**PRELIMINARY RESEARCH REPORT**

## Mathematics Undergraduates' Experience of Visualisation in Abstract Algebra: The metacognitive need for an explicit demonstration of its significance

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*Abstract Algebra is considered by students as one of the most challenging topics of their university studies. Our study is an examination of the cognitive, social and emotional aspects of mathematics undergraduates' learning experience in Abstract Algebra. Our data consists of: observation notes and audio-recordings of lectures and group seminars of a Year 2 course in the UK; student and lecturer interviews; and, coursework and examination papers. Data analysis is currently in progress. For the purposes of this paper, following some of our preliminary observations on the students' apparently diminishing engagement over the ten weeks of the course – and, particularly, their comments on the effect that the abstract, not easily visualisable nature of Abstract Algebra has on their relationship with the topic – we scrutinize the data sources listed above for evidence of their perceptions about / attitudes towards / employment of visualization in Abstract Algebra.*

Abstract Algebra is one of the mandatory courses taught usually in the second year of a degree in Mathematics and is typically considered by the students as one of the most challenging. Often, after their first encounter with Abstract Algebra, students avoid third-year or further courses in this area of mathematics. Previous research (e.g. Nardi, 2001) attributes student difficulty with Abstract Algebra to its multi-level abstraction and the less-than-obvious, to students, *raison d'être* of concepts such as cosets, quotient groups etc. Furthermore Abstract Algebra is amongst the first courses in which students are not able to cope with by just memorising formulas or by "just learning 'imitative behavior patterns'" (Dubinsky et al, 1994, p268).

### The aims, context and methods of the study

The ongoing study we draw on here is the doctoral study of the first author and aims to examine closely as many facets – cognitive, emotional and social – as possible of Year 2 mathematics undergraduates' learning experience in Abstract Algebra. At the centre of our data collection is a course currently taught in a well-regarded mathematics department in the UK. Data collection took place in the Spring Semester of a recent academic year. The course was mandatory and attended by 78 students (10 weeks, 20 hourly lectures, 3 cycles of seminars, in Weeks 3, 6 and 10). The role of the seminars was mainly to support the students with their coursework. There were 4 seminar groups, with about 20 students in each. In each seminar group there was a seminar leader, a full-time faculty member of the department, and a seminar assistant who was a PhD student. All members of the teaching team had related research interests. The students submitted the coursework at the end of the semester. This was marked and returned to them soon after.

In the lectures the lecturer, a very experienced mathematician, was writing extensively on the chalkboard and was commenting orally alongside. In the seminars the students were expected to work on problem sheets, distributed to them earlier in the preceding weeks, and arrive having prepared questions. They had the opportunity to ask the seminar leaders and assistants anything they had encountered difficulty with and to receive help. The lecturer was also available during 'office hours' for the same purpose.

Data was collected by the first author and consists of:

- **Lecture observation notes**. These covered: record of student attendance; instances of interaction between the students and the lecturer; verbal, body or other evidence of student (dis)engagement and emotional response to the lecture; and, general observations of lecturer and student behaviour.
- **Lecturer notes**: notes of what the lecturer was writing on the blackboard.
- **Audio-recordings** of the 20 **lectures**.
- **Audio-recordings** of 24 **seminars** (2 recorders in each of the 12 seminars; one on the seminar leader and one on the seminar assistant) in which we have captured all conversations with students during which they predominantly discuss difficulties with certain items in the problem sheets.
- **Interviews with** 13 out of the 78 **students** who made themselves available on a voluntary basis, the 4 **seminar leaders and assistants and the lecturer**. There were three cycles of student interviews, at the beginning, the middle and the end of the course, in which students discussed their learning experience in Abstract Algebra. The discussions with the lecturer, who was also one of the seminar leaders, covered learning and teaching issues as well as institutional and administrative issues. Interviews with the seminar leaders and assistants were mostly about their discussions with the students during the seminars, and their general views on pedagogical issues.
- **Student coursework**. Students were given three problem sheets in Weeks 2, 5 and 9. They had to work on these before the seminars on Weeks 3, 6 and 10. They had to work on all problems, but they had to hand in only a selection of these in Week 12. The selection of problems to be assessed was announced to them after each seminar.
- **Marker (seminar assistant) comments on student coursework.**
- **Student examination scripts** collected at the end of the academic year.

