Mathematicians' Mathematical Thinking for Teaching: Responding to Students' Conjectures

Estrella Johnson, Sean Larsen, Faith Rutherford
Portland State University

Introduction

As part of a curriculum development and research project in abstract algebra, we have been investigating teachers' implementation of an inquiry oriented abstract algebra curriculum (Larsen, 2004, 2009; Larsen, Johnson, Rutherford, and Bartlo, 2009). To understand the ways the instructors might interact with the curriculum, including the challenges they may have during implementation, we have been video taping three mathematicians teaching with our curriculum. In total, we have videotaped four classes over three years. A key feature of the curriculum is that the formal mathematical ideas are developed by building on the students' informal reasoning and strategies. Thus, the way that the instructor responds to student contributions is of particular importance.

As we analyzed the video we began to identify instances in which the teachers were not able to successfully capitalize on students' contributions. Sometimes, the teachers seemed to ignore student contributions or quickly moved on after a brief acknowledgement. In other cases, teachers worked to understand the ideas that the students presented. Some times they were unsuccessful in doing so, other times they were successful in making sense of students' thinking, however they did not make use the contribution to further the development of the mathematics, and occasionally, the teachers were able to capitalize on the student contributions – sometimes refining the instructional sequence in response to potentially productive ideas. Our goal was to understand and explain the various ways the mathematicians responded to student contributions.

Methods

This paper will focus on one mathematician, James, during his second term implementing the curriculum. James is a research mathematician specializing in combinatorics and graph theory and is a co-investigator on the project. During this implementation we videotaped all eighteen regular class sessions. We also received email reflections from James throughout the term.

Consistent with Lesh and Lehrer's (2000) iterative video analysis technique, we made four passes through the videotapes of James' class. In the first pass of analysis we identified episodes in which students made contributions and made notes on how teachers responded to those contributions. Secondly, we categorized both teacher responses and student contributions. During the third pass of analysis we reanalyzed these episodes using framework for teacher listening (Davis, 1997; Yackel, Stephan, Rasmussen, and Underwood, 2003) and a content knowledge for teaching framework (Ball et al., 2008). Finally, we coordinated these analyses to identify connections between listening and content knowledge for teaching.

Listening

Background

To gain insight into how James was responding to student contributions and conjectures, we analyzed how he was *listening* to the students. This analysis was guided by a composite framework that includes constructs described by Davis (1997) and

Rasmussen (from a paper by Yackel, et al., 2003). The framework includes three types of listening: evaluative, interpretive, and generative.

The term *evaluative listening* is characterized by Davis (1997) as one that "is used to suggest that the primary reason for listening in such mathematical classrooms tends to be rather limited and limiting" (p. 359). When a teacher engages in evaluative listening the goal of the listening is to compare student responses to the "correct" answer that the teacher already has in mind. Furthermore, in this case, the student responses are largely ignored and have "virtually no effect on the pre-specified trajectory of the lesson" (p. 360).

When a teacher engages in *interpretive listening*, the teacher is no longer "trying simply to assess the correctness of student responses" instead they are "now interested in 'making sense of the sense they are making'" (Davis, 1997, p. 365). However, while the teacher is now actively trying to understand student contribution, the teacher is unlikely to change the lesson in response.

Finally, *generative listening* can "generate or transform one's own mathematical understanding and it can generate a new space of instructional activities" (Yackel et al., 2003, p. 117) and is "intended to reflect the negotiated and participatory nature of listening to students mathematics" (p. 117). So, when a teacher is generatively listening to their students, the student contributions guide the direction of the lesson. Rasmussen's notion of generative listening draws on Davis' (1997) description of *hermeneutic listening*, which is consistent with instruction that is "more a matter of flexible response to ever-changing circumstances than of unyielding progress towards imposed goals" (p. 369).

Note, that in the above description, these types of listening are primarily characterized in terms of *intention*, which is not directly observable. Thus, in addition to trying to infer intent from body language, tone of voice, and the teacher's history of interactions with students, we also attempted to infer the teacher's intent by analyzing the impact of the teacher's listening on the classroom (i.e. we inferred the intent from the function). For example, if it was clear that the teacher was trying to make sense of the student's thinking but ultimately did not utilize the student's contribution we categorized this as interpretive listening rather than generative (in the absence of other evidence suggesting generative listening).

Data

We will now look at examples of how each type of the listening framework played out in James's class. Also, for each example we will briefly discuss the opportunities that were gained or lost due to the type of listening used.

Episode 1: Evaluative listening.

In this episode, while the class was reinventing the quotient group concept (Larsen, Johnson, Rutherford, & Bartlo, 2009), James was trying to get the students to re-voice a coset algorithm that he had heard a student come up with in the previous class. However, instead of the algorithm James was expecting, a student gave a different idea, one that focused on the order of the elements. When it became clear that this student's contribution did not match the one that James was expecting, James quickly redirected the class, looking for other contributions.

