Symbol Sense in Linear Algebra: A Start Toward Eigen Theory

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In this report we detail linear algebra students' interpretation of expressions and equations that are fundamental to eigen theory, prior to any formal instruction on eigenvalues and eigenvectors. Data for this analysis comes from semi-structured problem solving interviews with 13 undergraduate students as part of a semester-long classroom teaching experiment in linear algebra. We identified three main categories of student reasoning about the two equations: 1) students who used superficial algebraic cancellation, 2) students who correctly solved the system but were unable to interpret their result, and 3) students who correctly solved the system and correctly interpreted their result. More broadly, the research reported here lays the groundwork for characterizing student symbol sense in linear algebra.

Purpose and Background

An enduring challenge in mathematics education is to create instructional sequences in which students generate, refine, and extend their intuitive and informal thinking to more sophisticated and formal ways of reasoning. The design of such learning sequences typically begins with careful analyses of students' existing or informal knowledge that can be leveraged for the development of formal or conventional mathematics. In this report we continue in this tradition by detailing linear algebra students' interpretation of equations and expressions that are fundamental to eigen theory, prior to any formal instruction on eigenvalues and eigenvectors.

Prior research points to the many difficulties learners have with the formal and abstract concepts in linear algebra (e.g., Carlson, 1993; Dorier, Robert, Robinet, & Rogalski, 2000). Few of these studies, however, reveal promising ways that student reasoning can be built on to develop concepts and techniques for solving problems, especially in the mathematical terrain of eigen theory. In the context of differential equations, Rasmussen and Blumenfeld (2007) found that students were able to create their own, novel algebraic techniques for finding eigenvalues and eigenvectors before any instruction on these ideas. Inspired by this prior work, this research sought to reveal linear algebra students' informal or intuitive algebraic ways of interpreting and key algebraic equations and expressions central to the formal development of eigenvectors and eigenvalues. Revealing the range of ways that students interpret such expressions and equations have powerful implications for teaching and instructional design. More broadly, the research reported here contributes to a general characterization of symbol sense (Arcavi, 1994) in linear algebra.

Theoretical Framework

Arcavi (1994) describes a number of different facets that characterize what he refers to as "symbol sense" in the domain of high school algebra. An explicit definition of symbol sense, he argues, is not viable due to the complexity and variation in what is needed to effectively use and reason with various symbolic forms in algebra. He therefore describes several themes, the following five of which are particularly pertinent for this analysis of students' work with symbols in linear algebra: (1) An aesthetic feel for the power of symbols including an appreciation of what they can and cannot do, (2) A feeling for when to abandon the use of symbols and turn to other representational forms, (3) The ability to read symbolic expressions and equations, (4) The ability to engineer symbolic forms, and (5) A sense for the different roles that symbols play in different contexts. As will be described in the results section, these themes resonate with our findings of how students make sense of various symbolic forms in linear algebra.

Symbol sense could be studied in many different courses, so why study symbol sense in linear algebra? We point to two main reasons. First, linear algebra offers a particularly complex case of a need for symbol sense because of the change in symbolism that students encounter with this course. From high school courses or single-variable calculus, students are accustomed to equations such as $cx = d$, where all three variables take their values from the real numbers. However, consider the complexity in the following example, offered by Harel (2000). In the matrix equation $R\vec{x} = \vec{0}$, R is a matrix and both $\vec{x}$ and $\vec{0}$ are multi-component columns vectors. To find the solution (if one exists) to the aforementioned equation involves determining the values for each component of the vector $\vec{x}$ such that the associated system of linear equations, embedded with this matrix equation, has solutions. As this demonstrates, the commonly used symbolism in linear algebra entails a new level of complexity for students. Second, there is an abundance of research showing students have difficulty with linear algebra (e.g., Hillel, 2000; Sierpinska, 2000; Stewart & Thomas, 2009), and hence there is a need for research that details the subtleties that students often find problematic. There is a perhaps even greater need for research that investigates students' informal or intuitive reasoning that can be leveraged to develop more formal or conventional mathematics. The research reported here makes a contribution to both of these lines of research.