Data analysis is currently in progress. At the time of writing, interview recordings have been transcribed and the contents of the lectures and seminar recordings have been summarised. In the spirit of Data Grounded Theory all interview transcripts and lecture/seminar summaries have been scrutinised for the purpose of identifying preliminary themes (Strauss & Corbin, 1998) and – as part of the first author's apprenticeship in educational research methods – appended with short reflective accounts on methodological issues. Here we report initial analyses regarding one of the themes that emerged in the above process: *Visualisation in Abstract Algebra*.

## Visualisation and Engagement in Abstract Algebra

Our motivation to focus on this particular theme originates both in the literature – regarding students' typically problematic relationship with visualisation (e.g. Presmeg, 2006)

– as well as the first, strong impressions from our data. During an initial scrutiny of the lecture data (audio recordings and notes) amongst the eight themes that emerged (Ioannou & Nardi, 2009b) were: the **diminishing student engagement** over the ten weeks of the course; and, the variable responses to uses of **visualisation** in the lectures. Evidence on the former (lecture observation notes and the student and lecturer interview transcripts) suggested what we listed in (ibid, p37) as: a *pathology of absence* (decreasing student attendance); a *pathology of presence* (increasing disengagement of those still attending); and, *explicit student expression of emotion* (statements of disengagement increasing gradually in frequency and power over the three cycles of interviews).

Amongst the 89 explicit and substantial statements of emotion, made in the 39 interviews conducted with 13 of the 78 students were several statements (ibid, p39-40) concerning the students' – often self-pronounced as futile – attempts to bestow meaning to the new ideas they are being introduced to. These statements often focused particularly on attempts to do so through constructing 'appropriate visual imagery' (Zazkis et al, 1996) and contained evidence on **how the students' relationship with (and achievement in) Abstract Algebra was being affected by the difficulty to visualise**. In some of the students' statements cumulative 'local-affect' (Goldin, 2000) experiences (initial optimism followed by moments of increasing frustration) result, particularly towards the last interviews, in an overall sense of hardship and frustration. In this sense the difficulty to construct pictures-in-the-mind can be seen as having an adverse emotional impact on the student's engagement with the subject (Ioannou & Nardi, 2009a).

As we are particularly interested in the intertwinement of cognitive and socio-affective aspects of the students' experience – as Goldin (2000) proposed, affect is 'critical to the structure of competencies accounting for success or failure' (p211) – we are currently examining data from across the eight sources listed above to trace evidence of this relationship. Our emphasis – much like Keith Weber's (2008) – is on student affective responses and learning behaviour (written and verbal; in interviews, coursework and exams). In Goldin's (2000) terms the emotional states recorded in the interviews we refer to above, while localised in terms of time (they are about specific moments) and in terms of context (they are about specific aspects of Abstract Algebra activity), may evolve into longer-lasting, globally 'unfavourable' emotional structures. Our current analysis aims to trace these structures, and their impact on students' competence in Abstract Algebra, across all other data sources (e.g. coursework and exam papers).

Specifically, while students (as evident in the interviews we refer to above) repeatedly express – often with intense emotion – **a need for 'pictures'** that will illuminate the nature of the novel to them Abstract Algebraic objects, our overall impression, accrued during data collection, was that they were **at the same time reluctant to attempt a construction of such images** (e.g. as evident in the scarcity and limited quality of the visual imagery in their written work) or **engage with the images on offer** in the lectures and the seminars (e.g. as evident in the lecture observations). To substantiate and elaborate – or, if necessary, refute – this impression we sought further evidence in the data and identified: 42 episodes related to visualisation in the 39 student interviews with 13 of the 78 students (including 31 instances of visualisation occurring mostly during the second cycle of interviews); 19 instances of visualisation in the 10-week coursework of the 78 students; and, 18 instances of visualisation in the exam papers of the 78 students.

