remaining members $n - m$ from the committee by ask each one if they either *(illegible)* in or off the committee in 2^{n-m} ways.

By the mult princ. we have $\binom{n}{m} \cdot 2^{n-m}$.

L.H.S: $\binom{n}{k}$: Let us choose k leaders from the committee and from the leaders we want to choose m special group in $\binom{k}{m} = \binom{n}{k} \cdot \binom{k}{m}$

By the Addition princ., we have $\sum_{k=m}^{n} \binom{n}{k}\binom{k}{m}$

Since we count the same idea in both sides

$\sum_{k=m}^{n} \binom{n}{k}\binom{k}{m} = \binom{n}{m}2^{n-m}$

Ishmael's midterm proofs show a great amount of difficulty. In the combinatorics midterm, Ishmael struggles even with the meaning of the choose function, as evidenced by his use of the singular noun "person" throughout. His discrete mathematics midterm, written one week later, shows some improvement, but like Russell, Ishmael does not seem to have any clear idea why the addition principle should be invoked, or why the two combinatorial arguments count the same set.

However, by the final exams, Ishmael shows improvement. His proof on the combinatorics final exam is below:

$$\binom{3n}{3} = 3\binom{n}{3} + 6n\binom{n}{2} + n^3$$

$S = \{1, 2, 3, \ldots, n, \ldots, 3n\}$ Let S be a set with n people (n boys, n girls, n teachers)

L.H.S: From the total set S choose 3 people in $\binom{3n}{3}$ ways.

R.H.S: Case 1: Choose a subset of n boys with size 1 $\binom{n}{1}$ and choose another subset of n girls with size 1 $\binom{n}{1}$ and choose another subset of n teachers with size 1.

By the mult. princ. we get: $\binom{n}{1}\binom{n}{1}\binom{n}{1} = n^3$

Case 2: Choose a subset of n boys with size 3: $\binom{n}{3}$ or choose a subset of n girls with size 3: $\binom{n}{3}$ or choose a subset of n teachers with size 3: $\binom{n}{3}$.

By the addition princ. we get: $\binom{n}{3} + \binom{n}{3} + \binom{n}{3} = 3\binom{n}{3}$

Case 3: Choose 6 subset of n boys with size 2: $\binom{n}{2}$

$6\binom{n}{2}$ and choose spe. person from the set in $\binom{n}{1}$ ways.

By the mult. princ. we get $6n\binom{n}{2}$

Consider all cases: case 1 + case 2 + case 3 $= 3\binom{n}{3} + 6n\binom{n}{2} + n^3$

$\therefore$ The LHS counts the same as the RHS.

Ishmael's discrete mathematics final exam:

$$\binom{3n}{3} = 3\binom{n}{3} + 6n\binom{n}{2} + n^3$$

Let S be a set with $3n$ $\{1, 2, 3, \ldots, n, \ldots, 3n\}$ where n girls, n boys, and n teachers.

L.H.S: $\binom{3n}{3}$ Choose a subset of size 3 from the total set $(3n)$.

R.H.S: Case 1: $\binom{n}{1}$: Choose a subset of size 1 from n girls,

$\binom{n}{1}$: Choose a subset of size 1 from n boys and

$\binom{n}{1}$: Choose a subset of size 1 from n teachers.

By the Mult. princ. we get $\binom{n}{1}\binom{n}{1}\binom{n}{1} = n^3$

Case 2: $\binom{n}{3}$: Choose a subset of size 3 from n girls or

$\binom{n}{3}$: Choose a subset of size 3 from n boys or

$\binom{n}{3}$: Choose a subset of size 3 from n teachers.

By the addition princ. we get $\binom{n}{3} + \binom{n}{3} + \binom{n}{3} = 3\binom{n}{3}$

Case 3: $2n\binom{n}{2}$: Choose 2 subset of size 2 from n girls with a sp. girl or

$2n\binom{n}{2}$: Choose 2 subset of size 2 from n boys with a sp. boy or

$2n\binom{n}{2}$: Choose 2 subset of size 2 from n teacher with a sp. teacher.

By add. prin. we get $2n\binom{n}{2} + 2n\binom{n}{2} + 2n\binom{n}{2} = 6n\binom{n}{2}$

Consider all cases: case 1 + case 2 + case 3 $= 3\binom{n}{3} + 6n\binom{n}{2} + n^3$

Thus The RHS = The LHS

Ishmael's final exam proofs are much improved; he is clearly counting the same sets in two ways in both of these proofs, with the exception of the final "Case 3" in each proof. In this third case, Ishmael is unable to see any reason for the multiple of n in the identity other than picking a "special person," which does not appear in the argument for the left hand side of the identity. However, even with this error, Ishmael's argument on the discrete mathematics exam, written two days after the combinatorics exam, is somewhat closer to a correct argument.