Methods

Data for this analysis comes from semi-structured problem solving interviews (Bernard, 1988) with 13 undergraduate students. The students were primarily engineering majors at a large southwestern university. The interview was the second of a series of three interviews that was part of a semester-long classroom teaching experiment (Cobb, 2000) in Linear Algebra. This particular interview was conducted after students had discussed geometric and algebraic interpretations of linear transformations, but before they had begun a unit on eigen theory. Each interview was videotaped and subsequently transcribed. We began analysis by writing detailed descriptions of each student's response to every question. In the spirit of grounded analysis (Corbin & Strauss, 2008), we then examined these thick descriptions for cross-cutting themes and patterns, going back to original videorecordings as necessary. We concluded the analysis by examining the similarities of our findings to Arcavi's characterization of symbol sense.

One of the first tasks the students were asked to respond to was the following: "Students read expressions in math differently, I'm curious to know how you read this expression $\begin{bmatrix} 1 & 3 \\ 6 & -1 \end{bmatrix}\begin{bmatrix} 2 \\ 3 \end{bmatrix}$." The intention of this question was to provide insight into how students think about matrix multiplication, which previous research has shown to vary substantially (Larson, Zandieh, Rasmussen, & Henderson, 2009). The next prompt asked students to explain how they thought about the equation $A\begin{bmatrix} x \\ y \end{bmatrix} = 2\begin{bmatrix} x \\ y \end{bmatrix}$, where A is a 2 x 2 matrix. The third prompt followed up from the second one as follows: "Now suppose A was this particular matrix $A = \begin{bmatrix} 4 & 1 \\ 2 & 3 \end{bmatrix}$. Now how do you think about $\begin{bmatrix} 4 & 1 \\ 2 & 3 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = 2\begin{bmatrix} x \\ y \end{bmatrix}$?" With this question students moved away from merely reading symbolic expressions and were prompted to engage in solving the equation. These questions form a foundation for eigen theory and provide essential foundational ways in which students think about expressions and equations in the domain of linear algebra.

Results

The analysis presented here focused on how students interpreted and worked with the equations $A\begin{bmatrix} x \\ y \end{bmatrix} = 2\begin{bmatrix} x \\ y \end{bmatrix}$ and $\begin{bmatrix} 4 & 1 \\ 2 & 3 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = 2\begin{bmatrix} x \\ y \end{bmatrix}$. Grounded analysis of student responses led to identification of three main categories of student reasoning about these two equations: 1) students who used superficial algebraic cancellation, 2) students who correctly solved the system but were unable to interpret their result, and 3) students who correctly solved the system and correctly interpreted their result. These distinctions led us to wonder if there was any correlation between how students responded to these questions and their final course grade. We conjectured that students in Group 1 would have lower course grades than students in Groups 2 and 3. Table 1 suggests that this conjecture has some validity, with those students who fell within Group 1 receiving lower final course grades than students who fell into Groups 2 and 3.

Student	Group	Final Grade
Bethany	1	B
Aaron		C+
Dave		C+
Zander		C
Jason	2	B
Darren		B-
Mark		C+
Mitchell		B+
Henry	3	A+
Josiah		A
Darnell		A
John		B
Martin		B

Table 1. Final grade for students per group

A Group 1 response was characterized as a superficial algebraic cancellation. These students tended to over generalize the use of algebraic cancellation. They did not have a sense of the different roles these symbols had within the context of Linear Algebra. They also had more trouble in interpreting the two equations as compared to the students in Groups 2 and 3. Students in Group 2 could make progress in interpreting the two equations and actually make progress in solving for x and y on the third question. However, despite finding the correct relationship between x and y in the equation $\begin{bmatrix} 4 & 1 \\ 2 & 3 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = 2\begin{bmatrix} x \\ y \end{bmatrix}$, they were unable to makes sense of their result. Students in Group 3, on the other hand, made progress similar to that of Group 2 and they had appropriate ways to interpret their results after solving the equation $\begin{bmatrix} 4 & 1 \\ 2 & 3 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = 2\begin{bmatrix} x \\ y \end{bmatrix}$.