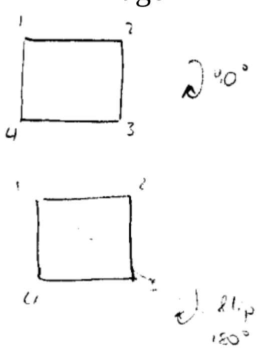
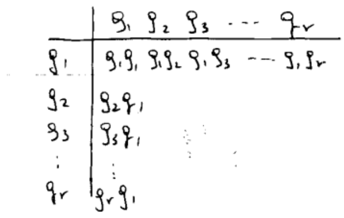
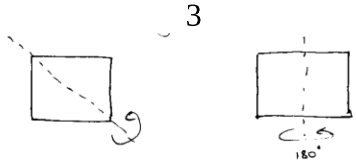
Here we exemplify from the visualisation data listed above and comment on these data examples with regard to the following questions:

- Do the lecturer/seminar leaders employ/refer to/encourage visualisation in their Abstract Algebra teaching? If so, how? If not, why?
- Do the students use/refer to visualisation in their writing/speaking? If so, how? If not, why?

For the purposes of our study we adopt the definition of visualisation given in Zazkis et al (1996, p441) describing visualisation as "an act in which an individual establishes a strong connection between an internal construct and something to which access is gained through the senses." Additionally, these authors emphasize the duality of visualisation, for it may consist either "of any mental construction of objects or processes than an individual associates with objects or events perceived by her or him as external" (p441), or "of the construction, on some external medium such as paper, chalkboard or computer screen, of objects or events that the individual identifies with object(s) or process(es) in her or his mind." (p441) They also note that visualization is not the final product of what is done, but rather a process of doing; visualization consists of those constructions that make the association between mental and external objects possible.

**Lecturers' use of visualisation: substantial but implicit**

Scrutinizing, first of all, the lecture notes we noticed that during the 20 lectures in which the course was taught, the lecturer used a total of 14 illustrations. We classify these illustrations into two distinct categories. The first one, which included rotations of a square (twice), multiplication table of a group, rotations of a regular pentagon and a regular hexagon as examples of dihedral groups and symmetries of a cube, is a list of external objects that were represented graphically in order to facilitate students' seeing of these objects (Fig. 1).

