Student Misconceptions of the Language of Calculus: Definite and Indefinite Integrals

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Abstract

Many mathematical terms are also used in everyday English. We say things like origin, derivative, sum, tangent and we mean very specific things when we are inside a mathematics classroom. The problem here is that when we step outside a mathematics classroom, these words take on a whole new life; sometimes they mean the very same thing, and sometimes they are entirely different entities. In this study, twenty-five students in an introductory calculus course were interviewed about their knowledge of integration. Participants were asked to discuss various integration problems, both definite and indefinite, as well as defining the terms “definite integral” and “indefinite integral.” Students provided many different kinds of responses, but most interestingly, that the definite integral is more “precise” than the indefinite integral and the indefinite integral is “vague.” Additionally, one student when asked what an indefinite integral was, responded “I don’t know, opposite of a definite integral, obviously.” These types of responses are indicative of not only poor understanding of mathematical concepts, but also conflict between the students’ knowledge of mathematical terms and their everyday English counterparts.

Introduction

Interviewer: "Do you know what an indefinite integral is?"
Student: "Opposite of a definite integral I suppose."

It's hard to call logic like this a misconception; actually, it's hard to even call it logic, speaking from a mathematical standpoint. Yet, considering solely the words involved and not the concepts described, I would argue this is actually a very logical lingual assumption. One would guess that the lingual split between definite and indefinite parallel other similar pairs, capable vs incapable, appropriate vs inappropriate, efficient vs inefficient, etc. In all these examples, it is true that one is, in fact, the opposite of the other. So the fact that a definite integral is NOT the opposite of an indefinite integral seems to be the illogical piece to this whole puzzle. The issue is that although the concepts are different in many ways, the language pulls the definite and indefinite integral together. Student response data will show that this phenomenon is present and debilitating for some students.

Problems such as these arise when words and phrases that have one meaning in our natural language have a different meaning in the language of mathematics. Language plays an important role in mathematics, in fact, many have argued that mathematics is itself a language (Krussel, 1998). In addition to the issue outlined above where mathematical concepts and lingual relations are at odds, students may have a hard time with some definitions since terms that live in the mathematics register are ported from our everyday language register. We take Halliday's (1978) meaning of register in general and specifically the "mathematics register" here in that,

> A register is a set of meanings that is appropriate to a particular function of language, together with the words and structures which express these meanings. We can refer to the "mathematics register," in the sense of the meanings that belong to the language of mathematics (the mathematical use of natural language, that is: not mathematics itself), and that a language must express if it is being used for mathematical purposes. (p. 195)

Take the word *tangent* for example, mathematically speaking *tangent* has a specific definition concerning a specific spatial relation between two objects; however, in our everyday register we say that someone "goes off on a *tangent*" when they deviate from their original train of thought.

The current landscape of research on student understanding of integration is sparse. A handful of studies have been done in this field and all deal primarily with the definite integral. These studies have shown that students have difficulty understanding limits as they pertain to Riemann sums (Bezuidenhout & Olivier, 2000), when given an area representation of the integral they are unlikely to be able to tie it to a conception involving Riemann sums (Sealey, 2006), and students generally have issues handling functions which cross the x-axis and/or involve negative area (Orton, 1983). The question at hand then is what, if any, impact does language have on students when they discuss definite and indefinite integrals?

Background

Mathematics is a "Language"

Learning mathematics is much more than filling one's mind with various tools and formulas that can be whipped out whenever the need for calculation arises. Mathematics is a sophisticated subject which employes logic and critical thinking as well as complex terms and definitions. The way mathematics educators require students to read and understand definitions and syntax makes learning mathematics similar to, if not the very same as, learning a foreign language (Krussel, 1998).

The importance of terms and definitions in the learning of mathematics cannot be overstated. As Raiker (2002) asserts, "...mathematical concepts are to a large extent connected and hierarchical, it is crucial for the understanding of sound concepts and the subsequent development of mathematical thinking that the precise mathematical meanings of mathematical words are established" (p. 45). When students cannot properly conjugate verbs in French class, they can hardly be expected to form grammatically correct sentences. Likewise, if a student cannot sufficiently define *slope*, how can they be expected to graph an equation with any kind of understanding?

Some may argue, much like Hersh (1997), that mathematics is not truly a foreign language, but more of a "lingo" which borrows certain aspects of the natural language of which it is a subset. For the purposes of this argument, it is not necessary to agree that mathematics is a language, merely that learning terms and definitions in mathematics is very similar to learning terms and definitions in a foreign language.

