An Investigation of Post-Secondary Students' Understanding of Two Fundamental Counting Principles

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Theoretical Perspective and Motivation

Although counting problems are often simple to state, they can be surprisingly difficult for students to solve. This fact is corroborated both by authors of combinatorial texts (e.g., Martin, 2001; Tucker, 2002) and by researchers on the teaching and learning of combinatorics (e.g., Batanero, Navarro-Pelayo, & Godino, 1997; Eisenberg & Zaslavsky, 2004; English, 1991; Kavousian, 2006). In particular, there is evidence of low student success rates on a variety of counting problems (e.g., Batanero, et al., ibid; Eisenberg & Zaslavsky, ibid), and there are documented pitfalls that plague counters even at the post-secondary level (Hadar & Hadass, 1987). In spite of this existing confirmation of the difficulty of counting, however, there has overall been relatively little research conducted on the teaching and learning of counting problems. The study described below seeks to contribute to the existing body of research on combinatorics education by exploring and elucidating specific domain-specific issues that arise for students as they count.

The addition principle and the multiplication principle are foundational ideas in elementary and advanced counting. They are presented early on in many popular combinatorics textbooks (e.g., Bona, 2005; Tucker, 2002) and are generally considered to be fundamental to counting. They are not, however, explicitly addressed in the mathematics education literature. Because of the essential nature of these principles, and due to their absence in the literature, this study was designed to enhance researchers' knowledge base of how students understand these principles and how they implement them as they solve counting problems.

The theory of learning presented by Hiebert and Lefevre (1986) has been adopted as the theoretical perspective for this research, providing a backdrop from which the study was designed. Hiebert and Lefreve's theory focuses on the complementary relationship between conceptual knowledge and procedural knowledge, and it has been chosen primarily because of its emphasis on the interconnectivity of conceptual relationships. Such connectedness seems to be foundational to the learning of combinatorics (see, for example, Batanero et al., 1997; English, 1993), and in particular to the learning of the two counting principles with which this research is concerned. Hiebert and Lefevre define conceptual knowledge as "knowledge that is rich in relationships…a connected web of knowledge, a network in which the linking relationships are as prominent as the discrete pieces of information" (p. 3). It is therefore the connections and relationships between ideas and concepts that are of utmost importance in this framework. They define procedural knowledge as being composed of the formal language, symbols, and algorithms that students use as they complete mathematical tasks. They maintain that, "mathematical knowledge, in its fullest sense, includes significant, fundamental relationships between conceptual and procedural knowledge" (p. 9). They thus propose that linking conceptual and procedural knowledge has advantages both for procedural and conceptual knowledge.

Upon preliminary analysis of student work on counting problems, it became clear that while Hiebert and Lefevre's (1986) framework appropriately addressed students' mathematical knowledge of counting problems, it was not sufficient to describe all of students' counting activity; their definition of procedural knowledge did not properly describe all that students were doing. Unlike some mathematical disciplines, in which problems can be solved predictably according to algorithms, many counting problems require robust problem solving skills. Indeed, it is difficult, if not impossible, to separate successful counting from problem solving. Given the importance of problem solving in the domain of combinatorics, the researcher chose to utilize the Multidimensional Problem-Solving Framework of Carlson & Bloom (2005). This framework proposes four problem solving attributes (resources, heuristics, monitoring, and affect) and four problem solving phases (orienting, planning, executing, and checking) and provides appropriate language with which to describe much of students' counting activity.

Conceptual Analysis of the Addition and Multiplication Principles

Prior to the implementation of the study, the researcher developed an initial conjecture about what specific mathematical ideas might play into an understanding of the addition and multiplication principles. Because of the preliminary nature of the report, this conjecture was based primarily on informal discussions with a combinatorialist, as well as the author's personal experience as both a teacher and a student of these two counting principles. In addition to this, combinatorial texts were used as resources in order to get a sense of the kind of mathematics that underlies the two principles. Because these principles are not mentioned in the combinatorics education research, this conceptual analysis is not rooted in mathematics education research, but rather is a result of mathematical brainstorming based on personal experience of teachers and students of combinatorics. The main points of the conceptual analyses are briefly outlined below.