| Image 1 | Image 2 | Image 3 |
| --- | --- | --- |
|  |  |  |

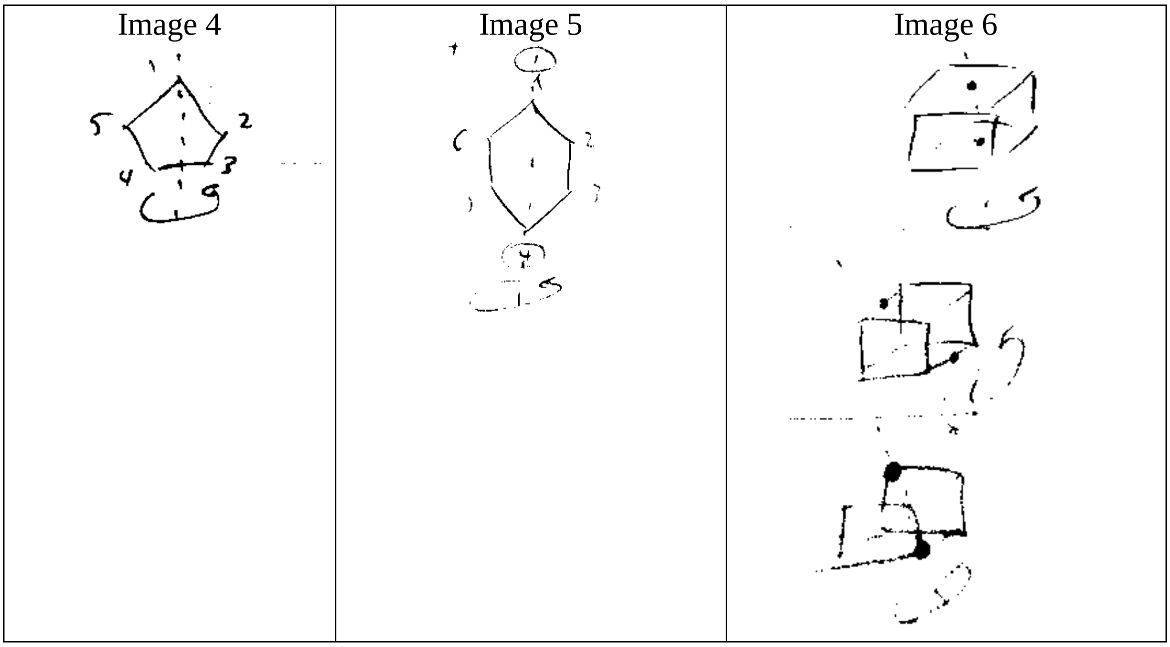
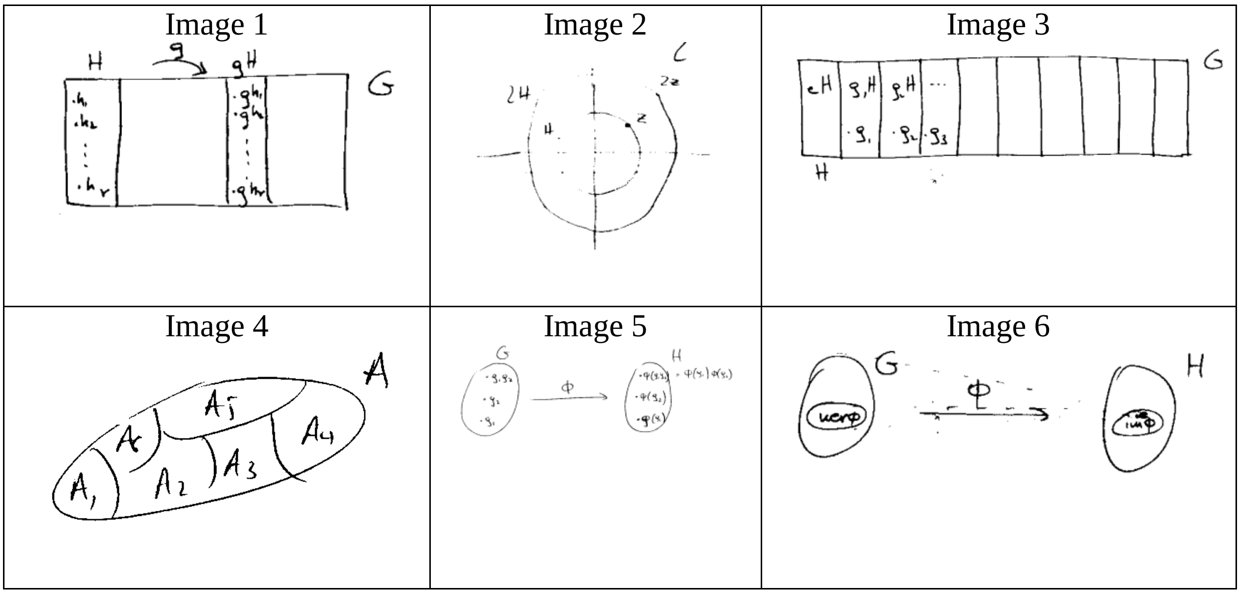



Fig. 1 Images used by the lecturer, Category 1 (external objects)

The second category, which included an image of coset, example of a coset involving complex numbers and Argand diagram, a Venn diagram as part of Lagrange's Theorem's proof, an illustration of equivalence classes, a representation of homomorphism, a representation of homomorphism including the kernel and image and an image of a quotient group, is associated with the visualisation of abstract concepts of Group Theory (Fig. 2).



| Image 7 | Image 8 | |
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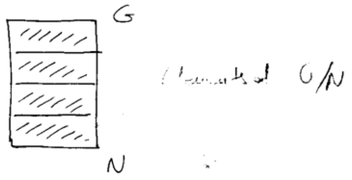
|  |  | |

