Do They Really Get It? Evaluating Evidence of Student Understanding of Power Series

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Abstract

Most teachers agree that if a student understands a particular mathematical topic well, she will be able to do problems correctly. The converse, however, frequently fails: students who do problems correctly sometimes still hold significant misconceptions about the topic in question. In this paper we explore this phenomenon in the context of power series, one of the most challenging topics in the Calculus curriculum. We report on clinical interviews with students, many of whom arrived at correct answers to questions about series, explaining their answers in appropriate terms, despite having significantly flawed ideas about those series. Implications for further research and teaching power series are discussed.

Keywords: student thinking, calculus, power series, teacher knowledge

Introduction

In this paper we report on exploratory work to understand how students interact with the mathematical topic of power series. While extensive research informs how students think about other calculus topics, little work elucidates students' thinking about the convergence of power series – or even what a power series is. We report on an interesting case from an interview with two students as an example of the types of issues that might be explored with further research. While this case may not be representative of student thinking, it provides an example of the conceptions of the actions students take when presented with power series questions and some of the reasoning behind those actions.

Literature Review

Many of the key topics in Calculus have been studied extensively, and we can now catalog how students think about topics such as limits (Roh, 2008; Tall & Vinner, 1983; Williams, 1991), derivatives (Zandieh, 2000), and integrals (Oehrtman, 2009). In some of these areas, research has progressed to the stage where researchers are now testing instructional strategies specifically designed to help students move beyond the naive conceptions and fill in the conceptual gaps found in earlier studies (e.g. Boester, 2008; Carlson, Jacobs, E., Larsen, & Hsu, 2002; Roh, 2008; Sealey, 2008).

In contrast, although they occupy a significant portion of many calculus courses, very little is known about how students think about series. Our interest is in student understanding of series in general but for this project we narrowed the focus to power series. This topic makes use of and builds on many ideas from calculus (and earlier courses) and lays the groundwork for students' further study of topics in analysis. Despite the richness of the topic (and the anecdotal evidence of challenges students face in understanding the ideas), we have not yet amassed a research base that provides insights into student understanding of this topic and that can be used to inform and improve instruction. In this section, we review the handful of studies that do touch on this topic directly.

Martin (2009) tackles one particularly common example of power series, Taylor Series. Through interviews with students who had just studied power series, students who had learned them a year or two earlier, and experts who had taught the subject, he determined that experts tended to "use a wider range of images that they efficiently and effectively employed as different situations prompted" (p. xix). The students' thinking about Taylor series generally lacked the well-developed graphical sense that experts possessed. Prompts for a graphical representation of the first few Taylor polynomials elicited some mistaken ideas, with several novices appearing to apply knowledge of translating functions in inappropriate ways.

In a teaching experiment, Kidron and Zehavi had students use Mathematica to program an animation showing how Taylor polynomials approximated a function (2002). Their study indicated that some students' conceptions of this approximation process were helped by the animations. For instance, one student offered that "the higher degree of the approximating polynomial, the bigger is the interval in which f(x) and the polynomial coincided" (Kidron, 2003, p. 318). Other students, however, were unable to complete the programming task and did not benefit from the instruction in the same way.

In a series (or sequence?) of articles about real analysis, Alcock and Simpson detailed much student thinking about convergence in general, but little work specifically about power series (Alcock & Simpson, 2000, 2004, 2005). They do report asking students this intentionally ambiguous question:

"When does this series converge: $\sum \frac{(-x)^n}{n}$?" (Alcock & Simpson, 2004, p. 7)

However, they report very little about students' responses to this question, in part because many students were simply confused about what the question was asking for (Alcock, 2010). In general, their work focused on classifying students by whether they used visual representations as they worked through analysis problems or not.

While much of the literature on sequences, series of numbers, and limits in general sheds light on how students might interpret power series, a review of this literature goes beyond the scope of the current work.

The work discussed in this paper is an attempt to build on the limited work described above, and should be seen only as an exploratory first step toward understanding students' thinking about power series, developing curriculum materials that help students improve their thinking, and eventually improve student learning of this extremely challenging topic.

Methods

In this paper we report on only one case but the data come from a larger project involving 13 participants. Given the dearth of literature on students understanding of power series, there were very few examples of data collection and analysis methods on which we could model our research design. As a result, our approach was quite exploratory both in terms of how the data were gathered and how they were analyzed.