Effects of Linguistic Complexity

This discussion of the importance of language in the mathematics classroom is all well and good, but what evidence is there that these terms and phrases have a measurable impact on a student's learning? Abedi & Lord (2001) conducted a study in which they tested nearly 1,200 8th grade students with released items from the National Assessment of Educational Progress. In order to test what effect the linguistic complexity of the examination questions had, they rewrote some of the questions with simpler language. Extensive research into linguistic complexity and its effects was cited as were previous efforts in the task of simplifying mathematical phrasing. The researchers made changes which familiarized or even personalized the setting; conditional and relative clauses were reworked or replaced. Two individual studies were conducted, one in which 36 students representing a wide range of backgrounds and abilities were interviewed after solving 10 regular NAEP items and 10 modified items, and the second where 1,174 students took 10 regular NAEP items, 10 modified items, and five control items.

In the first facet of their study, the interviews showed that students significantly preferred the modified items with 63% of students from the first batch of interviews and 83.1% of students from the second batch stating that they felt the modified item sounded easier and would choose it over the item worded in the NAEP if under duress. Many students commented that the revised items were easier to comprehend.

In their second study, the researchers found that English language learners (ELL) and students of low socio-economic status (SES) performed significantly lower on the standard items then their native English speaking or higher SES counterparts. In terms of the modified items, there were some very promising findings. First, on average, ALL students benefited from the modified, linguistically simpler terminology with an average of 2.9% improvement over standard items. Secondly, it was observed that the modified items seemed to close the gap between native English speakers and ELL as well as between students of low SES with more affluent groups. All stated differences were found to be statistically significant.

While this study speaks volumes about the major cultural riff that exists in the mathematics classroom, it also speaks more broadly to the power of language. Understanding the questions that are asked is crucial, as this study has shown; if students are having a hard time defining the terms they encounter, then they very well may be confused during assessment.

Varied Use of Mathematical Terms

Thus far it has been asserted that mathematics is a language (or at least close enough), that terms and phrases may be a tripping point for some students, and that language has been shown to have significant power on examination questions. The question still remains, though, as to what role language has in the classroom. With a group of students aged 7-10, Raiker (2001) used "discourse analysis" (p. 3) in order to study how many mathematical terms students and teachers used in regular classroom setting. Using the National Numeracy Strategy's mathematical vocabulary, Raiker studied how many terms the teacher used in front of the class, how many terms students used during the whole class session, and also how many terms students used in special small group settings without the teacher present.

What Raiker found probably wouldn't surprise most readers: teachers, in a whole class setting, used terms the most (29-55 different terms); students, in the whole class setting, used significantly fewer then the teachers (17-30); and, students in the small group settings used significantly fewer than they did in the whole class setting (3-22). What we see is that the teacher's presence, and possibly their use of mathematical terms, incites students to use more terms than they feel obliged to on their own.

An important implication of this study is that students don't view the mathematical register as their natural one when in a mathematics class. When in their small groups, they used as few as three mathematical terms in their discussions. While it must be observed that it was not made clear what the students were discussing or how productive their interactions may have been, it is true that these students used fewer terms then they did in the whole class setting, and far fewer then their teacher. It would seem that students are not thinking about language mathematically, but from a natural-speaking standpoint.

Student Understanding of Integration

Integration, as a whole, is a complex subject; the concepts which appear under the hood of integration are numerous and multifaceted. In considering all that is involved within integration, students are required to discuss and understand a great many topics including definite integration and indefinite integration, the foci of this research. Digging deeper, inside definite integration we face issues including, but not limited to Riemann sums, limits, area, function, covariational reasoning, and accumulation. As with all mathematics, reading definitions of these issues is not sufficient to develop deep understanding. Research in these areas inform us of common misconceptions students hold about integration and the ways in which students tend to view integration. Unfortunately, research on indefinite integration is far more scarce, thus, there will be no discussion of student understanding of the antiderivative as a stand-alone concept.

Area and the integral.