A notion of a solution set. One aspect of the addition and multiplication principles that appears significant is the notion of a solution set – the collection of objects that are being counted, conceived as a whole, a mathematical entity. That is, it would appear that an understanding of the principles entails an ability to recognize that counting problems ask for the cardinality of a set, a perspective which allows the student to bring to bear manipulations of that set as an aid in determining its number of elements. In particular, it is significant for students to be able to visualize/imagine/conceive of even a single solution to the problem. If they cannot even imagine what a single solution looks like, this indicates that they are likely unclear on the membership criteria for the solution set, which reflects a poor understanding of the problem constraints. Without clarity on this point, students cannot be expected to reliably count the total number of solutions. Therefore, part of the underlying idea of a solution set is the fact that students can visualize a solution, and consider that their task of solving the problem is to count all such solutions. Doing so can give them a sense of the magnitude of the solution, it can offer insight into effective problem solving procedures, and it can potentially address more advanced issues such as over- or under-counting. Therefore, in interviewing the students for the study, special attention was paid to the extent to which they conceived of a solution set.

The addition principle. Tucker (2002, p. 170) offers the following definition of the addition principle:

> "If there are r_1 different objects in the first set, r_2 different objects in the second set, …, and r_m different objects in the *m*th set, and if the different sets are disjoint, then the number of ways to select an object from one of the *m* sets is $r_1 + r_2 + ... + r_m$."

An equivalent definition is that for any subsets S_1, S_2, …, S_m of a set *S* if $S_i \cap S_j = \phi$ is empty for all *i* and *j*, and if $S_1 \cup S_2 \cup ... \cup S_m = S$, then $|S_1| + |S_2| + ... + |S_m| = |S|$. Even though the word *addition* is typically in the name of this principle, it is much more of a set theory concept than it is a concept about the operation of addition. Indeed, the addition principle could perhaps more accurately be called "the principle of disjoint sets." The principle offers an alternative way of counting the number of elements in a set; in particular, this is accomplished by breaking the original set into subsets with particular properties, and then adding up the cardinalities of those subsets. In order for this to work, the subsets of the original set must form a *partition* of the original set. A partition has two important properties: the subsets must be disjoint from one another (the subsets cannot overlap at all), and there must not be any elements in the original set that are not included in the subsets (the subsets must not "miss" any of the elements in the original set). If these two properties of the subsets are met, then the size of the original set can be found by adding up the sizes of the subsets. This principle is a powerful counting tool, then, because in many cases, the solutions are not all of a homogeneous nature, and so it can happen that counting smaller subsets is more efficient than counting the original set. Indeed, in some problems, such a breakdown into subsets is a critical feature of the solution.

Although the addition principle has been discussed above in terms of sets, and while most textbooks define it as such, some combinatorial textbooks vary in the extent to which they employ set theoretic language when presenting the addition principle. Particular wordings of the principle may differ across textbooks, but the underlying mathematical ideas are the same. In addition, the term "case breakdown" is often used to describe the process of defining the subsets. Thus, partitioning the solution set into disjoint subsets could be described as breaking the problem down into disjoint cases, each of which can be counted separately.

The Multiplication Principle. While combinatorial textbooks primarily define the addition principle as a set-theoretic idea (as described above), they tend to define the multiplication principle in terms of a *procedure* involving independent stages. For example, Tucker (2002, p. 170) defines the multiplication principle in the following way:

"Suppose a procedure can be broken into *m* successive (ordered) stages, with r_1 different outcomes in the first stage, r_2 different outcomes in the second stage, …, and r_n different objects in the *m*th stage. If the number of outcomes at each stage is independent of the choices in previous stages, and *if the composite outcomes are all distinct*, then the total procedure has $r_1 \times r_2 \times ... \times r_m$ different composite outcomes."