Fig. 2 Images used by the lecturer, Category 2 (mental objects)

The lecturer overall did not give any particular emphasis on these illustrations; he treated them merely as tools of clarification and demonstration. During the twenty lectures he made no explicit comments about the significance of these images and the potential assistance they could be to the students in their efforts to understand the newly introduced notions of Abstract Algebra Nowhere in the lectures. The only **allusion to this significance** was **implicit** in his use of them. It seems to us that it is partly because of this implicitness that this significance may elude the students who – in an apparent contradiction – make no use of any of these images in the interviews (even after being explicitly requested to do so) while they explicitly express a desire to have and use images of the newly-taught concepts. We discuss this issue in more detail below.

During the seminars, the majority of the seminar leaders/assistants referred to the symmetries of cube and encouraged the students to make a cube and play with it. At least one of the seminar leaders was more graphic in his descriptions and employed several metaphorical expressions which often amounted to vivid images, for instance, concerning the formation of cosets and the way quotient structures work.

**Students' use of visualisation: limited but explicitly needed**

Students' need to visualise concepts such as group, subgroups, coset etc. was quite evident in the interviews. As we mentioned above, when they were asked to draw on paper any images they have in their minds of certain notions in Abstract Algebra, most of them could produce none and could not recall any of the illustrations used by the lecturer in the lecture and in his notes. Asking to draw on paper their images of Group Theory concepts quite often would cause hesitation, even anxiety, and stress. The majority of students offered an image of group and subgroup, most of the time using Venn diagrams and, more rarely, using multiplication tables which were not always correct or complete. Only three gave a representation of homomorphism; one offered both an illustration with an example and the other two just an illustration (Fig. 4). All were similar to the one given by the lecturer.

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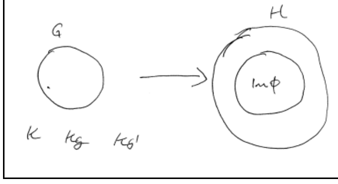
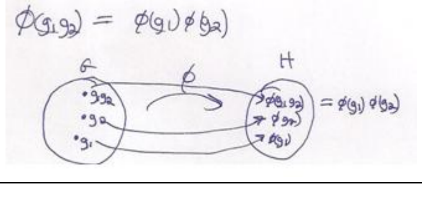
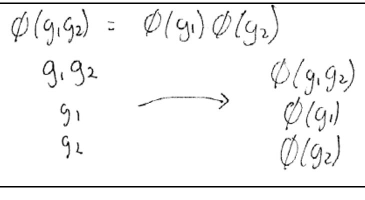
|  |  |  |

Fig. 4. Students' illustrations of a homomorphism in Group Theory

Even more reluctant were the students to give illustrations of the concept of coset although an image was given in the lectures. Only 2 of the 13 students gave a substantial, but not always clear, image of cosets (Fig. 5):

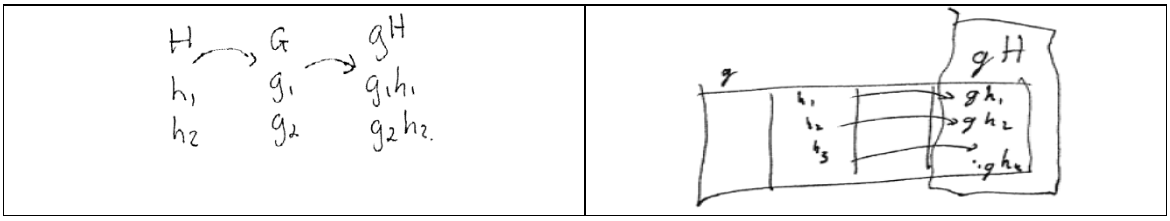



Fig. 5. Students' illustrations of a coset in Group Theory

Scrutinizing further the student coursework and exam scripts we deduced that **the image** of cosets given by the lecturer **did not have the impact he intended**, since it was **neither noticed nor adopted** by the great majority of students.