Sample responses from Group 1 students

With respect to the two questions that we analyzed, four of the 13 students primarily operate from a rote algebraic computational view. This view relies upon mechanical cancellation in order to simplify the expression for interpretation. By this we mean that students, when presented with $A\begin{bmatrix} x \\ y \end{bmatrix} = 2\begin{bmatrix} x \\ y \end{bmatrix}$, cancelled the vector $\begin{bmatrix} x \\ y \end{bmatrix}$ and then tried to make sense of how the matrix A could equal the number 2. In several cases, the students interpreted the new expression, A=2 as meaning that the determinant of A equals 2. For example, when Bethany was asked how she interprets $A\begin{bmatrix} x \\ y \end{bmatrix} = 2\begin{bmatrix} x \\ y \end{bmatrix}$, she responded, "A would be equal to 2, because $\begin{bmatrix} x \\ y \end{bmatrix}$ is equal to $\begin{bmatrix} x \\ y \end{bmatrix}$. And I guess it's the determinant of A." Her rationale for canceling the vector $\begin{bmatrix} x \\ y \end{bmatrix}$ makes her a classic case of over-generalizing the use of algebraic cancellation. In this case, Bethany used computational methods from algebra on the real number line to simplify the expression. She had no problem with the fact that there is no division defined for transformation in her class or in operations on matrices and hence the operation is illegal. Instead she relied on her intuitions and finds a linear algebra explanation, the determinant, that may explain her simplified result.

Bethany's rationale exemplifies many aspects of group 1's approach to the first question. However, she is somewhat extreme in her sole reliance on computational methods to make sense of the questions given. Dave also responded similarly when given $A\begin{bmatrix} x \\ y \end{bmatrix} = 2\begin{bmatrix} x \\ y \end{bmatrix}$, however, Dave shows that his interpretation of the det A being 2 is more problematic for him. Dave responded, "Well, I think. The first thing I think of is A = 2, hey, how simple! Then I realize that a matrix is not a single integer but I know the determinant of a matrix can be that." Dave's response is similar with respect to using division to simplify the expression. Dave's response differs from Bethany's in that Dave analyzed his response and realized that the result of the simplification does not match what A is defined as, a 2 x 2 matrix. He stated that this is why he decides on the determinant being 2 as his answer. Dave recognized that there was a problem in his cancellation that required him to craft an interpretation of what A=2 means, and his solution was to mean that a function was being applied to the matrix A.

Although both individuals exemplify group 1's algebraic cancellation approach, the subtle cognitive dissonance that occurs when interpreting the result of the simplification makes a difference in responding to the next question of how to interpret the previous equation when given the matrix A: $\begin{bmatrix} 4 & 1 \\ 2 & 3 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = 2\begin{bmatrix} x \\ y \end{bmatrix}$. Bethany stated, "Can I find the determinant [computes determinant] 10 times $\begin{bmatrix} x \\ y \end{bmatrix}$ equals 2 times $\begin{bmatrix} x \\ y \end{bmatrix}$…because A equals 10, and so you replace the A with the 10…I just think, does not equal".

$$10\begin{bmatrix} x_1 \\ y_1 \end{bmatrix} \neq 2\begin{bmatrix} x_2 \\ y_2 \end{bmatrix}$$

It is important to note that Bethany does not in fact abandon her notion that A must equal two and this must mean that the determinant equals two. When she finds the determinant of the provided matrix is 10, she attributes this to the equality being invalid. She reinterprets the question to ask if the equality was in fact possible given this particular matrix A. This allows for her to maintain her conviction that the cancellation is correct and still have the determinant for A not be two. The result of the determinant not being two is thus not problematic