We find it interesting to note that Ishmael's handwriting is very poor in his written proofs on the midterm exams, but his handwriting on the final exams was much clearer. Although we are not trained in handwriting analysis, it appears that Ishmael's poor handwriting on the midterms may indicate a lack of confidence, while the much stronger handwriting on the final exams shows that his confidence has increased.

Kathy

Kathy was enrolled in both the combinatorics and discrete classes. Her proof as written on the combinatorics midterm is as follows:

$$\sum_{k\geq 0}\binom{n}{k}^{\circledast}\binom{k}{m}^{\oplus}=\binom{n}{m}^{\star}2^{n-m\,\text{☺}}$$

Let S be a set of n committees.

LHS: We have a committee of n where we want to pick k special people $^{\circledast}$. Out of the k special people we want to pick m VIP people $^{\oplus}$. The sum is determines how many k people are picked. The lowest # of k people that can be picked is 0 and the highest number of k people that can be picked is n. The sum determines every possibility of picking k up to n, which is including everyone from the n committee.

RHS: From the n committee, we pick m VIP people*. The number of people not picked for the VIP people $(n-m)$ have a chance to be in or out of the committee, which comes from 2^{n-m}. ☺

In the proof above, Kathy has not written a question, but she has indicated with symbols (such as ☺) how each statement she wrote corresponds to some symbol in the given identity. She has the big idea of the proof that she needs to be looking for a subgroup of a subgroup.

On Kathy's second midterm below, we see that she is consistent in the context (committees) she chooses. A big difference between the proof she wrote for the discrete midterm a week later is that she attempts to write a "how many?" question. In fact, she writes two questions - apparently one for each side of the identity she has been asked to prove.

> A question: How many ways to form a committee w/n people where k people are picked to form a subcommittee, and from the k subcommittee to pick m special people where k can be up to m people.
>
> Question: How many ways are there to pick from an n committee people picking m people for a subgroup.
>
> LHS: From n people, we want to form a committee of k special people, and from these k special people, we want to pick m super special people. $\binom{n}{k}\binom{k}{m}$ The possible way to do this is to pick one special k person at a time until you get m super special people which calls for the sum until you've placed all n people into the groups, which is $\sum_{k=m}^{n}\binom{n}{k}\binom{k}{m}$.
>
> RHS: From n people, we can pick m super special people which is $\binom{n}{m}$. After picking m super special people from n people, the remaining people can be in or out of the group, which is 2^{n-m}, so we use the mult. principle to get $\binom{n}{m}2^{n-}$.

In each of Kathy's midterms, she struggles to articulate why the addition principle is necessary.

For Kathy's combinatorics final exam problem below, we observe her still using the committee context to frame her proof.

$$\binom{3n}{3} = 3\binom{n}{3} + 6n\binom{n}{2} + n^3$$

> LHS: We have a <u>committee</u> of n people.
>
> We have a <u>group</u> of 3 committees, so we have in a group: $3n$ people.
>
> Out of the group we want to select 3 leaders. There are $\binom{3n}{3}$ ways to do this.

RHS: Case 1: From a committee of n people, we can pick our 3 leaders in $\binom{3n}{3}$ ways. But since there's 3 groups of n committees there must be 3 ways to pick, so there is $3 \cdot \binom{3n}{3}$ ways to pick.

Case 2: From a committee of n people, we can pick our 3 leaders and from a different committee we can pick our 1 leader, this is done in $\binom{n}{2}\binom{n}{1}$ ways. WE can switch the pick of 2 and 1 leaders from the committee of n people so we multiply by 2. We multiply by 3 to have 3 different ways to pick 2 & 1 leaders. So we have $2 \cdot 3 \cdot \binom{n}{2}\binom{n}{1}$ ways.

Case 3: From a committee of n people, we pick one leader from each committee in the group. Since the group we have 3 committees of n people, we can do this in $\binom{n}{1}\binom{n}{1}\binom{n}{1}$ ways.

By the addition principle we add all the cases, so we have
$3 \cdot \binom{3n}{3} + 2 \cdot 3 \cdot \binom{n}{2}\binom{n}{1} + \binom{n}{1}\binom{n}{1}\binom{n}{1} = 3\binom{n}{3} + 6n\binom{n}{2} + n^3$

Finally, on Kathy's second final we notice that she is again consistent in her choice of context (committees) and how she proceeds with the proof.

$$\binom{3n}{3} = 3\binom{n}{3} + 6n\binom{n}{2} + n^3$$

LHS: We have a group of n people, and a committee of $3n$ people, meaning there are three groups of n people. The group of n people are all the same. We want to pick 3 leaders from the committee of $3n$ people, so there are $\binom{3n}{3}$ ways to do this.