The participants were all students enrolled in three different second semester calculus classes at a small public college in the eastern United States. The interviews were conducted by one of the author's of the paper with an undergraduate assistant in the room to provide technical support.

Near the end of the spring 2009 semester the students participated in clinical interviews that lasted approximately one hour. Twelve of the 13 students were interviewed in pairs and one student was interviewed individually. During the interviews, students were presented with a task and then given additional tasks as time allowed. The students completed one to three tasks (depending on how many they were able to work through in the allotted time).

The tasks were designed to elicit student thinking about ideas related to convergence of power series. The data we report on in this paper come from the interview of a student pair as they worked on the first task:

For what values of x does the following series converge: $\sum_{n=1}^{\infty} \frac{x^n}{n2^n}$?

This type of problem is a common exam-type question, and we asked it specifically to probe the thinking behind what students might answer. We anticipated that a student with an extremely thorough understanding of the topic might complete the ratio test, test the endpoints, and be able to explain that the interval of convergence meant that exactly those values of x have the property that if they are substituted back into the series, the series would converge to some number. An expert might also describe why the ratio test works – that it compares an unknown series to a geometric series. We reviewed the work of all the students and we selected this pair as the focus of this exploratory analysis because they appeared to have some understanding that enabled them to make progress on the tasks but also demonstrated some weaknesses in their understanding of the underlying concepts. Unlike some of the other students who either completed the tasks with ease or struggled to make any progress at all, we suspected that this pair, with their somewhat developed, yet imperfect understanding of the ideas would be a productive focus for our analysis.

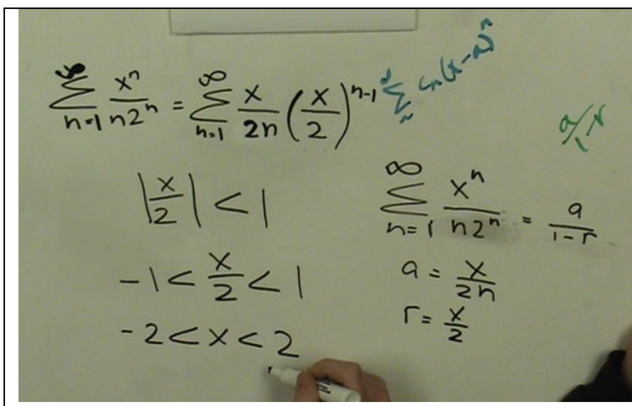
The data we report on here were generated from the discussion that Michael and Peter (pseudonyms) had while working on this task. This portion of their interview lasted approximately 30 minutes. Their discussion and written work were captured using two video cameras. The two students worked side-by-side and one camera was position in front of them and captured their faces. A second camera was suspended above the work surface and captured their written work and gestures and they discussed the problem.

As they worked on the task, in keeping with the spirit of clinical interviews (Hunting, 1997), the interviewer asked clarifying questions and probed their understanding of various terms and concepts the students mentioned but endeavored to not influence the direction or substance of the students' solutions.

Because of the exploratory nature of this endeavor, we did not approach the analysis with codes, categories, or other tools of analysis in mind. Instead, after identifying this pair as an interesting case, our goal was to understand and describe the student pair's written and spoken work on the task and to then consider the questions that this work generates about student understanding of power series and concepts connected with series and convergence. During this process, we generated transcripts of the interview as well as still frame images of their written work.

The Case of Michael and Peter

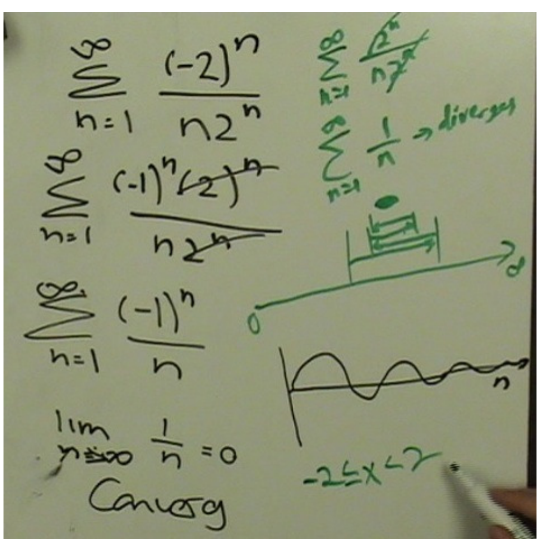
We report on a particularly interesting piece of work done by two students who decided (incorrectly) that it was a geometric series. Here we see a still shot of their work along with the transcript of the accompanying dialog.