This can be contrasted with Dave's response to the second question. Dave has already demonstrated some cognitive dissonance with the algebraic cancellation. His choice to make the determinant being the operation acting on A that makes it equal to 2 is a rationalization of his finding, and so can be used or discarded depending on if it fits in other scenarios. The consequence of his flexibility with his rationalization are clear in his response. "Clearly, there is some other operation that's going to make this turn into 2, because the determinant of this is about 10…something, whatever we're doing to A, makes it 2. I don't know what it is, it's not something I've done before, so that's my variable for it".

$$☺A = 2$$

At this point, his rationalization for how A could equal 2 is what is problematic. This result arises from his original assertion that there is no way that the matrix A can equal two. The determinant is an operation on the matrix A, and so there must be another kind of operation that will in fact make A equal to two. He resolves this by naming a function "☺" to be the "something" that makes A the number 2. Similar to Bethany, Dave did not abandon his result of the simplification. Unlike Bethany, Dave decided that it is the operation on A that is incorrect and not the equal sign. Dave attested to this fact by deciding that there must be something that he does not know or understand that is creating the mismatch between his actual result and his desired result. Dave interprets that his explanation from the previous result must be wrong and that is what is causing the mismatch. Bethany, on the other hand, chose to maintain the correctness of the response to the first question and then used that assumption to maintain that the equality in the equation must be the incorrect.

In both cases, the students used simplification based upon cancellation of the vector $\begin{bmatrix} x \\ y \end{bmatrix}$ to get their result. These students show a symbol sense for linear algebra that assumes operations defined in linear algebra are similar or the same to those used in earlier courses using algebra on the real line. Rather than question there sense of the symbolic expressions and operations on those expressions in the questions, these students instead reinterpreted the question to fit their symbol sense, as is the case with Bethany. Or they assumed that there was some unknown knowledge that facilitated a fit between their symbol sense and the results that they were attaining, as was the case with Dave. In either case, these students displayed a lack of knowledge of the new meanings for the symbolic expressions at play in their linear algebra class.

Sample responses from Group 2 students

Another four of the 13 students interviewed were able to correctly solve the associated system of linear equations, but could not interpret what their solution meant. In several cases it caused students to second guess themselves and believe they must have made an algebraic mistake. Group 2 students were able to get beyond the idea that A should be a single number and into interpreting the system of equations associated with the matrix equation. However, these students tended to be confused by finding that $y = -x$ and were not able to further interpret this result.

When asked how to interpret the equation $A\begin{bmatrix} x \\ y \end{bmatrix} = 2\begin{bmatrix} x \\ y \end{bmatrix}$, Jason responded, "I don't know if that means the determinant of A = 2? I don't know. [pause] Somehow you're just transforming the xy so it's double the size." Jason's initial reaction, that the determinant of A=2, was similar to the algebraic cancellation suggested by students in Group 1. However, he showed symbol sense beyond that shown by Group 1 students by disregarding this idea and reinterpreting the symbols in terms of a transformation that doubles the size of a vector. Jason continued, "That matrix A which is a 2 by 2…so you just have to find values for a,b,c,d so then it comes out so you have 2x and 2y. So c would equal 0, would equal 0…and a would have to be 2, then d would have to equal 2." In this way he explained that the matrix $\begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix}$ would make this equation true. Note that although Jason did not say this, $\begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix}$ is a matrix that works for all *x, y*. It was not explicit in the statement of the question whether that the matrix had to work for some *x, y* or all *x, y*. The next question to students however, provided a specific matrix, that would not work for all *x, y*. For the specific matrix, Jason applied his knowledge of how to convert a matrix equation into a system of equations. "So I solved for one of the variables. So $y = -x$. But yeah, the equation, it works, but I don't know what x and y is, which is weird; I don't know how to find them. But your y would change, depending on x. Which means there's not a unique solution, I guess."