RHS: Case 1) From one group of n people we can pick 3 leaders. There are $\binom{n}{3}$ ways to do this. But since there are 3 groups of n people, then we multiply by 3 to have the number of ways to pick 3 people from a group of n people: $3\binom{n}{3}$.

Case 2) From one group of n people we can pick 1 leader and from another group of n people we can pick 2 leaders. This is done in $\binom{n}{1}\binom{n}{2}$ ways. But since we have 3 groups of n people, we need to permute the picks among the 3 groups, so there are $3!\binom{n}{1}\binom{n}{2}$ ways to do so.

Case 3) From each group of n people we can pick one leader, and since there are 3 groups we account for each group so we have $\binom{n}{1}\binom{n}{1}\binom{n}{1}$ ways to do this. Since there is only one way to pick 1 leader from each of the 3 groups, we multiply by 1. so we have $1 \cdot \binom{n}{1}\binom{n}{1}\binom{n}{1}$.

By the addition principle, we add up all the cases. So we have $3\binom{3n}{3} + 3!\binom{n}{1}\binom{n}{2} + 1 \cdot \binom{n}{1}\binom{n}{1}\binom{n}{1} = 3\binom{n}{3} + 6n\binom{n}{2} + n^3$. Since the LHS=RHS, then the statement holds true.

There definitely appears to be growth in Kathy's understanding of combinatorial arguments between the midterm and final. On her final, she appears to have the big ideas about how the proof should go figured out in her head. However, she still appears to be struggling with how to articulate those ideas.

Conclusions

Our data indicate that when students explicitly write out the question they are trying to answer, they are more successful in completing a combinatorial proof. When students were unsuccessful, the errors made by the students tended to fall into four broad (and overlapping) categories:

1. Language mimicking
2. Latching on to one idea that may not work for every situation
3. Misunderstanding the choose function
4. Failing to indicate how each side counts the same thing

Language mimicking was particularly evident in Arlene, Russell and Ishmael's proofs. Each of these students had clear ideas about the kind of language that had been used to prove similar statements in class and in homework. However, when trying to construct an appropriate justification for the given identity, these students tended to try to replicate that language without a clear understanding of that language. Therefore, these students were not successful in writing proofs on the midterm. By the time of the final, Russell and Ishmael showed improvement, but

Arlene never managed to construct a combinatorial proof on her own.

Arlene's proofs provided the strongest evidence for how a student may latch onto a context and try to use it for every possible combinatorial argument. She had clearly seen a counting argument that involved assigning people to jobs and tried to make that context work for every identity she was asked to prove. Contrast this with Daisy's arguments: Daisy saw each identity to be proved as an opportunity to "write your own word problem" which allowed her to think flexibly about what the identity could represent. From what we observed, the students who relied on language mimicking for their proof production strategy would likely be classified as using a syntactic proof production by Weber and Alcock's (2004) definition whereas Daisy's strategy would be classified as a semantic proof production. We believe that exploring this framework further in regards to combinatorial proof may have significant implications for our work.

An understanding of combinatorial language is critical to constructing combinatorial arguments. When describing what the notation $\binom{n}{k}$ represents, it is imperative that it is understood that one is choosing a group of size k from a group of size n. In Ishmael's work in particular, we saw evidence that he was thinking about choosing a particular person with some kind of designation, "k", rather than a group of size k. Arlene used language that indicates she has seen a combinatorial proof before, but nothing that suggests she has an understanding of what she is counting.

In many cases, particularly when a question had not been written, the students frequently concluded the proof with a statement such as "Since the LHS=RHS, they count the same thing." This statement belies a logical flaw; the two sides of the identity are equal *because* we count the same set in two ways, not the other way around. Even so, from what they had written it was not clear at all to the reader that they had in fact counted the same thing in two distinct ways. For example, aside from Russell referencing committees in the argument for each side of his midterm exam proof, there was no obvious way to see how, or even if, he was counting the same thing.

By identifying some of the common errors made by students in writing combinatorial proofs, we hope to be able to guide students to write better proofs. The technique of writing a question to be answered by both sides of the identity seems to have some value in helping students in avoiding the fourth type of error. Further work in this area will help to develop other techniques to help students to be more successful in writing combinatorial proofs.

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Table 1

Student proofs rated by level of success and written question.

Midterm:	4	3	2	1
Question written	13	4	1	1
Question not written	11	9	11	12

Final:	4	3	2	1
Question written	13	1	1	0
Question not written	16	12	9	5