A definite integral will compute the area between a curve and the *x*-axis between two points. This fact makes it a powerful geometric and computational tool; it also gives students a viable fixation point when attempting to grasp the difficult concept of integration. While it is true that a definite integral will compute area under a curve, it is only powerful if the student also has a sophisticated view of area. As was stated by Bezuidenhout and Olivier (2000), "Such an inappropriate '*area-conception*' [*sic*] of integral stems from the special case, $f(x) \geq 0$, as its construction is based on a generalisation of the special case" (p. 87) and defining a concept through such a special case can be dangerous. Without the ability to view area as something other than a physical measurement of space, students will have a very narrow and limited view of what integration can do. As we will see there are many problems with using a focus of area under the curve to teach definite integration and as we shall discuss later, this idea permeates even to the indefinite integral.

Area and Riemann sums.

Sealey (2006) conducted a study looking at the area representation of definite integration particularly focused on how students understood area under a curve with respect to Riemann sums. Sealey's research "...provide[s] data that shows that when solving real world problems, students need to understand why this relationship between area and the definite integral holds" (p. 47), a concept that has been known to elude many students of calculus (Orton, 1983). Sealey observed students trying to work through a problem involving force and energy; when the students graphed the force equation they were given, one group stated that the area under the curve would be the energy. Sealey discusses this instance:

> When I asked the students to explain to me *why* [sic] area under the curve was equal to energy, they could not explain, and were never confident that they were correct in graphing force, instead of energy. Their only justification was that they had gotten confirmation from one of the research assistants that this was an acceptable method. When I pushed them to explain *why* [sic] this was an appropriate method, they were unable to do so. (p. 51)

Sealey was focused on the students' understanding of Riemann sums and commented that she "hypothesize[d] that one of the reasons the students struggled with explaining area [was] because they did not understand the structure of the Riemann sum" (p. 51). Sealey concluded that the students in this study "may be proficient in dealing with area under a curve, but may not be able to solve other accumulation problems without thinking about area under a curve, or may not be able to relate the area under a curve to the structure of a Riemann sum" (p. 52). Here, the author noted the students' abilities in computation, but also found a significant problem with the students' attempts to rationalize those computations.

Additionally, Sealey noted that students who lacked an in-depth understanding of Riemann sums were not very adept at explaining area under a curve within the context of the problems they were working on. For example, one group of students were working with a problem involving force and energy and the researchers frequently heard that the "summation of forces equals energy" from the students. As Sealey points out, "it is not just the summation of *forces* [*sic*] that equals energy, but it's the summation of the *products* [*sic*] of force and distance that equals energy" (p. 51). This concept of accumulation was studied extensively by Thompson and Silverman (2000) but will not be reviewed in-depth here.

Negative area.

When a function crosses the x-axis within the limits of integration, the negative contributions will subtract from the total area, introducing the concept of "negative area." This term has no physical meaning that can be explained using physical area and students have a hard time even computing problems of this nature (Orton, 1983).

Tall and Rasslan (2002) included a question on their survey involving a definite integral in which the function crossed the x-axis. Upon analysis, they

> ...hypothesized that the students did not necessarily know how to calculate the area when the function changed its sign…it follow[ed] that 15 students out of 41 [did] not explicitly evoke a change in sign when it occur[ed] in a given interval [*a,b*]. (p. 6)

The power of the area representation is lost in such cases since students who have been describing the definite integral as area under the curve must adjust their definition to include these cases where the function is not always positive. What we see from these types of results is that students are utilizing an image of *physical* area rather than an abstract concept. Since they cannot visualize negative area in a physical sense, students have a difficult time with problems involving negative function values.

Along these same lines, Bezuidenhout and Olivier (2000) found that students, who demonstrated their understanding of the definite integral as an area, were more apt to provide incorrect assessments of their problems. They discuss how the area interpretation can be confounding in instances where the function in question is not strictly positive. The students were asked at which point on the interval [*a,g*] would $p(x) = \int_a^x h(t)dt$ take on its maximum value, to which one student responded, "*p(x)* will increase … it will increase up to *g*." (p. 87). When probed as to which point will result in the maximum value for *p(x)*, the student stated that it would be *p(g)* since "it gives the total area" (p. 87). It should be noted that *h(t)* was not positive over the entire interval and therefore it is obvious the student is not dealing with the negative contributions that would occur within the given interval. Another student with a seemingly similar understanding of the definite integral as an area function responded that the maximum value of *p(x)* would occur at *c* because “here (student points at the interval [*a,c*]) the area gets larger, larger and larger” (p. 87). This student seemed to have conflicting ideas even within area as a definition for definite integral.

