Modeling Mathematical Behaviors;
Making Sense of Traditional Teachers of Advanced Mathematics Courses Pedagogical Moves

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Abstract: This paper considers a set of techniques that can be used to understand and improve the context of a traditional (i.e., lecture-based) math classroom, and which could help students develop higher order thinking skills. These techniques, which I call "Modeling Mathematical Behaviors" (MMBs), consist of the instructor demonstrating good mathematical habits by thinking out loud and questioning students during whole-class discussions. I draw mainly on observations of abstract algebra courses. I contend that a class taught using MMBs may, in part, dampen the standard criticisms of lecture-based teaching. It appears that mathematicians already incorporate MMBs into their teaching, and asking them to be more explicit about the fact that they are doing so, and explicitly describing to the students what behaviors they are modeling, would be far less onerous than expecting mathematicians to adopt an inquiry-oriented pedagogy.

Abstract algebra is considered an essential course in the undergraduate mathematics curriculum, yet research literature suggests that many students have difficulty meeting instructors' goals (Cuoco, Goldenberg, & Mark, 1996; Dreyfus, 1999; Dubinsky, et. al, 1994; Hart, 1986; Leron & Dubinsky, 1995; Leron, Hazzan, & Zazkis, 1995; Weber, 2001). Advisory reports issued by the National Science Foundation (1992) and the Mathematical Sciences Education Board (1991), drawing upon advances in our understanding of how students learn, have called upon faculty to move away from the lecture format and towards investigation-based class sessions. As a result, in the past few years there has been significant attention paid to classroom instruction and ways to improve student learning. This has led to the development of new pedagogical practices; in particular, there have been a number of projects focused on teaching abstract algebra to undergraduates (Leron and Dubinsky, 1995; Larsen, 2004; Weber, 2006). Most of these new projects draw upon the theoretical framework of Realistic Mathematics Education (Freudenthal, 1973) and have sought to create rich mathematical contexts in which students can explore and develop mathematical ideas and methods for themselves (Larsen 2002, 2004; Rasmussen and King, 2000). There is a concurrent cycle of research that guides further curriculum design and revision based on an analysis of the students' work. In abstract algebra, this research cycle has described how students might learn the concepts of group, subgroup, partition, and quotient group within the context of a particular instructional approach (Larsen 2002, 2004; Larsen, Johnson, Rutherford & Bartolo 2009).

Yet, asking or expecting all mathematics faculty to transition to this new type of instruction would require them to develop significant new knowledge and skills (Wagner, et al., 2007). They would need to know, at the least: the role of faculty in an inquiry-based course, how to determine appropriate learning goals and how to assess student progress towards those learning goals, and they would need to have an understanding of how students will develop the mathematical ideas, as well as the potential mistakes students will make. As Kline (1977) argued, there is no institutional incentive for most faculty to develop this new knowledge. Thus, most courses will continue to be taught in a traditional, lecture-based format.

Let me specify what I mean by traditional, lecture-based courses as compared with inquiry-oriented or reform-oriented classrooms. Rasmussen and Marrongelle (2006) described a scale of teaching that ranges along a continuum from "pure telling" to "pure investigation." They claimed that a well-designed inquiry-oriented course would fall close to the middle of this continuum and suggested that a lecture-based course would be closer to "pure telling." Similarly, McClain and Cobb (2001) characterized the spectrum of teaching as running from "non-interventionist" to "total responsibility." The class described in this study is one where the instructor had almost total responsibility for daily classroom activities and the mathematical content and was closer to "pure telling" than an inquiry-oriented class.

There are many critiques of traditional, lecture-based instruction. In particular, there are claims that students are responsible for recalling answers and prescribed procedures as opposed to the emphasis on problem solving, and recognizing and constructing arguments that students are responsible for in less traditional classes (Wood & Turner-Vorbeck, 2001). Davis gave a similar critique:

> But when I am done, most students will have acquired nothing but a collection of skills, and may soon even begin to forget or distort many of these. They will not have learned *why* things are done in a certain way, nor *what else* might be done instead, nor how one could *modify* these procedures, nor what the *cues* might be that would help you decide which procedures to use, and which to avoid. They surely will not have acquired much ability to *invent their own original algorithms* in order to deal with novel problems. Nor will they have learned how to *explore* a truly novel problem in order to get some notion of how it might be dealt with (Davis, 1994, p. 17). (italics in original).