James initiated the episode by asking how people were finding the elements of a quotient group after having chosen which subgroup would be the identity element of the quotient group. A student tried to describe his approach, one that might be related to thinking of cosets as translations of the subgroup:

> *S: … you take the order of the non-identity elements in your identity group, and then the difference between, let's say we're doing two element subgroups, subsets, the difference between, if we picked one element for the first for the subset and then we would have to find an element that was exactly…*

The students' strategy is unclear from this description, and the student was unable to complete his thought before James interrupted. Apparently as soon as he recognized that this student's statement did not match the response he expected. James then redirected the class after brief acknowledgment.

> *James: Oh that's right. Yeah, ok. So I like how you described that. There was another way of describing that. Someone have a different way?*

In doing so, James seems to have missed an opportunity to make sense of an alternative approach, one that might be related to thinking of cosets as translations (a view that James has repeatedly expressed holding himself).

Without knowing more about what this student was thinking, it is unclear whether the rest of the class could have benefited from making sense of his idea or whether the coset idea could have been developed based upon it. Perhaps more significant, routinely engaging in dismissive evaluative listening (a type of evaluative listening that James did not do here) may send the message to the students that the teacher is not interested in their mathematical ideas. This of course would be inconsistent with the establishment of social and socio-mathematical norms (Yackel & Cobb, 1996) that could support an instructional approach in which the students have ownership of and responsibility for the development of the mathematics.

Episode 2: Interpretive listening.

In this next example the class was trying to prove that the identity element of a group must be a member of the identity set of the quotient group (as part of their efforts to prove that this identity set must be a subgroup). A student, Eric, offered an unexpected (and complex) proof of this conjecture. James worked diligently to make sense of Eric's proof. However, once James was satisfied that Eric's proof was valid, rather than engage the rest of the class with Eric's ideas, James asked for another proof, one which better matches his thinking and what was expected in the curriculum.

To show that the identity element of the group is in the identity set of the quotient group, Eric assumed, by way of contradiction, that the identity of the group, e_G, is in the set X, where X is not the identity set, E. He then claimed that, if the partition does in fact form a quotient group, then there must be a set X^{-1}, such that $X * X^{-1} = E$. It was at this point that James became aware that Eric's proof was not the proof that he was expecting (the expected proof does not involve introducing and inverse of X). However, in contrast to the previous episode, James decided to explore Eric's reasoning and contribution.

James: I'll catch up to you once I'm convinced you're gonna give me a good contradiction that I like.

Eric then continued with his proof, looking at $X * X^{-1} = E$. The class has been defining this operation as an operation on sets. Therefore, $X * X^{-1}$ is the set of all elements of the form $a * b$, where a is in X and b is in X^{-1}. So, Eric argued, that for b in X^{-1}, b must also be in E, because e_G is an element in X and b is in X^{-1}, so $e_G * b = b$ must be an element of $E = X X^{-1}$. However, because b is in E and b is in X^{-1}, that means E must equal X^{-1}, because the elements of quotient groups must be disjoint. During Eric's explanation James often stopped him to ask clarifying questions and to re-voice Eric's reasoning.

> *James: Ok, alright. I'm glad that I let you run with it. So, let me back up a little bit for review.*

However, when Eric then tried to finish his contradiction argument, showing that X must also equal E, James cut him off.

> *James: … Ok, I believe you'll get a contradiction. Anyone got a different way? I think that is a complex one, but it works.*

Another student, Mike, then offered a different proof by contradiction, still assuming that e_G is in X and X does not equal E. Saying that, instead of looking at $X * X^{-1}$, you can just look at $X * E$.

> *Mike: I was just gonna say an easier way is to just take the identity in X and then you multiply in by something in the identity set. You're gonna get something in the identity set. So, X times the identity is the identity.*

It is clear that this was the proof that James was expecting, and he uses Mike's proof to continue with the lesson.

> *James: That's kind of along the same lines but it matches the handout a little more … Let's try to write that one up over here.*

Here, even thought James took the time to understand Eric's argument, he did not deviate from the planned lesson trajectory to allow the class to work on understanding Eric's argument. Instead, once he was convinced that Eric's proof was valid, he reverted back to the proof that he had expected to be generated by the students, even before Eric had a chance to conclude his argument.

So, while James did validate Eric's mathematics, he did not let Eric's mathematics guide the trajectory of the lesson. This type of listening can instill a classroom culture in which students feel that, while their mathematical thinking is important, their mathematical ideas are not ultimately shaping the development of the course. Further, by not giving the classroom a chance to make sense of the more "complex" proof offered by Eric, James missed an opportunity for the students to explore this alternate proof. In particular, the class was not given an opportunity to compare the two arguments, which are in fact closely related and involve different approaches to coordinating the properties of the groups involved (the original group and the quotient group that is being constructed).