In addition to these results, Orton (1983) found that in regards to student understanding of definite integrals when the function in question crossed the *x*-axis, “many students appeared to know what to do, but, when questioned about their method, didn’t really know why there were doing it” (p. 8). In this case, the survey asked for the area of the shaded region and students needed to split their integral into two parts, a slightly different task than solving for the definite integral of a function in which the function crosses the *x*-axis, however similar thinking is involved.

In each of the previous cases it is clear that the students involved had some issues in describing the definite integral of a function in terms of area when that function crosses the *x*-axis. Since many students seem to understand area as a physical measurement and lack the ability to see it as an abstract concept, it seems unwise to use area to define the broad concept of definite integration which necessarily employs such an abstraction. Sealey (2006) recognized this disconnect when she stated, “I do not in any way claim that area under a curve is a bad representation or that it should not be taught. Instead, I claim that area under a curve is not *sufficient* [*sic*] for understanding the definite integral” (p. 52). Definite integrals are complex and multi-faceted and thus require an equally dynamic definition.

Limits.

Orton (1983), one of the first education researchers to tackle the concept of integration, conducted a study in which 110 British students aged 16-22 were interviewed on their knowledge of integration. Orton saw that students had many issues with integration ranging from their algebra skills to their knowledge and use of limits. Concerning the latter, Orton’s results,

> ...suggested that most students had little idea of the procedure of dissecting an area or volume into narrow sections, summing the areas or volumes of the sections, and obtaining an exact answer for the area or volume by narrowing the sections and increasing their number, making use of a limiting process. The vast majority of students could not complete an explanation for this item. (p. 7)

Additionally, Orton observed that, “students were expected to understand that the limit of the sequence of sums of rectangular areas, as the number of rectangles $\rightarrow \infty$ and the widths of the rectangles $\rightarrow 0$, would give the exact area under the curve,” (p. 4) and found that “many students interviewed in the study had problems …generally understanding the relationship between a definite integral and areas under the curve” (p. 12). Basically, Orton saw that students were fairly adept at computing integrals but were unable to discuss integration’s most basic components.

Research Methods and Data Analysis

Near the end of the Spring semester of 2009, 25 students in an introductory calculus course at a liberal arts university in the Northeastern United States were offered extra credit in exchange for their participation in half-hour long interviews. The interviews were designed to probe their knowledge of basic integral computation as well as their understanding of the terms "definite integral" and "indefinite integral." Students were encouraged to write, draw, and speak about anything they thought was pertinent to the topic at hand. The purpose of the interview was to get the student engaged and to follow his or her own train of thought, rather than be strictly led by the researcher. The interview questions were prepared by the researcher and were informed by initial analysis of data collected in a previous, unpublished study based on second semester calculus students' understanding of the integral.

Interviews.

The interview consisted of three main parts. First, students were asked to discuss four integration problems, two definite integrals and two indefinite integrals. Students were not asked to solve these problems, but merely to explain what that particular set of symbols meant. Next, they were asked to do similar exercises to the first, but this time with the terms "Definite Integral" and "Indefinite Integral." They were also asked to come up with examples of where these two concepts could be used in mathematics. Finally, the student was asked to complete a short worksheet of integration problems and describe their thinking while they solved them. As a conclusive statement, interviewees had to talk about any confusions they had while learning integration and/or whether they felt integration was "harder" or "easier" then other math they had learned in the past.

The interviews were conducted in the concluding weeks of the semester so the students had learned all the integration material they would learn. Some students were interviewed before the final exam, some just after. All interviews were audio recorded and transcribed. A grounded theory approach to analysis was taken. Student responses were reviewed and grouped for similarity. For example, a response that described the indefinite integral as "antiderivative" was categorized with responses that described it as "inverse derivative" since these two statements describe similar mathematical ideas. After the responses were categorized and major themes emerged, the data was analyzed again in order to code the interviews according to these new classifications. The same procedure was followed for the definite integrals, where the actual responses the students gave were used to create the classes of responses used in analysis.

Student responses were not categorized by question type in order to provide the most inclusive data available. In other words, in terms of the total counts of student response data, it will not be explicitly stated to which question the student was replying to. While the researcher believes that this may be a point of further study, this report was interested instead on the students generic knowledge of integrals regardless of the context in which the response was derived.

Computation of integral problems.

