Conceptualizing Multivariable Limits: From Paths to Neighborhoods

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There has been significant study into the question of how students transform mathematical processes into coherent objects. The limit concept plays a special role in these studies since it is among the first mathematical processes students encounter which is infinite in nature. In this paper I report on a study using task-based interviews to describe how students conceptualize multivariable limits. The results suggest that some students struggle to understand multivariable limits due to weaknesses in conceptualizing infinite processes. I look closely at the process utilized by students to understand multivariable limits and how the infinite nature of this process created an obstacle to further understanding. I also describe how overcoming this obstacle created a fundamental change in the way several students view the concept of limit.

Introduction

It is well accepted that the limit concept plays a foundational role in present-day undergraduate education. Its role in calculus is so critical that James Stewart (2005), in his popular textbook, defines calculus as "the part of mathematics that deals with limits" (p. 10). At the same time, there is widespread agreement among both educators and researchers that most students struggle to develop a solid understanding of the limit concept (e.g., Vinner, 1991). This is due in part to the conceptual complexity of the concept. Tall (1992), in reference to Cornu (1983), observed that "this is the first mathematical concept that students meet where one does not find the result by a straightforward mathematical computation. Instead it is 'surrounded with mystery,' in which 'one must arrive at one's destination by a circuitous route'" (Tall, 1992, p.501).

The study described in this paper develops a description of how students may understand the concept of limit in multivariable calculus. This description stems from an infinite process utilized by several of the students in the study in order to reason about multivariable limits. I argue that the infinite nature of this process creates an obstacle for some students' understanding of multivariable limits, and that, for those students described in this study, overcoming this obstacle fundamentally changed the way they viewed the limit concept.

Related Literature

Over the past twenty-five years, there have been many studies on how students understand the limit concept. They vary in many ways. Some focus on limits as understood in an introductory calculus class (e.g., Williams, 1991) while others focus on limits of sequences and series (e.g., Alcock and Simpson, 2004). Some attempt to characterize common misconceptions (e.g., Davis and Vinner, 1986) while others attempt to describe how students come to understand the concept (for example, Cottrill, et al., 1995). However, there is currently very little research describing the concept of limit in multivariable calculus. In this paper, I begin the process of creating such a description of how students may reason about multivariable limits and how reasoning about multivariable limits may impact their understanding of the concept of limit.

Theoretical Perspectives

This study was conducted from two primary theoretical perspectives: The use of reflective abstraction to internalize mathematical processes, and the use of conceptual metaphors to construct meaning when encountering new phenomena.

Mathematical Processes

Much of the current discussion of mathematical processes has as its origin Piaget's work on reflective abstraction (see, for example, Piaget 1985). Piaget characterizes reflective abstraction as an entirely internal process, as opposed to empirical abstraction or pseudo empirical abstraction, which develop from properties of objects and actions on those objects, respectively. Piaget considered four different kinds of mental constructions that could take place during reflective abstraction: *interiorization*, or the construction of an internal process in order to represent a perceived phenomena; *coordination* of two or more processes into a single new one; *encapsulation* of a dynamic process into a static object; and *generalization*, or the application of existing knowledge to new phenomena. Even though Piaget's work was used to describe the cognitive development of logico-mathematical structures in children, several researchers have meaningfully adapted his work to develop theories of learning relevant to undergraduate mathematics. I will describe two such theories here: Dubinsky and his colleagues' development of APOS theory (1991), and Sfard's description of reification (1991).

Dubinsky (1991) developed a theory of conceptual development based on the creation of actions, processes, objects, and schemas. In this model of student learning, the student first understands a mathematical concept as an *action* to be performed. After some experience with the action, the student is able to perceive it as a *process* and speak of its result without being required to perform the action. Eventually this process may be encapsulated into an *object* which can in turn be used to create more sophisticated mathematical actions. The student then gathers these related actions, processes and objects into a coherent collection called a *schema*.

Sfard's (1991) description of the development of mathematical processes takes place in three steps. She describes the adoption of a familiar process as the *interiorization* of that process. Once interiorized, the process can be compacted and understood as a whole. Sfard refers to this development as the *condensation* of the process. To Sfard, a condensed process is still operational and the individual will interact with the process in an operational manner. It is the process of *reification* that transitions the individual from dealing with an operational process to a structural object.