William Thurston (1986) gave a rather cynical description of what such an undergraduate mathematics classroom might look like:

> …we go through the motions of saying for the record what we think the students "ought" to learn, while the students are trying to grapple with the more fundamental issues of learning our language and guessing at our mental models. Books compensate by giving samples of how to solve every type of homework problem. Professors compensate by giving homework and tests that are much easier than the material "covered" in the course, and then grading the homework and tests on a scale that requires little understanding. (p. 343)

In short, these critiques, and others, claim that students who learn within a lecture-based classroom are likely to have little understanding of what constitutes mathematical activity and see few if any connections between the various pieces of mathematics they have studied through the course of their schooling (Cuoco, 2001; Dreyfus, 1991; Kline, 1977).

Despite all of the effort put into developing inquiry-oriented curricula, reports and research studies, they are based primarily upon the perception of the personal experiences of the authors, based on what is happening in the lecture-based classes that they are teaching (Smith, Speer & Horvath, 2007), even in the face of repeated suggestions for study of teaching (Harel & Sowder, 2007; Harel & Fuller, 2009). The study reported here investigates a traditional, lecture-taught abstract algebra class from a researcher's perspective, rather than a first-hand account.

Pedagogical Content Tools

As one means of making sense of undergraduate teaching, and especially the teacher's pedagogical moves, I will draw upon the lens of *Pedagogical Content Tools* (PCT) as articulated by Rasmussen and Marrongelle (2006). The construct of a PCT was created to help researchers explain how teachers connect to student thinking while also moving the mathematical agenda forward. They suggested that PCTs could include graphs, diagrams, equations, gestures and even verbal statements that will help a teacher achieve a stated classroom goal.

Rasmussen and Marongelle (2006) theorized two broad categories of PCTs: *transformational records* and *generative alternatives*. Transformational records are defined as "notations, diagrams, or other graphical representations that are initially used to record student thinking and that are later used by students to solve new problems;" generative alternatives are defined as "alternate symbolic expressions or graphical representations that a teacher uses to foster particular social norms for explanation and that generate student justifications for the validity of these alternatives" (Rasmussen & Marrongelle, 2006, p. 389). Although Rasmussen and Marongelle were basing their work on inquiry-oriented classes, their model is flexible enough to offer insight into the teaching of more traditionally taught courses as well. However, we will also need a new tool.

Modeling Mathematical Behavior

In this paper, I introduce the construct of *modeling mathematical behavior* (MMB). MMB is an instructor action that is purposefully done in order to demonstrate particular mathematical behavior(s), such as making sense of a definition or example, or the process of proof-construction and active monitoring. MMBs are combinations of gestures, speech and possibly writing that serve to advance students' understanding and connect to students' ways of thinking; thus, an MMB could be considered a type of Pedagogical Content Tool. MMBs are one way for a lecture-based teacher to illustrate mathematical habits and thinking. The key is that the instructors are modeling appropriate mathematical habits that they know, based on their experience, which students are unlikely to develop or deploy without specific instruction. Instructors often model these behaviors as a means to help students develop good habits in mathematical thinking, and to help students avoid the most fundamental errors and misconceptions.

I propose that there are at least four distinct types of MMBs.

i) Proof-writing (PW) (these are a super-set of the proof-creation heuristics found in texts and Weber's (2001) article).
ii) Definition Exploration
iii) Knowledge Organization
iv) Example exploration and generalization

The present article will confine itself to an exploration of the Proof Writing (PW) type of MMB. PWs consist of a host of techniques and habits that aid in proof writing.

The skills needed to write proofs

In order to describe what it means to model proof-writing strategies, it is important to describe the knowledge and skills that are needed to successfully write proofs. Selden and Selden (2009) have begun describing these skills, and I draw upon their work. They explain that in order to write proofs, a person must know:

1) The Proof Framework (PF) encompasses the conventions of proving things in mathematics, but does not require understanding the meaning of any of the terms.
2) The Hierarchical Structure (HS) includes knowing what the proof has to accomplish and coordinating any sub-proofs, including lemmas.
3) The Construction Path (CP) is the means for actually creating the proof (as distinct from the way that the proof is actually written for publication)
4) The Formal-Rhetorical (FR) aspect and the Problem-Centered (PC) aspects of proof-writing. The FR part includes knowing that if a theorem says, "for all real numbers …" that the proof should start by introducing an arbitrary real number, "let x be a real number…" (Selden & Selden, 2009, p. 343) The FR aspect also includes the ability to do algebraic and technical symbolic manipulations within the structure of the proof-system. The PC aspect includes determining the key idea(s) and coordinating aspects of the proof (especially if they include non-standard argument structures).