$$4x + 2y = 2x \qquad 2y = -2x \qquad y = -x$$
$$x + 3y = 2y \qquad x + 3(-x) = 2(-x)$$
$$-2x = -2x$$

Jason's solution and his interpretation -- y would depend on x, there is not a unique solution -- are algebraically correct. However, he was not able to further interpret his solution in terms of the original matrix equation. In other words, although in the $A\begin{bmatrix} x \\ y \end{bmatrix} = 2\begin{bmatrix} x \\ y \end{bmatrix}$ problem he realized that A is a transformation that is doubling the size of the vector, in this problem, he did not recognize that $y = -x$ describes the set of vectors that are doubling under this specific matrix transformation.

Mitchell provides another example of a Group 2 response. Initially Mitchell stated "'I'm just going to cancel out and say A = 2,' but A is a 2 by 2 matrix, how can a matrix equal a single value?" Similarly to Jason, he considered the problem in a way consistent with Group 1 student, i.e., to consider that A=2. However, he recognized that that interpretation did not match his symbol sense for matrix equations. He continued, "So that doesn't make sense. Basically what we're saying is that this [points to matrix A] somehow or other this is causing xy, basically doubling in magnitude."

$$\begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 2x \\ 2y \end{bmatrix}$$
$$A_{11}x + A_{12}y = 2x$$
$$A_{21}x + A_{22}y = 2y$$

He then wrote out the associated system of equations and found the matrix $\begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix}$. Mitchell's answer to the next question provides an example of symbol sense where a student engineers a new symbolic form. For some reason Mitchell did not initially write out a system of equations, but used a non-standard augmented matrix form for the system.

$$\begin{bmatrix} 1 & 3 & 2y \\ 4 & 2 & 2x \end{bmatrix}$$

The standard augmented form should not have a *2y, 2x* in the final column but rather only numeric values. Mitchell then completed a series of calculations to put his augmented matrix in row reduced echelon form. Interpreting his echelon form, he found $y = -x$.

$$x = 2y - \frac{3}{5}(4y - x)$$
$$y = \frac{1}{5}(4y - x)$$
$$5x = 2y - 12y + 3x$$
$$5y = 4y - x$$
$$y = -x$$

We label Mitchell's calculation as an example of engineering a symbolic form because he was not simply following a standard algorithm but rather creating a variation on that algorithm that turns out to work for his situation. Upon finding $y = -x$, Mitchell wondered, "Why would I have that? Why does that...did I make an arithmetical error?" We take this cognitive dissonance as an indicator that Mitchell was unsure how to interpret his result. Similarly to Jason, Mitchell had answered the first problem with the realization that the matrix caused the vector to double in magnitude, but did not apply this type of interpretation to the result of his calculation to the question where the matrix was given.

Sample responses from Group 3 students

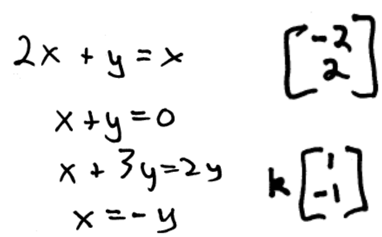
Finally, we found that five of the 13 students were able to not only solve for $y = -x$, but—in contrast to Group 2 — were also able to interpret this result in terms of the matrix equation. Interestingly even Group 3 students began their response to the $A\begin{bmatrix} x \\ y \end{bmatrix} = 2\begin{bmatrix} x \\ y \end{bmatrix}$ problem with the type of thinking evidenced by Group 1.