Michael: If it is a Geometric sequence... It's going to be…
Peter: You've got an x out here. Isn't that going to matter?
M: This is an a right? I don't think it's going to matter that much because this is-
P: So it's not to a power?
M: If we need to find an a which is the sum, to find the sum it's going to be a over one minus the absolute value of r, right?
Interviewer: Can you write out what you're saying?
M: Yeah, I think that sum…
Instructor: You're saying you have some formula?
P: Yeah, the a over one minus r where a equals something and r equals something else.
M: n 2 to the n equals a which is this for us divided by the one minus—I think it's the absolute value of r I'm not sure, maybe not.

	P: No.
	M: It’s just r. Yeah one minus r in our case a equals the x over 2n and r equals the x over 2n, right? And for it to converge, r has to be this region, however we also have to substitute it in and test using some kind of test of convergence to see whether the points- the end points are going to be included in the interval of convergence or not.

We interpret the students’ work to mean that, after deciding (for unknown reasons) that this series is a geometric series, they have worked to identify which symbols in this power series match up with which parts of the known formula for a geometric series. Since the geometric series they have seen begin a + a r + a r^2…, but the series in questions starts with n=1, they have rewritten the series so that the expression for r is more clearly equal to (x/2). This leaves them with a first term of a = (x/2n). Armed with these values of r and a, they calculated the correct radius of convergence, and went on to correctly check both endpoints, as seen in the still shot below. Thus despite their serious error in identifying the series as geometric, they ended up calculating the correct interval of convergence.



In the middle of Michael and Peter's discussion of the endpoints, Michael said, "For some reason I think the ratio test could be incorporated somehow, but it's usually for ones that are not geometric but this does look like a geometric sequence. Ya know, that's why—it seems to be reasonable. I don't know." After they finished checking the endpoints, the interviewer returned to this point:

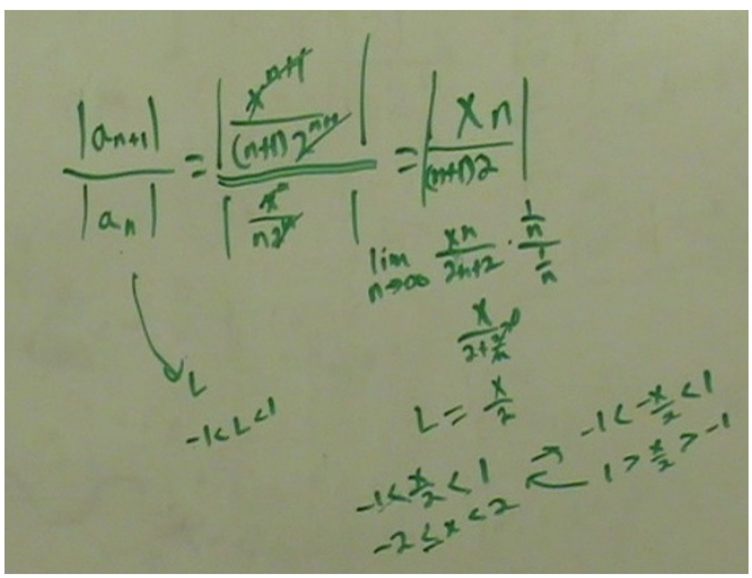
Interviewer: Somewhere in there one of you mentioned doing a ratio test on this?

Michael: Yeah we thought that would work.

Interviewer: Do you mind if I ask you to do that?

Peter and Michael: No that's fine.

They went on to complete the ratio test correctly, as seen in this still:



Here we see evidence that these students are able to carry out the tasks demanded by the ratio test, correctly setting up the ratio, taking the limit as n tends to infinity, determining the radius of convergence, and writing the interval of convergence. At this point they referred to their earlier work to determine the endpoint behavior, coming to the same answer as before.

What We See

As an exploratory study, this work yields very few concrete conclusions. However, here we discuss our tentative initial conclusions drawn not only from the single case described above, but from all 13 participants.

Series are hard. Watching students struggle with the complex ideas of power series leaves no room for doubt that series are extremely difficult – not that anyone who has taught the subject has ever thought differently. In many instances, students confused sequences with series, got lost in the notation, and appeared to quickly reach a state of cognitive overload.