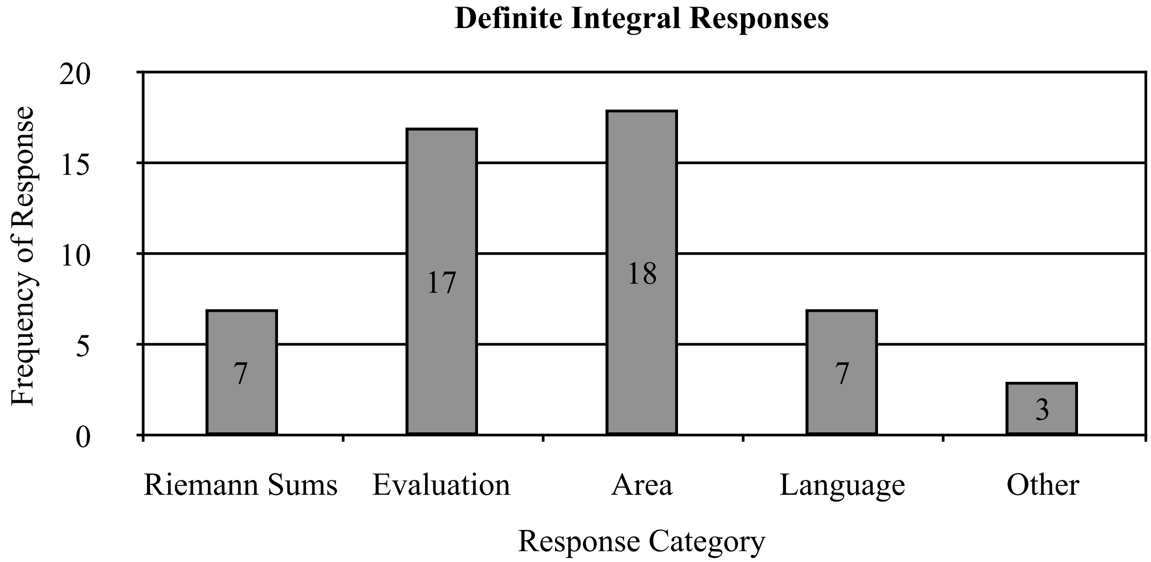
The last page of the interview consisted of six basic integral problems that the students were asked to solve. Each problem was specifically modeled after problems from the students' calculus textbook to ensure that they were not too difficult for the students to solve. In order to score the students' performance on these problems, a rubric was created which scored each problem on a 0-4 scale with zero being a blank response and four being a correct response. Responses which included some appropriate information but showed no significant attempt at a solution were scored with a one; responses involving some aspect of an appropriate solution method but also had a significant flaw in the logic were scored with a two; and, those responses which employed correct solution methods but had minor computational or notational flaws were scored with a three.

Findings

Definite Integral Response Categories

Student responses concerning the definite integral were categorized via the following five categories, "Riemann Sums," "Evaluation," "Area," "Language," and "Other." Responses coded as "Riemann Sums" were those which associated the definite integral with the concept of Riemann Sums, either implicitly by talking about the idea of summing up the area of rectangles or explicitly by connecting the ideas of definite integration with a limit of Riemann sums. Responses coded as "Evaluation" were those in which the student, when asked about a definite integral, discussed how to evaluate a definite integral; this usually took the form of talking about the various tools used to solve a definite integral problems. "Area" responses connected the definite integral with area in some way; whether the student explicitly said a definite integral computes area under a curve or made an implicit connection, these responses indicated that the student had area in mind in connection with the definite integral. "Language" responses were those which seemed to be rooted in the students' natural language instead of demonstrating an understanding of the mathematical terms and concepts involved; these are the responses which this study is primarily concerned with. Finally, there were a few responses which didn't seem to fit into any of the previous categories but didn't warrant a category of their own. Responses of this nature were generally an instance where a student started to mention a topic but failed to make any clear connections between concepts.