By drawing upon this list, as well as Rasmussen and Marrongelle's (2006) concept of Pedagogical Content Tools, I argue that we have a powerful way to understand traditional, lecture-based instruction at the advanced undergraduate level. In the following pages, I draw upon this lens to analyze the teaching of an abstract algebra class. I will, in particular, analyze how the instructor modeled the MMB of proof writing during class meetings.

3. Methods

3.1 The teacher and institution

When the study began, Dr. Tripp (a pseudonym) was an assistant professor working towards tenure at a mid-sized doctoral granting institution in the mid-west. Dr. Tripp earned a doctorate in algebra and had taught abstract algebra a number of times prior to the study. In the context of this study, Dr. Tripp's most important characteristic is her self-description as a traditional teacher of abstract algebra. For her, this meant that lecturing in front of the class would be the predominant method of instruction, that "proofs form the backbone of this course," and that her organization and presentation of material would closely follow that of the text, *Abstract algebra; An introduction* (Hungerford, 1997). That is, considering teaching on the continuum from pure telling to pure discovery, Dr. Tripp claimed that she would be closer to pure telling.

3.2 The class

The class met four times per week for 50 minute sessions in the spring semester. Students had frequent homework assignments from the text, two in-class exams and one final exam. Typically, the students are juniors who have completed a two-semester calculus sequence and an introductory course on mathematical proof; thirteen such students were enrolled in the course section being studied, as well as one sophomore and one senior. The course featured a significant amount of ring theory, beginning with the definition, including quotient rings, and culminating with the construction of roots of irreducible polynomials. The class transitioned to the study of group theory with approximately one week remaining in the semester.

3.3 Methodology for analysis

I observed 18 of the class meetings taking detailed field notes, and made video recordings of 15 of those classes. I also transcribed all dialogue as well as text on the board. The video camera was primarily recording the instructor as she was the principle focus of all class activities. In particular, I reviewed all classroom video recordings and made a log of all episodes that included proof-writing or presentation. I made the choice to use classroom dialogue as a means of making sense of the classroom activity, instead of just examining the mathematical content of the proofs. My reasoning was that the mathematical content alone did not capture the feel of the summary memos in my notes, which included significant commentary on Dr. Tripp's questions. With the focus on classroom dialogue, I further refined my coding system to focus on language that might represent a PCT. My iterative process led to identification of MMBs, particularly in *proof-writing*, as the most compelling aspect of Dr. Tripp's class.

Drawing on the field notes and transcriptions of the video data, I constructed a narrative, taking note of all proof writing incidents. Criteria for proof-production or presentation was rather straightforward. An incident was logged as such when any member of the class community was writing or showing a formal mathematical proof that drew on symbolic notation and logical reasoning. This did exempt any informal justification that was offered if it was not then connected to formal proof. Work with particular examples of structures was read with a bias towards categorization as proof as long as the instructor or the students were using algebraic techniques.

I first distinguished between who was writing or presenting the proof; classifying them either as student-authored or teacher-authored. I further refined the categories by manner of proof-production. In my notes, I had repeatedly described Dr. Tripp as writing proofs, and, while doing so, asking the students a large number of questions. The way that she asked them, as well as very similar rhetorical questions, suggested that she was modeling what she believed to be appropriate mathematical behavior. Once I identified that Dr. Tripp was modeling mathematical behavior while writing proofs I closely read each of the proofs that were produced or presented in order to refine the codes I had used to describe what happened during the class. The code that was consistently repeated was, "modeling proof writing." I reread and refined my codes looking for, in particular, examples of MMBs that the teacher might have been employing (e.g., words, gestures or written work that furthered the mathematical learning of the class) in other aspects of her classroom work as well. From there, I selected a set of representative examples that captured both the concepts of MMBs and proof-production/presentations in order to analyze and document one such example in the present article.

4. Results

During the time that I observed the class, either the students or the teacher wrote 29 analytical verifications of properties or other types of results. Seven of these proofs were given entirely by students, and each of these was a property-verification argument. Dr. Tripp wrote one proof without any dialogue with students. The remaining 21 proofs were of a very similar pattern: when writing a proof during class, Dr. Tripp asked a large number of questions that solicited student feedback rather than delivering a monologue-type lecture. For that reason, I call this type of interaction a *participatory proof writing* pedagogical scheme. Dr. Tripp seemed to use the participatory proof writing with students to accomplish multiple ends, but in most cases she was modeling proof-writing skills.