It is important to notice the relationship between Dubinsky's and Sfard's two descriptions. They each describe a three-step process which begins with the adoption of a mathematical procedure – an action according to Dubinsky and an interiorized process according to Sfard – and ends with the development of a mathematical object which may be used to develop further mathematical procedures. However, the two descriptions emphasize different elements during the middle stage. Dubinsky and his colleagues emphasize the ability to envision the *result* of the process without carrying out the individual steps, while Sfard emphasizes the *oneness* of the process and the ability to speak of the process as a whole. In my study, the oneness of the process that students experience plays an important role in the students' development of multivariable limits. For this reason, this paper will adopt the language developed by Sfard when describing student reasoning about mathematical processes.

Conceptual Metaphors

My study was conducted from the perspective of conceptual metaphors in the sense of Lakoff and Nunez (2000). From this perspective, our understanding of mathematics is considered to be a result of our embodied experiences, and this understanding is built upon conceptual metaphors which project meaning onto new, abstract domains from previous experiences.

In the past ten years there have been several studies that indicate that this perspective can be used to effectively describe mathematical concepts in general, and the concept of limit in particular (Oehrtman, 2009). Of particular interest to my study is the work by Nunez (1999) which describes the metaphors students may use to reason about the concepts of limit and continuity. In his work, Nunez describes two inherently different metaphors for the concepts of continuity: natural continuity and Cauchy-Weierstrauss continuity. According to Nunez, natural continuity arises from the metaphor that "a line IS the motion of a traveler tracing that line" (p. 56); whereas, Cauchy-Weierstrauss continuity is built upon three conceptual metaphors: "A line IS a set of points; Natural continuity IS gaplessness; Approaching a limit IS preservation of closeness near a point" (p. 57).

Methods

The data for this study was collected through a series of task-based interviews with seven undergraduate students. These seven students were each enrolled in a traditional, lecture-based multivariable calculus course. Each student took part in three fifty-minute interviews outside of class. The First interview gathered background information about how the students understand the concept of limit in single variable calculus. The second and third interview sessions gathered information about how the students understand the concept of limit in different multivariable settings. The activities involved in the second and third sessions are described below:

- Session 2, Part 1: Students were given several multivariable functions represented symbolically and asked to determine whether or not the limits exist at (0,0).
- Session 2, Part 2: Students were given several multivariable functions represented graphically using the computer program Maple and asked to determine whether or not the limits exist at (0,0).
- Session 3, Part 1: Students were given several multivariable functions represented symbolically and were asked to convert these functions using polar coordinates. Upon converting the functions, students were asked to determine whether or not the limits exist at (0,0).
- Session 3, Part 2: Students were given several multivariable functions represented with contour graphs and were asked to determine whether or not the limits exist at (0,0).

During the study, the interviewer took on two primary roles. The first role was as a *facilitator*, administering the tasks to be encountered by the students. The students were not expected to be familiar with the tasks, yet the role of the interviewer was simply to introduce and observe the task and not teach new concepts to the participants. The second role of the interviewer was as a *questioner*, probing the thoughts of the students. The interview sessions were based on a loose outline for questioning which gave the interviewer the opportunity to explore in more detail student behaviors that were unclear or unexpected.

All interviews were videotaped and the conversations were transcribed for analysis. During the analysis, student descriptions were broadly classified according to the primary metaphors used by the students. In particular, student responses were classified into the following three categories: metaphors based on topographical features of the function, metaphors based on dynamic motion, and metaphors based on proximity or closeness. The metaphors based on dynamic motion and closeness will play an important role in the description of student activity presented in this paper. These two metaphors are described below and share a close resemblance to Nunez's (1999) description of natural and Cauchy-Weirstrauss continuity:

Dynamic Motion: In this metaphor, the limit is the result of motion along a path.

Closeness: In this metaphor, the limit is the preservation of closeness near a point.

Results

The primary result discussed in this paper was observed in three of the interview participants. These three students all described an interiorized infinite process by which they understood the concept of multivariable limits. For each student, the infinite nature of this process created a cognitive obstacle, and overcoming this obstacle resulted in a fundamental shift in the language used to describe the limit concept. The experience of one of these students, Josh, will be described in detail in order to demonstrate the shift in thinking experienced by these three students.

Josh's multivariable limit process

At the beginning of the second interview session, Josh was asked to describe the multivariable limit concept as he understood it from the lecture.

Josh: “*When we were doing one variable limits what we were doing was, ‘do we approach from the left’ or ‘do we approach from the right’ and we compared those two, whereas now, we can approach from an infinite amount of directions.”*

It is not immediately clear from this statement which metaphors Josh possesses to conceptualize the multivariable limit concept; however, it is important to note that, to Josh, finding a multivariable limit is equivalent to applying the process of evaluating a single variable limit along different “directions.” It is important to notice that Josh is very aware of the infinite nature of his multivariable limit process and that, as we will see later, he becomes aware of the obstacle that this creates.