Episode 3: Generative listening.

Finally, in this example the class was working to develop a minimal list of criteria that will guarantee that a subset of a group is a subgroup. While discussing what criteria may be needed, James noticed that it was unclear to students that the identity of the subgroup must be the same as the identity of the group and that the inverse of an element in a subgroup will be the same as the inverse of that element in the group. In the last episode we saw that James was hesitant to deviate from his original trajectory, however in this instance, James detoured from his planned lesson to address these issues.

During a whole class discussion, one student conjectured that, in order to know that a subset of a group is a subgroup, it is enough check that, for every element in the set, the inverse of that element of is also in the set, and that the set is closed under the operation of the group. This student argued that, if the operation is closed and the set has an inverse element for each of the elements, then the identity of the group must be an element of the subset.

> *James: So, any element with its inverse, when multiplied together gives you the identity, yeah. That's from closure and from the definition of inverses.*
>
> *S1: And if the identity isn't in your subgroup, then closure would be wrong.*
>
> *James: So that's kind of the general thinking. It's not a full-blown proof. But if you just check closure and you just check the existence of inverses then it seems like maybe that will be enough to get that the identity is in the subgroup. So, that's an interesting idea that may in fact save us an axiom.*

This is a common conjecture for students to develop while they work through the curriculum, thus even through it is not the standard subgroup theorem, James was expecting it. However, it is likely that James was not expecting the following question.

> *S2: How do you check inverses without knowing the identity?*
>
> *S3: He's saying check the original.*
>
> *S2: I'm not sure.*
>
> *James: So, talking about checking inverses for… ok, so it's a good point, the identity of the whole group vs. the identity of the subgroup.*
>
> *S1: I said that the identity of the subgroup has to be the identity of the group, because if they're different then you have two elements that are the identity of the main group.*
>
> *James: Ok, this group was harping on these two things as well. This is probably something we should talk about, clear the air on, and then forever more be happy about. If you have a subgroup, is it or isn't it, let's actually detour for just a moment. Let's take a few minutes and try to write, everyone, some sort of a proof or something, that if you have a subgroup inside of a group, true or false, the identity of the subgroup has to equal the identity of the group. And, if you can crank that out then, consider the second question. Does the inverse of an element in H then have to be equal to the inverse of the same element in G?*

In this case, James was able to use the students' comments and ideas to shape the direction of the lesson. Not only did he identify the issues that some students were having but he also adjusted his lesson to address these issues. This is a much different teacher move than what we saw in the last two episodes. Here James built off of the student contributions and questions and diverged from the planned lesson, something that we saw him hesitate to do before. In this case, his decision gives the class the opportunity to resolve an issue that a number of students were unsure about, and one that would represent a significant step toward finalizing and proving their subgroup characterization theorem. Note that this move also makes explicit a subtle but important distinction between checking the inverse and identity axioms for a group and checking these same axioms for a subgroup. In the case of the identity axiom, for a group one must generally establish that there is an element that has the identity property whereas, for a subgroup, one merely needs to establish that the element already known to have this property is in the subset under consideration.

Content Knowledge

Background

To further gain insight into how James was listening to students, we looked to see what types of *content knowledge for teaching* (Ball, Thames, & Phelps, 2008) was needed for each of the different types of listening. We will begin by giving a brief description of the different types of content knowledge for teaching (figure 1).

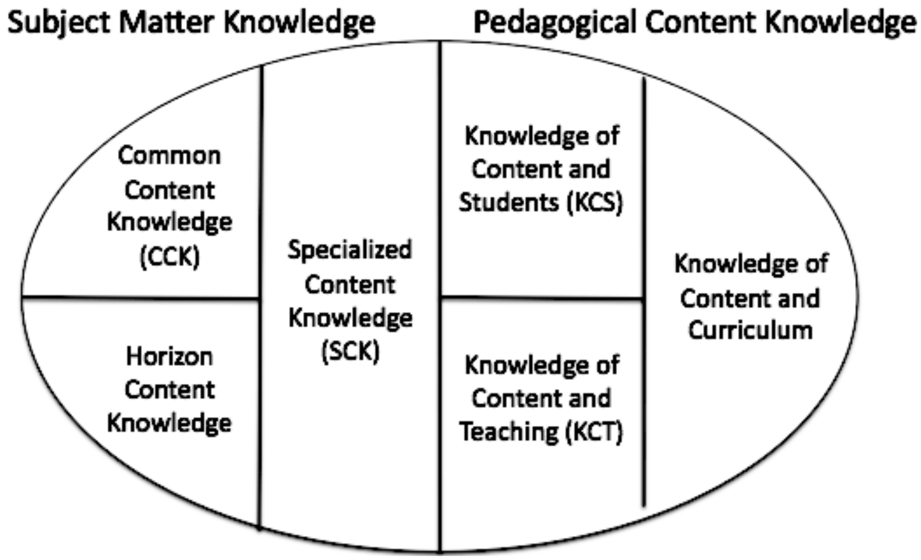

Figure 1. Content Knowledge for Teaching (Ball et al., 2008)

Subject matter knowledge.