Below is a graph which depicts the total number of students responding in each category listed above. It should be noted that the most frequent types of responses were those dealing



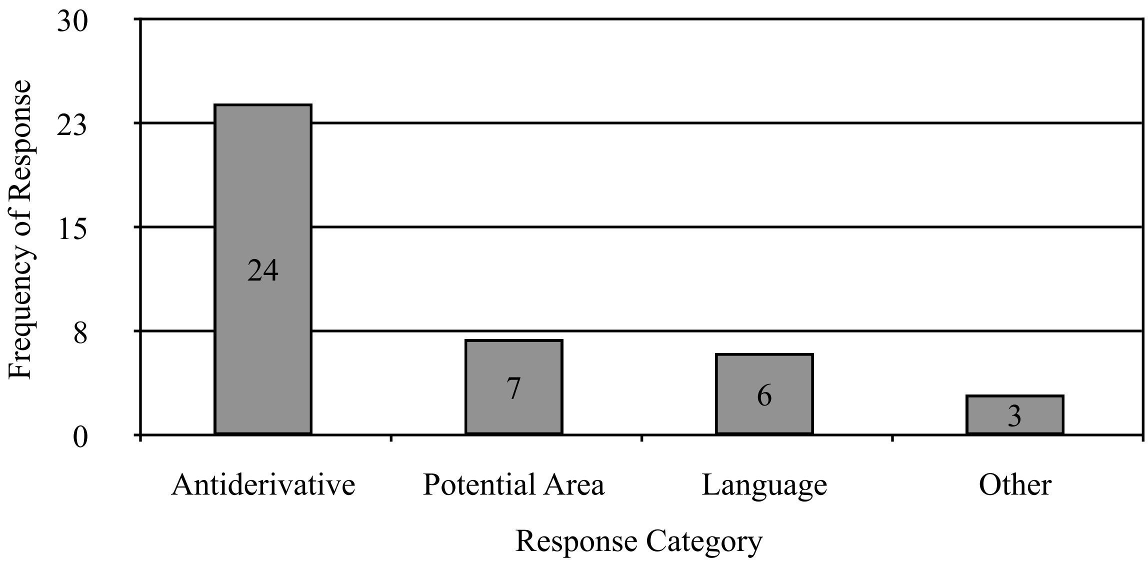
with an evaluation or a mention of area. Additionally, there were seven students who made language-related statements, a sample of these responses will be discussed in-depth later.

Indefinite Integral Response Categories

There were four response categories which were developed from student responses concerning the indefinite integral: “Antiderivative,” “Potential Area,” “Language,” and “Other.” Responses deemed as “Antiderivative” associated the indefinite integral with the antiderivative or some equivalent mathematical concept. Responses categorized as “Potential Area” were those in which the student discussed how the indefinite integral could compute area if bounds were determined or how the indefinite integral computed an “infinite” area. “Language” and “Other” responses were defined in much the same way as the categories from the definite integral responses.

Within the student responses to the indefinite integral, a trend appears that wasn’t apparent in responses to the definite integral questions. When discussing the indefinite integral, an overwhelming majority of students responded in a way that was categorized as “Antiderivative,” meaning that almost every student had very similar conceptions of what an indefinite integral is. Student response figures are shown below.

Indefinite Integral Responses



"Language" Responses

The responses of main import to the present study are those seven "Language" responses from the definite integral questions and the six from the indefinite integral questions. Due to some expected crossover, we are left with 10 out of the original 25 students who, when discussing either definite or indefinite integration, made statements that were related to issues with the language of integration. It would be most beneficial to simply review a few representative examples of the type of lingual complication that's being discussed.

The first example comes from a student who was asked to tell what they thought an indefinite integral was. It was obvious that the student had some misconceptions about integration in general, but in particular it's interesting to note that the student added a comment at the end of their statement about the lack of "precision" of the indefinite integral:

WLH: "What is an indefinite integral?"

Student 1: "Well the definite integral is the limit of the indefinite integrals, pretty sure. And then, the indefinite integral, I mean, it's more...it's more open-ended, less precise, obviously."

This response seemed to be pointing less to a misconception on the student's part and more toward their use of language. In our natural language, indefinite could very well be taken to mean "less precise," but within a mathematics context, the indefinite integral has nothing to do with precision of any kind.

The next example parallels the first, but introduces a new facet to the discussion. This student was asked about the difference between definite and indefinite integrals:

WLH: "So why would we do this [definite integral] as opposed to this [indefinite integral]?"
Student 2: "It's more accurate, I guess."
WLH: "Okay, in finding what?"
Student 2: "I'm not sure."
...
Student 2: "So like, a definite integral would maybe be the um, like, when it shows you two different points on the...and then indefinite would be just, a more, open, I guess."