Throughout this portion of the interview, Josh consistently uses the same process to evaluate multivariable limits. His process can be summarized as follows:

1. Select a path that passes through the limit point, (a, b).
2. Algebraically substitute that path into the multivariable function to create a function of a single variable.
3. Evaluate the limit of the resulting single variable function as (x, y) approaches (a, b). This will be a single variable limit in either x or y.
4. Repeat this process selecting different paths until two paths are found with different single variable limits.

Step four in this process is problematic for two reasons. First, there is no clear ordering of the paths, making it difficult to determine which path to select for the next iteration of the process. Second, there is no means by which this process can be ended should the limit exist.

Josh soon explored a function whose limit does exist and noticed that the first few iterations of the process all yielded a result of zero. Josh commented:

> Josh: *"I kind of found myself in the same boat again that I appear to be getting the same number… I have to go through the same steps again to say, 'well is there something I can plug in here that I'll get a number different than zero?' So… I'm just going to keep throwing darts at the board and see if I hit anything,"*

At this point Josh becomes aware that he has no systematic method of choosing paths to "plug in" and evaluate the limit. As he tries a few more paths he begins to believe that every path will yield the same result. He believe that the limit exists, but his process does not allow for him to show that this is true:

> Josh: *"I'm taking different things and trying them and I'm ending up with the same answer… I need to show that this limit exists, but this is where I kind of fall off."*

As this portion of the limit ended, Josh became aware of the fact that his process of evaluating multivariable limits was not useful for analyzing limits that do exist; however, he possessed no means of constructing such an alternate process of approaching the problem.

Josh reinterprets his multivariable process in the context of three dimensional graphs

When Josh moves on to the three-dimensional graphing portion of the interview, he encounters the surface shown in Figure 1.

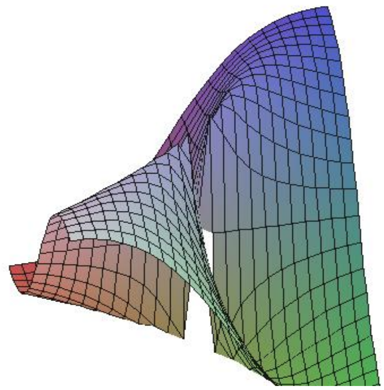

Figure 1

> Josh: *"I'm kind of new to the whole looking at 3D functions and determining visually, but I kind of take that same approach of 'I'm going to look at it from different ways and see if I approach the same point.'… I'll try to look at it from a different perspective. If I approach from this direction, I'll get a point… and if I look at it from the other side, they appear to be different points."*

In this statement we see that Josh goes back to his four-step multivariable limit process when encountering multivariable limits in a new environment. This indicates that Josh has interiorized this process and is using this process to define the multivariable limit concept. At this point, he still describes the process using one path at a time; however, when he encounters a limit that does exist (figure 2), he begins attempting to visualize the process in its entirety by visualizing all possible paths simultaneously.

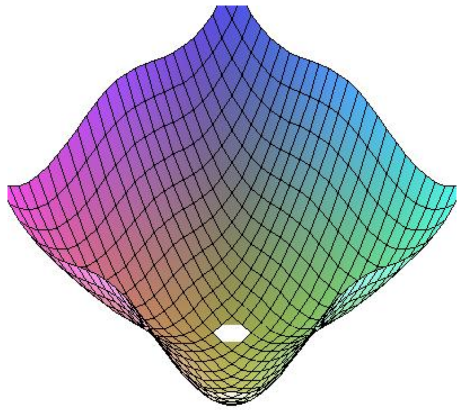

Figure 2

Josh: *"If I take one approach it appears that I am coming to this point here, if I take the left approach, it's the same thing. I'm kind of approaching this same value here in the middle. If I look at it all around, it appears to be coming to this single point…"*

At this point we see Josh beginning to condense this infinite process into a single process. At the same time, we see Josh's language begin to change. Josh has transitioned from using "different approaches" to looking "all around" the limit point. In the prior case, Josh tended to use a metaphor based on motion projected onto the shape of the surface; whereas, his new language uses a metaphor of closeness to describe the shape of the surface near the limit point.

Josh reinterprets his multivariable process in the context of contour graphs

In the following interview session, Josh encountered multivariable limits using contour graphing. He began this session as he began the session emphasizing multivariable graphing, by analyzing the graph using "different approaches," and analyzing these "approaches" one at a time. Figure 3 shows Josh's markings on the contour graph.