In all *participatory proofs,* the dialogue began with a teacher-initiated question, most often directed at the whole class, followed by a student response that was relatively short, generally followed by an evaluative comment by Dr. Tripp, conforming to the Initiate-Response-Evaluate pattern identified by Mehan (1979).

An example of the participatory proof-writing scheme.

I have chosen to show a proof that the kernel of a ring homomorphism is an ideal of the domain. By drawing upon the research literature on instructor's use of proof it is possible to imagine that Dr. Tripp may have had multiple reasons for presenting this proof. She might hope to:

1) convince students that the kernel is an ideal,
2) explain why the kernel is an ideal,
3) demonstrate how to write a proof that something is an ideal,
4) demonstrate how to write a proof that something is a subring, or
5) multiple other reasons.

To better understand Dr. Tripp's actions, let us consider what the students had previously done and what she said and did immediately before presenting the proof. She introduced the claim while discussing ideals in rings and noted, "So far, we've only looked at a couple of examples of things of the form R mod I, namely polynomial, quotients of polynomial rings like this and zee modulo n. Where else do these things show up?" Then, she reminded the students that they had previously done homework determining the kernel of a specific function, treated many specific cases of homomorphisms of polynomials (for example, in the construction of roots of irreducible polynomials) and she announced that on the previous exam the students were to show that the kernel was a subring of the domain. Finally, before beginning the proof, Dr. Tripp worked through a proof that, when considering the standard homomorphism between the integers and the integers modulo three, that the kernel is an ideal.

It was only after this discussion and example that Dr. Tripp was ready to begin the proof that the kernel of any ring homomorphism is an ideal of the domain. This set of activities before presenting the proof suggests that Dr. Tripp was primarily using this proof presentation to model the skills of writing a subring and ideal proof. She had likely already achieved a high level of conviction on the part of the students, and the worked example just prior to the proof should have explained to the students why a kernel is an ideal. Thus, while there may have been some need for conviction and explanation, it is likely that her principal purpose was to illustrate the appropriate proof-technique, which is corroborated by Alcock's (2009) interviews with mathematics faculty. They claimed multiple reasons for presenting proof, but the most common was demonstrating or modeling proof techniques.

Finally, let us examine the actual proof that the kernel of a ring homomorphism is an ideal of the domain:

> **Dr. Tripp**: Let's see why the thing called *K*, which has a name, it's the kernel, is an ideal all the time. So, we need to get back to this ring homomorphism. If we have any ring hom *f* from *R* to *S*, let's show *K* is an ideal. What do we have to do to show it's an ideal? [pause] You have to show it's closed under addition, closed under multiplication, it's non-empty, every element has an additive inverse. What do those four things tell us?
>
> **S**: Subring.
>
> **Dr. Tripp**: Subring.

In this section of the proof presentation, Dr. Tripp introduced the goal of the proof and then asked a rhetorical question, "What do we have to do to show it's an ideal?," a question that structured the proof. It is an example of modeling one way to make sense of the proof hierarchy (Selden & Selden, 2009), as Dr. Tripp was explicitly stating what must be proved. She finished the comment with a factual question of the students that did not require any understanding of the context of the proof. This question can be understood as a *product* question following Mehan's (1979) categorization.

> **Dr. Tripp**: [continuing] And then, I need what? I multiply element in *R*, I take anything in *K* and I multiply by any element in *R*, and that product comes back into *K*. That's the ideal condition. That last condition actually includes, like we said, that multiplication is closed. So, then we just need to check addition, that it's non-empty, that additive inverses work out, and the ideal condition. So, let's do that.

In this comment, Dr. Tripp completed modeling the creation of the hierarchical structure by stating that the proof must check the ideal condition. Then she noted a possible change to the proof, verifying the ideal condition also shows multiplicative closure, which can be understood as possibly modeling a proof-construction route (Selden & Selden, 2009).

Dr. Tripp begins the proof by demonstrating a verification of the non-empty property of subgroups. This is traditionally the first property verified both because it is often the easiest and because verification of this property before proving others is important mathematically.

> **Dr. Tripp**: How could I show it's non-empty? How do I show that there's something that goes to zero?
>
> **S**: [Inaudible]

Dr. Tripp: Yeah. So, we know that $f(0r)$ equals $0s$, so there's something there. So, $0r$ is in K. So, it's got something in it.