This response also speaks to the student claiming some kind of, in this case, "accuracy" involved in integration, yet in this case there was no trace of understanding whatsoever. The student provided no clear reasoning as to why they thought a definite integral is more accurate since they couldn't even discuss what it is more accurate in finding. Again, this seems to be a case of the words being defined rather than the concepts.

In the final example, a student was asked to simply tell what they thought definite and indefinite integrals are:

WLH: "What is a definite integral?"
Student 3: "The definite integral, like, you KNOW [emphasis by student], like it's in-between these two, like it's definite."
...
WLH: What is an indefinite integral?"
Student 3: "Indefinite integral is where you don't...you're not sure, like, they don't give you as much information, like, they don't give you 'it's between here and here.' Like, you have to find it, so there's like the...it's less defined."

Responses such as these are very telling about a student's thought process: this student specifically stated that indefinite integrals are "less defined," which seems to be a direct association between the terms and the concepts.

Computational Fluency of "Language" Students

In order to compare the students who made some kind of lingual association in their interview with those those who did not, statistical analysis was performed on the page of integral computations completed by each student. What was found was that students who were categorized within the "Language" category in either definite or indefinite integrals scored, on average, 2.567 out of 4.000 on the computations whereas those who were not categorized within the "Language" category scored a 3.250 out of 4.000. Upon completion of of a two-tailed, unpaired T-test, we observed a p-value of 0.1137 which does not meet the accepted 0.05 level for significance. We therefore fail to reject the null-hypothesis that the means are equal. Even though it seems like there might be something more to investigate within the two groups, as far as this study is considered we cannot assume that the two groups are significantly different in terms of computational fluency.

Discussion

In answering the research question, “what, if any, impact does language have on students when they discuss definite and indefinite integrals,” it seems that the language of integration had an effect on some of the students’ ability to discuss the definite and indefinite integral. The fact that the students who seemed affected by the language of calculus performed as well computationally as those who did not leads us to believe that affected students are going largely unnoticed since computations are the mainstay of many calculus assessments.

So what does this mean in the grand scheme of integration? While student understanding of the topics is most definitely a factor and instruction methods can certainly remedy some of this confusion, students must also be able to communicate their ideas clearly and thoughtfully. Taking into consideration the results of this study, which showed the linguistic complexity of the terms surrounding integration, as well as the findings of Abedi and Lord (2001) concerning the simplification of examination questions, it may be that the various student misconceptions noted by the studies cited earlier (Bezuidenhout & Olivier, 2000; Orton, 1983; Sealey, 2006; Thompson & Silverman, 2000) have roots in, and therefore possible solutions within, the language of calculus. This is not to suggest that alleviating the problems seen in this study will solve every student’s calculus misconceptions, but that fluency in the language of calculus may provide a foundation upon which those solutions are built.

Secondly, we must take great care when introducing new terminology in calculus class. As noted by Raiker (2002) the teachers of mathematics are using significantly more terms than the students; therefore, we should take care that we are not enabling students’ misuse of language, such as the linguistic crossover between natural and mathematical registers that was seen in this study. Krussel (1998) makes great connections between speaking a foreign language and communicating in mathematics, he claims, “it is not simply enough to recognize and use the words and feel a comfortable familiarity with them. Building deep structures requires reflective thought and time to integrate the surface words and symbols into existing cognitive patterns so that meaningful mathematics may be spoken” (p. 9). It is the job of mathematics instructors to provide this time and the impetus for students to study the vocabulary and syntax of mathematics. We must take care and include time for vocabulary in mathematics class and, more importantly, we must assess students on their understanding and their ability to communicate that understanding. With added pressure from educators perhaps students can fully integrate into their vocabulary the terms they only have a cursory knowledge of and thus become fluent in mathematics. We would not expect a child learning English to read sheets of definitions of words and then be able to use them in an hour so why do we expect this kind of performance from mathematics students? Learning to speak mathematically takes time and effort, and only through regular and repeated use will students be able to communicate effectively.

Further study

This project has opened many doors to further research. Future researchers may be encouraged to study more deeply the connection between computational competence and linguistic issues. This study was built as an investigative vehicle, not necessarily to prod into the students' abilities, and so a project developed specifically for that goal would be more able to make solid conclusions. More knowledge is needed on where students' thinking is derived from, as, though not mentioned within these pages, there is anecdotal evidence that asking students "What's an indefinite integral?" elicits very different responses then showing them an indefinite integral and asking "What is this?"

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