Within subject matter knowledge, there is *common content knowledge*, *horizon content knowledge*, and *specialized content knowledge*. Common content knowledge is "the mathematical knowledge and skills used in settings other than teaching" (Ball et al., 2008, p. 399) such as knowing how to perform a standard algorithm. While horizon content knowledge is "an awareness of how mathematical topics are related over the span of mathematics included in the curriculum … " (p.403) such as knowing that the basic properties of arithmetic are also ring theory properties.

For our analysis we found that specialized content knowledge may be the most important. Specialized content knowledge is defined as "mathematical knowledge and skills unique to teaching" (Ball et al., 2008, p. 400). This would include *knowing* whether a student's non-standard algorithm is valid or being able to *figure out* whether a student's non-standard algorithm is valid. It is important to notice that specialized content knowledge appears involve mathematical thinking in the moment, and not just the recall of previously acquired knowledge.

Pedagogical content knowledge.

Within pedagogical content knowledge there is *knowledge of content and students*, *knowledge of content and teaching*, and *knowledge of content and curriculum*. Knowledge of content and teaching "combines knowing about teaching and knowing about mathematics …" (p. 401), such as knowing "which examples to start with and which examples to use to take students deeper into the content". While knowledge of content and curriculum is knowledge of "the full range of programs designed for the teaching of particular subjects and topics at a given level" (Shulman, 1986, p. 10). An example of knowledge of content and curriculum would be knowledge about what "topics and issues that have been and will be taught in the same subject area in the proceeding and later years" (Ball et al., 2008, p. 391).

Again, one type of pedagogical content knowledge stood out in our analysis, knowledge of content and students. Knowledge of content and students is "knowledge that combines knowing about students and knowing about mathematics. Teachers must anticipate what students are likely to think and that they will find confusing" (Ball et al., 2008, p.401). Such as, knowing that students may reason about abstract group elements as if they were real numbers.

We will now look at which types of knowledge may be necessary for a teacher to engage in the kinds of listening we have been considering.

Listening and Content Knowledge, Preliminary Conjectures

For all three types of listening the teacher must make use of their common content knowledge. For example, in all three of the episodes, James relied on his basic knowledge of group properties. For evaluative listening, this may be the only type of knowledge necessary.

For interpretive listening, we believe that the teacher must also draw on their specialized content knowledge in order to make sense of students' unexpected and non-standard ideas. Additionally, the teacher may need to use their knowledge of content and students in order to compare student contributions to common student difficulties.

We suspect that generative listening is the most demanding of the three. In addition to the kinds of knowledge necessary for interpretive listening the teacher may need to rely differently on their knowledge of content and students to consider whether it would be helpful for other students to engage further with a given contribution. The teacher may also have to use their horizon content knowledge to understand how the student contribution may be related to further mathematics. Finally, since generative listening impacts the trajectory of the lesson, the teacher must also draw on other aspects of their pedagogical content knowledge to decide how to capitalize on a given contribution.

Conclusions and Discussion

In classrooms where the students are actively involved in developing the mathematical ideas, it is important for the teachers to be listening interpretively and generatively. The *listening* framework offers insight into how lessons unfolded during class sessions, and can be a powerful lens to help explain teacher responses to student contributions. In turn, the *content knowledge for teaching* framework provides insight into the demands of interpretive and generative listening.

Furthermore, as we conducted our analyses, we began to feel that *specialized content knowledge* (SCK) was more about mathematical thinking than mathematical knowing. This observation gives rise to a number of questions. Is one defining characteristic of SCK, the property that it is as much about a teacher's mathematical activity in the moment as it is about previously acquired knowledge? In this case, does it make sense to have SCK as part of the content *knowledge* for teaching framework, or should it be part of a mathematical *thinking* for teaching framework? While we recognize that there are theoretical perspectives in which knowledge is understood in terms of participation, we are concerned that the word knowledge is nevertheless often understood to represent something that one acquires rather than something that one does. This distinction may have consequences for professional development and teacher preparation. Perhaps the types of listening we have been considering would fit into a model of mathematical thinking for teaching. Perhaps listening is in fact an important kind of mathematical thinking for teaching.

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