Dr. Tripp modeled appropriate hierarchical strategy by stating that the sub-proof was complete. She then stated the next of the subring properties to verify and began.

Dr. Tripp: [continuing] That may be all that's in it, and if so, that tells us something very special about that map. Okay, let's take two things, not *r* and *s*, how about *a* and *b*. If *a* and *b* are in *K*, I want to show their sum is in *K*. How do I show their sum is in *K*? You have to use the definition of big *K*. The only thing you know about big *K* is, well, it consists of stuff that gets mapped to zero. So, what do I have to show about $a + b$ to show it's in *K*?

S: It gets mapped to zero.

In this set of comments, Dr. Tripp has modeled the formal-rhetorical aspect of demonstrating that a set is closed under an operation choosing two arbitrary elements. She has also modeled an aspect of hierarchical structure by stating the definition of closure, thereby telling students what to prove and what sub-proof they were completing. Finally, it is appropriate to note that when Dr. Tripp called into being two arbitrary elements she was careful in her choice of representatives, "not *r* and *s*" to avoid possible confusion with previous definitions of variables. I assert that she did this to model the mathematical behavior of thinking carefully about the choice of variables. In particular, she hoped to help prevent students from mistakenly assuming that the argument was about an arbitrary element of the rings *R* and *S* rather than elements of the kernel. This seems an explicitly pedagogical choice and is a clear demonstration of modeling proof-writing skills. She continued with her proof by modeling the remainder of the closure sub-proof.

Dr. Tripp: Ok, so, let's look at what *f* does to $a + b$. So, S, what can I say about $f(a + b)$?

S_2: It equals $f(a)$ plus $f(b)$.

Dr. Tripp: Is there anything I know about $f(a)$ now?

S_2: It equals zero.

Dr. Tripp: Great, and $f(b)$? And what do I know about zero plus zero? That's zero. Great. So, $a + b$ meets the condition it needs to be in *K*. [pause] So, that's the property of f preserving addition that we just used, and that gives us that the kernel is closed under addition.

Dr. Tripp suggested the way to begin the sub-proof, and asked students to make use of formal-symbolic rules to complete the next step of the proof. When she claimed that the subproof had been completed, she again modeled the hierarchical structure. She then, essentially, repeated the entire closure sub-proof creation process for multiplication, and modeled the same type of proof-writing skills while demonstrating the ideal property.

Dr. Tripp: And, let me change something… Also, how about we make this for any *a* in *K*, well, we already said *a* was in *K*, and how about for any *r* in *R*? Well, if I change *b* to be *r*, actually we could have just left this as *b*, but then we'd get zero. And that includes *a* and *b* being in *K*, as we were just talking about. Likewise, $f(ra)$, what is that going to equal?

S: $f(r)$ times $f(a)$.

Dr. Tripp: Great. And what is that going to equal?

S: $f(r)$ times 0, zero.

Dr. Tripp: So in other words, it doesn't matter, they're both the same. So, *ra* and *ar* are in *K*. So, what do I need left to check that this is an ideal?

S: Inverses

In this last set of exchanges between Dr. Tripp and the students, she appeared to be doing two separate things; again modeling the importance of choosing good notation, and modeling proof writing for the ideal condition in the same way that she did for both of the closure arguments. Dr. Tripp again demonstrated good proof-organization, clearly stating that the sub-proof was complete, "*ra* and *ar* are in *K*," before asking the students what sub-proof was left to complete. That is, she was modeling multiple aspects of the hierarchical structure of the proof. Dr. Tripp then finished writing the proof by cycling through HS and FR aspects of writing a sub-proof that the kernel of a homomorphism includes additive inverses. She began by giving a HS statement:

Dr. Tripp: Finally, let's look at what *f* does to negative *a*. For all *a* in *K*, *f*(-*a*), what can I say about that? What does *f* do to negative *a*?

S: Negative f of a.

Dr. Tripp: Yeah, this is another one of our properties of homomorphisms. *f* carries an additive inverse to the additive inverse of the image, to negative *f*(*a*), and that equals, negative zero! Yeah! And what's the additive inverse of zero? Zero, so we get zero. So, *K* is an ideal. [Underlines: *K* is an ideal.] So as a consequence, any time you have a homomorphism of rings, you get an ideal.