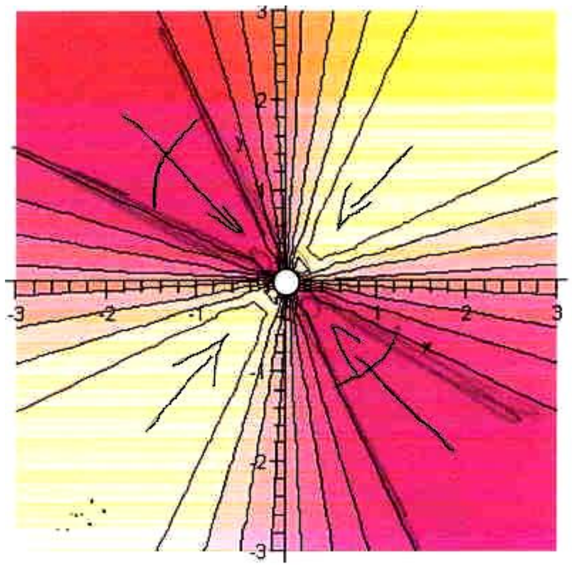

Figure 3

> Josh: *"If I approach from this diagonal here, I appear to be getting close to this low point on the surface; whereas, if I approach from the right, the top right, and the bottom left diagonals, it appears to be at a higher point."*

Similar to the three-dimensional graphing experience, once Josh encountered the contour graph of a function whose limit exists, he experienced a shift in the language he used to describe the multivariable limit process. His new markings and accompanying statements are found below.

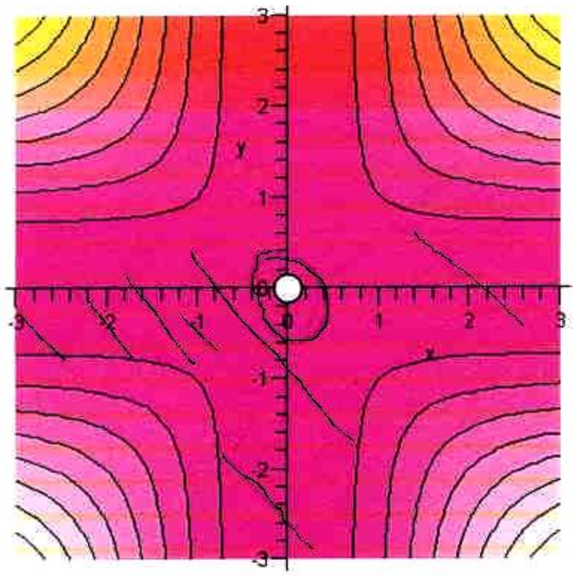

Figure 4

Josh: *"If I approach from any path, they're all going to converge to this central point…"*

…

Josh: *"So, I guess it would really be… 'is it the same color all the way around?'"*

At this point we see a major shift in the way Josh understood the multivariable limit concept. This shift occurred in three distinct ways. First, he condensed the infinite multivariable limit process by simultaneously visualizing all paths approaching the limit point. Second, the act of condensing this infinite process changed the focus of Josh's attention with regards to the contour graph. His language no longer emphasized motion along paths, but rather he began to emphasize the behavior of the graph near the limit point. In particular, he recognized that a limit which exists should have "the same color all the way around." In this way, the underlying metaphors which Josh is using shift from those involving motion imprinted on the shape of the graph to those involving the shape of the graph in a small neighborhood of the limit point.

Finally, this shift is reflected by the markings which Josh makes on the contour graph. Those markings no longer include arrows indicating motion towards the limit point; rather, they include a circled neighborhood of the limit point and shaded regions of the same color.

The interviewer asked Josh about this shift in his language and his use of color to analyze the multivariable limit.

> Josh: *"If all around the point that I'm taking the limit at is the same color, then it's going to be the same height at that point… it's the same color all around, so if I approach right, left, any approach whatsoever… it's going to be the same value."*

This indicates that the shift in thinking which Josh experienced is not a departure from his previous multivariable limit process, but rather the conclusion of it.

Conclusions

The primary result in this paper is that for several students, exemplified by Josh, the infinite nature inherent in their multivariable limit process created a cognitive obstacle when reasoning about multivariable limits which exist. I would argue that this cognitive obstacle is an epistemological obstacle in the sense of Cornu (1991) in that it arises from the nature of the concept of multivariable limit, itself. Furthermore, overcoming this obstacle resulted in a shift in the language used to reason about limits from metaphors emphasizing dynamic motion to metaphors emphasizing closeness. I argue that this shift in metaphors is a necessary precursor to formally defining the concept of limit, and as such there is great potential value in the use of multivariable limits as a transition from informal reasoning about the limit concept to formal reasoning.

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