> S: A coset... I don't really know... I don't – I'm not too sure in my head at the moment, about a coset, it's the thing I struggle most with, like, with um... it's cosets, when I – I don't think I can picture them, and I think – cos obviously I'm a visual learner, I learn – I'm better if I can picture in my head, but a coset, at the moment, I'm just... a bit... LH2

In our view the somewhat oxymoron phenomenon of **students' expressing an urgent need for visualisation but missing it when on offer** may occur because of two reasons, both identified in previous research (e.g. Nardi, 2008, p139-151): first, using pictures and representing concepts with images is not part of the typical way that this kind of course is taught. Having mental pictures and using them is not directly encouraged by the lecturers. Secondly, students do not feel confident to use their mental images in the 'formal' solutions of their coursework: drawing and using them as part of their solution is not perceived as adding rigour or formality in their writing and therefore they avoid doing so.

> S: Erm... see, there – this thing – I mean – the – statement, makes sense... I drew a little picture and like – I was just like – I mean – course that's going to be in it, but – how you prove that by actual kind of – prove that mathematically rather than just drawing a picture and just saying, it is true, it's just the actual showing that... *(He is talking about a certain exercise)* MH2

Proof is one of the most important parts of mathematics and especially of university level mathematics. Moreover, as highlighted in previous research (Hanna and Sidoli, 2007) and on evidence such as in the following excerpts, **the ability to visualise plays a significant role in comprehending and constructing proofs**.

> S: Yeah, and like... in the – in the proofs as well, it's like – oh, but the – that means this, and it's just trying to understand just why that means that, and because I can't see it in my head, and I can't visualise it, it just – I don't see why they're so you know – like it goes and therefore this, and I'm just like – but why? It just – I think that – more than sort of other courses it takes a lot more thinking about, and it's not laziness, it's just – it *(I: Definitely, yes)* there's just so much more to get your head round. And so I don't hate it, it's probably – it's quite useful to look at, on paper, but – it's just a lot

> – more thinking and getting my head round than some of the other things. And – it's just human tendency isn't it, to prefer things that you find easier, isn't it, to be honest, yeah! KL2

And specifically, on the proof of Lagrange's Theorem:

> S: Yeah, cyclic groups is ok, I can do that, um... I guess it's getting onto more of the sort of like – Lagrange's theorem, and things, and um, it just – yeah... trying to follow through this proof is just like –
>
> I: Lagrange's theorem...
>
> S: Yeah, it's crazy, it's just like – I'd – yeah... I mean that – that again, that made sense cos it's in the diagram, so you can kind of see it, like where it's all – where it's all split up, yeah? KL2

The students often highlighted that having an image in one's head (which sometimes can be just a hunch or just a vague impression) is **at quite a distance from having a detailed and workable image**. The ability to put this image on paper is a first step in this transformation towards achieving a workable and precise image:

> S: Yeah, definitely, vector spaces can be quite difficult when you try to visualise it in your head when you are working with it on paper because you can't always draw a picture and things like that... NT1

Being able to develop mental images does not necessarily mean that these images are workable and in use. One needs to fully encapsulate them.

> S: Yeah, it's like – drawing it is fine, but then you still have to see in your head like how it's – rotating, *(I: Rotating...)* and how it's going – one of um, the people that I sit near, in the course, he actually said that he was gonna think about making a cube, and like put string on it and just – rotating it so he could see it better, and…. – LH2

**Images of course are subjective, as is the choice to use them** or not. Some students prefer not to use, or express no need to use images. In the light of previous, categorisations (e.g. Zazkis et al, 1996: visualizers, verbalizers and mixers), we have the impression that – while overall the students we are examining more closely in this study have both abilities to visualise and verbalise in more or less equal measures, apply these abilities according to the situation and have no overt preference for one of the two ways of expression – there are some who distance themselves from visualisation quite intensely:

> S: No, I just – I'm really not a visual person, I find it really hard to draw stuff and... do it that way, I learn it by the words, like the definitions and stuff, like – even when we're actually doing stuff that is visual and people are drawing a graph on the board, I find it really hard to understand it. I don't like visual stuff. AH2

Having **pictures** and using them accordingly is **not always necessary** according to some students. Being able to visualise is beneficial for certain situations, but in some others it may be more convenient not to use these images and try to solve the task by using other strategies.