Written work can mask conceptual gaps. The case of Michael and Peter illustrates how student work might not be illustrative of their underlying conceptual reasoning. Both students appeared to erroneously believe that the series was geometric, while also being able to correctly complete the ratio test. Had an instructor only seen the latter work, full credit might have been given; had she only seen the former, a student might have earned only a few points. This case alone indicates that judging students' conceptual understanding solely from their written work on problems like these is fraught with difficulties.

Student may have mechanical views of series testing. Throughout the interviews, students took a very mechanical view of testing series, at times working through the steps of a particular test without being able to clearly articulate the underlying ideas, what they were testing for, or at times, even what the conclusion of the test was.

Many tasks are virtually concept-free. Like the task described above, many questions about power series appear to be virtually concept-free in the sense that they do not test for any real conceptual understanding of power series, what power series represent, what it means for a series to converge, or why a particular test yields a particular conclusion.

Research Questions

As this case reveals, there is much we need to examine about student understanding of series to build a literature base that can inform the design of curriculum and instruction for this topic. Here we provide some research questions, inspired by this case, which might be worth pursuing as future steps towards a more comprehensive understanding of students' thinking about these topics.

Our participants were asked to find the values of x for which the given series converges. The students carried out tests for convergences and worked to find intervals of convergences, but what meaning do students make of these processes? It may be productive to figure out what students' overall, bigger picture understanding is of this whole endeavor we call "testing for convergence." What does the "testing" description mean to the students? When a test is inconclusive, what do students think that means? If by using a test, they arrive at a conclusion of "converges," what sense are they making of this answer?

The idea of an "interval of convergence" or "radius of convergence" may also present some issues for students. For example, in a preponderance of cases during their study of mathematics, students have been asked to find single values that satisfy equations and not a range of values. In series problems, they are seeking multiple values, constrained by the equations that are generated during the testing process. This shares some similarities to the thinking and computations needed when working with inequalities but this may or may not be an area of the students' mathematical past to which they can connect and use as a resource. Further confusion might arise from the use of the word "radius" when no circles are present (at least when x is real as opposed to complex.)

From our data, it is not clear what features of the initial series prompted the students to think it was geometric. It would be productive to know more about how students interpret the symbols used to represent a generic geometric series and how they used that interpretation to recognize (or not) when a particular series is geometric. This also raises issues about what “geometric” might mean to students in the context of series and what connections they may be making between this use of “geometric” and the other contexts in which they have see or heard that term.

To fully understand student thinking about series convergence, we may also need to explore how students think about some of the sub-problems that they face as they work through the multi-step process needed to solve convergence tasks. Ideas about intervals, inequalities, and conversion of sequences are just three examples of the types of topics where understanding of them may be interacting with students’ thinking about convergence. Based on other researchers’ work that examines the influence of prior understanding on the learning of calculus (e.g.,(Carlson, 1998; D. H. Monk, 1994; S. Monk & Nemirovsky, 1994), it seems plausible that prerequisite topics are interacting with and shaping student learning of these ideas.

Teaching Questions

This work also raises a number of interesting and important pedagogical questions. While further work is needed before these questions can be answered, we hope that by elucidating them in this exploratory stage of our research, our later work will be more carefully targeted.

First, the case described above raises the question of how we can better assess students’ conceptual understanding of power series. What types of assessment questions are more likely to elicit conceptual responses?

While the students’ claim that the series was a geometric series is clearly false, a colleague saying a very similar thing would be quite convincing to us (e.g. “Well, it’s essentially a geometric series – I can ignore the n in the denominator – so the radius of convergence is 2.”) Should this type of informal, intuitive reasoning be encouraged? Should it be an explicit goal of instruction, or is it the end point of a long learning trajectory that Calculus students have only begun?

More broadly, what are appropriate pedagogical goals when teaching power series to second semester Calculus students? Is the goal of having students be able to do various series tests sufficient at this level, leaving the understanding of these tests for a later Real Analysis or Advanced Calculus course?

Conclusions

This first foray into the realm of student thinking about power series raises interesting questions that we hope to explore in future work. The present work indicates many of the challenges faced by students as they grapple with this most challenging of Calculus topics. A deeper understanding of their struggles, what they are thinking, and how they begin to gain a conceptual grasp of power series will eventually help us devise more efficient ways to teach this subject.

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