Darnell, when asked the first question responded, "I guess A must equal 2. No, A has to be a matrix where it only stretches x by 2 and stretches y by 2 [writes a matrix $\begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix}$]. Two times the identity matrix." Notice that although this is not mandatory for inclusion in Group 3, Darnell goes beyond the notion of the vector doubling that was mentioned by Jason and Mitchell and talks about the more geometric imagery of "stretching". In answering the second question with the specific matrix given, he stated, "I know I'm going to get a new matrix over here, a 2 by 1, that's going to be dependent on values of x and y. These look like equations and lines here [solves equations], so $x = -y$. I think that's going to be the relationship that makes this [points to $A\begin{bmatrix} x \\ y \end{bmatrix}$ from previous problem] equal to that [points to $2\begin{bmatrix} x \\ y \end{bmatrix}$]. Because whatever y is, x is the negative of y."

$$4x + 2y = 2x \Rightarrow y = -x$$
$$x + 3y = 2y \Rightarrow y = -x$$

In contrast to the Group 2 students who were confused by finding that $y = -x$, Darnell said, "yeah, these are the points that make the relationship true," and pointed explicitly back to the expression $A\begin{bmatrix} x \\ y \end{bmatrix} = 2\begin{bmatrix} x \\ y \end{bmatrix}$. Darnell was able to interpret the expression $y = -x$ not only in the context of solving a specific system but in terms of the matrix as a transformation of a vector. This more sophisticated symbol sense is what distinguishes the students in Group 3 from the students of Group 2.

A second Group 3 student, Josiah, began similarly by beginning with a Group 1 idea (i.e., A is 2) and then using his more sophisticate symbol sense to reinterpret the equation more appropriately for this context. "A is a matrix and 2 is just a number. So in my head, I was trying to think about how a number could be a matrix, but that's kind of from the fact that an identity matrix times a vector results in just the vector. So 2 times the identity matrix should result in 2 times the vector." So, like other some other Group 2 and 3 students, he came up with the matrix $\begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix}$ but in a distinctive way. "I think if you gave values to *x* and *y*, then other matrices could work." He was already seeing before working on the next problem that if it is for a specific *x*, *y* there could possibly be another matrix that satisfies the equation. For the second problem he rewrote the matrix equation as a system of equations to find $y = -x$. Then he interpreted this by saying, "I guess it means that if you want this matrix to double your vector, you have to use this vector [writes $k\begin{bmatrix} -1 \\ 1 \end{bmatrix}$]."



Like the other four students in this category, Josiah was able to correctly solve the system of linear equations and interpret what the solution $y = -x$ means in terms of the transformation indicated by the matrix equation. Josiah, even more so than Darnell, made explicit the set of vectors for which this matrix would double the vector.

The following table brings together examples of several symbol sense themes that were illustrated in the student work above.

Theme	Student illustration
The ability to "read" symbolic expressions and equations	"Somehow you're just transforming the xy so it's double the size." –Jason
	"Then x equals –y. I guess it means that if you want this matrix to double your vector, you have to use this vector." –Josiah

	$k\begin{bmatrix} -1 \\ 1 \end{bmatrix}$
The realization of the need to check symbol meanings	"The first instinct to say, 'I'm just going to cancel out and say A = 2,' but A is a 2 by 2 matrix, how can a matrix equal a single value? So that doesn't make sense." -Mitchell
Sensing different roles that symbols can play in different contexts	
The ability to "engineer" symbolic forms	$\left[\begin{array}{cc\|c} 4 & 1 & 2x \\ 2 & 3 & 2y \end{array}\right]$ -Mitchell