> I: Um... does it help you to have pictures, apart from symbols, in your mind?

> S: Sometimes, but sometime it's unnecessary, you know what I mean, it's like that list, I remember longer, or this group, remember it longer, or subgroup, you know, or subset, so you learn subset, or you learn subgroup, so yeah, it's some - and it's good to have a picture, but – sometimes some – like a symbol is better than words, you know what I mean, like that…like say last year we did – like um... limited, you know that converges or diverges, and instead of writing a lot of wording, we just do a bit of symbol, and you remember much longer than the wording bit, so yeah. Sometime's good, yeah... CL2

Furthermore **being unable to draw a picture does not imply that one does not have an adequate image of the concept**:

> S: No, I'm just thinking about how I actually don't really know, I know it by the axioms, rather than actually - the visual picture, I suppose just as a circle or something... like that. A group, and then when you've got subgroups, it's like – a bit of it. It's not a very exciting picture, but I don't really know how else I'd do it. AH2

Overall the students associated the difficulty to visualise with the level of abstraction of the course:

> S: Yeah. So sort of like – yeah... abstract is something you can't really see, it's just an idea, and that's what I find hard, cos I'm kind of like – I understand that idea, can't link it to anything, so when you kind of combine it all together, I just can't really – do it, cos I can't see the links between them NS2

Furthermore many stated that they prefer applied mathematics to pure mathematics because pure mathematics is more abstract and therefore harder to visualise:

> I: So you tend to like applied maths, not pure.
>
> S: Yeah…Because you can visualise what's going on with that in that situation whereas with pure you can't always. You can't always see what's going on with that. NS2

In sum in the above we offered the following observations on the ways in which students appreciate and use visualisation in their engagement with Abstract Algebra:

- the images given by the lecturer do not necessarily have the intended impact since they are rarely noticed or adopted by the students
- students express an urgent need for visualisation but often miss it when on offer
- the ability to visualise plays a significant role in comprehending and constructing proofs
- often images in students' minds can be at quite a distance from having a detailed and workable image
- images are subjective, as is the choice to use them, and not all students are willing to do so
- being unable to draw a picture does not imply that one does not have an adequate image of the concept.

In both student and staff interviews there is hardly any doubt about the usefulness of visualisation. The difficulty though, especially in this field of mathematics, lies in its abstract nature and the fact that usually the creation of mental images is left to the learners to achieve on their own. So while students are in urgent need of these images they are at the same time relatively unable to achieve their construction. According to Goldin this ‘failure’ is likely to cause discouragement and consequently, disengagement from these early stages – and even evolve into more substantial and global negative attitudes towards the subject subsequently.

In our view for the lecturer to offer images in the lectures and in the lecture notes and explain it briefly alongside the definition is simply not enough. A more overt attention to meta-cognitive matters (Schoenfeld, 1992) is needed: the lecturer needs to explain to the students how this image came to be, how it works and, most importantly, to convince the students why they should adopt such an image (or aim to construct suitable other images of their own). This is of course a delicate task: one does not wish to impose, through this explicitness, his/her own images on students who might as well be able to construct their own images; nor to convey as obligatory a visual approach to newly introduced concepts. One however needs to introduce visualisation as a potentially powerful tool in the mathematician’s repertoire (Nardi et al, 2005) and allow students to adopt it according to their own needs and preferences.

Our analyses are now turning to addressing the following questions:

- What patterns in student perceptions about/attitudes towards/employment of visualization in Abstract Algebra can we infer from the above data?
- Does – and how – student lack of experience/practice with visualisation and student uncertainty about the mathematical status of visualisation lie behind the reluctance/limited ability to visualize?
- How do these cognitive/epistemological issues intertwine with the above mentioned emotional ones?

The observations we offered in this paper are beginning to suggest preliminary answers to these questions.

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