Finally, she modeled one last type of HS in claiming that the proof was complete, claiming that demonstrating each required property was sufficient to guarantee that the Kernel is an ideal.

5. **Significance and directions for future study**

The first conclusion I draw from Dr. Tripp's teaching is that, at least in this traditional classroom, the students were likely to be participating in proof-writing or presentation activities. In 29 observed proofs, the students presented seven by themselves, Dr. Tripp engaged in participatory-proof writing schemes 21 times, and just once delivered the caricatured 'lecture' where students sit quietly and take notes without questions or comments while the professor presents the proof. This suggests that the vision of a lecture-based class as one where the teacher does all of the talking and the students almost none is not always accurate. While this style of teaching likely exists, it was not the norm in Dr. Tripp's traditional instruction.

Secondly, we should believe that Dr. Tripp's statements and questions were purposeful, and attempting to instruct students in algebra and proof-writing. The constructs of Modeling Mathematical Behaviors, and in particular, Proof Writing Strategies were observed and described based on the practice of a traditional abstract algebra teacher. Furthermore, examining Dr. Tripp's actions through the lens of Proof Writing Strategies allowed significant insight into the purposes of her questions and statements. This was the first time that the whole-class discussions of a lecture-based abstract algebra instructor have been analyzed.

Discussions with Dr. Tripp indicated that she, and the mathematics faculty more generally, had noted that students were generally ill-prepared to write and understand proofs when they began advanced mathematics courses. She had expressed, as a goal, that the students would "make significant progress" in their "ability to conceive of and write up proofs," and their "ability to generalize." As a result of her previous teaching and work with her current students, Dr. Tripp knew their strengths and weaknesses. I claim that Dr. Tripp used her knowledge of her students' thinking to better connect her pedagogy with their needs; one of her means of doing so was modeling the types of questions that one should ask while writing proofs, questions such as, "what does that mean," "what comes next," and "what do I still need to do." To draw on Selden and Selden's (2009) framework, she was modeling thinking about hierarchical structures, the formal-rhetorical aspects of proof, and the construction path that she hoped her students would appropriate. Thus, I have named this set of Modeling Mathematical Behaviors a Proof Writing Strategy.

This suggests that MMBs may offer a possible way that lecture-based instruction may not lead to the usual concerns about student learning of advanced mathematics quite as readily as suggested in the literature. However, no evidence of student adoption of MMBs has yet been documented. Many questions rightly remain, including whether and how other lecture-based instructors make use of MMBs, what other types of mathematical behavior are modeled, and what characterizes other MMBs. It is also worth investigating the relative frequency of each MMB as compared with the importance that faculty claim to assign those behaviors.

In terms of student appropriation of the modeled mathematical behaviors, I did not witness any. The observations of student proof-writing and presentation were constrained to in-class opportunities for students to write proofs, where they did not speak aloud while working. Thus, they may or may not have been using the modeled behaviors, as we were not noting them. Additional studies should be directed towards determining which, if any modeled behaviors students appropriate, and how the students make use of them.

Lastly, I see MMBs as one possible avenue for mathematics educators to work with mathematicians to make minor modifications of their teaching that may lead to increased student success. For example, by noting the frequent use of MMBs in Dr. Tripp's instruction, we might suggest that classroom professors make explicit statements, like, "listen to the types of questions I ask you and ask myself, these are the kinds of things *you* should be asking while you write proofs." Such statements could lead to greater student appropriation of the modeled PWS behaviors. Similarly, our examination of Dr. Tripp's class suggested that she already asks a large number of questions and expects students to be active participants, but the responses she expected were largely factual. If this type of questioning is already common in traditional classrooms, a small change in the type of responses that the instructors expect (and accept) could be easier to accomplish (as compared with a switch to an inquiry-oriented structure).

Conversations with mathematicians indicate that some of them are conscious of the fact that they model appropriate mathematical behaviors and do so purposefully. For instance, Alcock (2009) interviewed mathematicians about their teaching of proof and described different types of thinking that the mathematicians were attempting to promote in their classes. The construct of MMBs would give a means to ask mathematicians to be more explicit about their goals and, especially, the intended learning outcomes of classroom actions. The construct of Modeling Mathematical Behaviors may also allow for direct evaluation of the students' appropriation of the goal behaviors by examining both the students' in-class responses to questions and observing their proof-writing activities outside of class. Thus, I suggest that expansion of the domain of MMBs is both helpful to making sense of traditional classes and offers the possibility of affecting instruction for the better.

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