Arcavi describes the ability to read symbolic expressions not just as the ability to interpret a symbolic expression that is provided, but also the ability to interpret a symbolic expression that has been personally derived. An example of the former would be when students made comments interpreting $A\begin{bmatrix} x \\ y \end{bmatrix} = 2\begin{bmatrix} x \\ y \end{bmatrix}$ such as "Somehow you're just transforming the xy so it's double the size." An example of the latter would be students interpreting their solution to the system of equations in the second problem. For example, when Josiah says "Then x equals –y. I guess it means that if you want this matrix to double your vector, you have to use this vector," pointing to $k\begin{bmatrix} -1 \\ 1 \end{bmatrix}$. Another theme of symbol sense is realizing the need to check symbol meanings. We grouped this with the theme of sensing different roles that symbols can play in different contexts. In our data students illustrated these two themes together. For example, Mitchell said "The first instinct to say, 'I'm just going to cancel out and say A = 2,' but A is a 2 by 2 matrix, how can a matrix equal a single value? So that doesn't make sense" This is an example of these two themes. Here, Mitchell realized the need to check his symbolic meaning. First Mitchell thought, A is 2, then he checked whether or not that makes sense in the symbolic context. The realization that symbols have to make sense within this context is very important in Arcavi's framework. We infer, because the students did not explicitly state, that the students first consider A=2 because this equation is so similar to equations in high school algebra. In high school algebra $ax = 2x$ implies $a = 2$. However, realizing that the A in our context is a matrix illustrates a pivotal role of symbol sense. The final theme summarized in the table is the ability to engineer symbolic forms. Whenever people are creating symbolic forms you could call it engineering. Engineering can refer to both very unique uses of symbols and also to less unique student generated symbol use. Mitchell's augmented matrix, is an example of a unique engineering of new symbols, whereas other symbol creation by students in our data would be examples of engineering symbols in more standard ways, but ways which were nonetheless created by students to serve the students mathematical purposes.

Concluding Remarks

In this paper, we reported on students' various interactions with and interpretations of the equations $A\begin{bmatrix} x \\ y \end{bmatrix} = 2\begin{bmatrix} x \\ y \end{bmatrix}$, where A is a 2 x 2 matrix, and $\begin{bmatrix} 4 & 1 \\ 2 & 3 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = 2\begin{bmatrix} x \\ y \end{bmatrix}$. The mathematical symbols involved in these two seemingly simple equations—matrices, vectors, and scalars—add a layer of complexity to the interpretation of equations that is not present in examples such as $ax=2x$, where both a and x are real numbers. Investigating how students try to make sense of these matrix equations prior to any formal instruction regarding eigen theory provides vital information regarding improvement of the teaching and learning of linear algebra. Knowledge of common interpretations of mathematical symbols and equations, such as those presented in this report, are essential for developing a strong foundation for thoughtful instruction that builds on student thinking. Therefore, the analysis presented here provides critical information as we move forward in thinking about possible ways to account for differences regarding student understanding and dispositions in linear algebra.

The core analytical framework utilized in analyzing student responses was that of categorizing students' symbol sense with matrix equations, and we identified three main categories of student reasoning: 1) students who used superficial algebraic cancellation, 2) students who correctly solved the system but were unable to interpret their result, and 3) students who correctly solved the system and correctly interpreted their result. We conclude by suggesting additional ways to possibly account for differences in student responses. First, we contend that one way that student interpretations of the matrix equations were affected is whether or not the students were thinking about a matrix as a function that transforms one vector to another vector. This seemed to be a strong distinction between the different groups, especially between Group 1 and Group 3. A second possible way to account for distinctions in students' interpretation of the matrix equations seems to be students' ability or inability to interpret the situation geometrically. When asked, students in Group 1 reported that they had no or very few ways to think about the concepts geometrically, while students in Group 3 made comments such as, "I can think about this vector being stretched." Interpreting the matrix as a transformation acting on a vector to produce a geometrically transformed vector—one that was twice as long as the original—was an essential interpretation found in students from Groups 2 and 3.

Finally, a third possible way to account for student distinctions in symbol sense with matrix equations is by characterizing students' understanding of matrix multiplication, as this also seems to be a potentially important difference between the groups. In terms of matrix multiplication, some students had to convert a matrix times a vector into a systems of equations, which is a very computationally-driven approach. This view is very different than interpreting a matrix times a vector as the components of the vector as weights for the column vectors of the matrix, whose linear combination produce a resultant vector. These various interpretations of matrix multiplication and implications they may have for the learning and teaching of linear algebra are pursued in Larson (2010). In our own work, these conjectures provide an avenue of further research as we conduct larger scale analyses of our current semester-long classroom teaching experiment in linear algebra.

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