Using more set-specific language, Brualdi (2004, p. 46) defines the principle as follows:

"Let S be a set of ordered pairs (*a*, *b*) of objects, where the first object *a* comes from a set of size *p*, and for each choice of object *a* there are *q* choices for object *b*. Then the size of S is $p \times q$. That is, $|S| = p \times q$."

Under the first definition of the multiplication principle, the salient domain-specific mathematical idea is that the stages need to be independent of one another – the choice for a given stage must not depend at all on what was chosen in a previous stage. The idea is that in order to count the number of ways of accomplishing some procedure, the procedure can be broken up into a number of stages that are independent of each other. If they do not depend on one another, then any of the outcomes in a particular stage can be combined with any of the outcomes in another particular stage, and so the total number of ways of completing the stages together is the product of the cardinalities of each stage. This issue of independence is also articulated in the second, set-theoretic definition of the principle. In particular, the phrase "for each choice of object *a* there are *q* choices for object *b*" in Brualdi's definition indicates that for the multiplication principle to apply, the choices in the sets must not be dependent on one another. Therefore, that the underlying mathematical issue in understanding the multiplication principle is to recognize independence in the counting procedure.

The above conceptual analysis provided themes and ideas to which the researcher attended as she proceeded with the data collection for the study. In particular, she was attuned to the degree to which students: a) displayed solution set thinking; b) understood that the addition principle can be applied only when cases are disjoint; and, c) understood that the multiplication principle can be applied only when stages are independent. The study is described below.

The Study

Surveys were distributed to three different classes (an introductory statistics course for non-math majors; a 300-level discrete mathematics course; and a 400-level history of mathematics course designed for prospective high school teachers) and two students were chosen from each class. Tracking what coursework students had had allowed some insight into what counting principles might be available to them. This was not an essential aspect of the study, however, and there was not an attempt to control for what combinatorial instruction students had seen before. The rationale for not controlling for students' previous instruction was that a) it would be extremely difficult, if not impossible, to control for what counting principles students had learned (especially because discrete mathematics arises in K-12 curriculum), and b) the goal of the interviews was not to teach students how to count, but rather to provide an opportunity for the researcher to observe their counting activity.

The six students were each interviewed twice, and the data consists of twelve videotaped interviews, each about an hour long. The students were interviewed individually to get a sense of how the individuals, themselves, thought about and solved counting problems. In the interviews, the students were given several counting problems (they completed anywhere from four to eight in the hour) that they worked through on their own. The protocol for the interview sessions was to allow the students to work on the problem until they felt they had finished. During this time the interviewer would answer clarifying questions, but, in an attempt to see what students would do on their own, she generally did not interfere. Once the students were done, the researcher returned to the problem and asked questions of the students – these included general question such as "why did you do that?" or "what are you picturing?" but it also entailed specific aspects of the problem such as "what does that particular notation mean to you?" or "did you do that because you had seen it on a previous problem?". Therefore, the data includes what students did with problems when left on their own, and it also incorporates some explicit discussion and explanation by the students, as they were questioned more specifically about their activity. The former was important because it allowed some assessment of problem solving strategies and allowed for a chronicling of what students do naturally when faced with counting problems, but the latter was useful because it allowed the researcher to probe more deeply into students' counting activity. There was not a specific script that the researcher used; rather, she adapted her questions to the particular problems at hand, relying on her domain-specific knowledge to ask relevant questions. Lessons learned in reflecting on these interviews are meant to inform the structure of interviews in future research.

Analysis of this data consisted of re-watching the videos and making content logs while doing so. These content logs include time stamps of relevant problem solving activity and highlight mathematical and problem solving issues (using the language of Carlson & Bloom (2005)) that arose. Next the content logs were reviewed for salient themes, and, based on these emerging themes, several relevant sections were transcribed. From the data generated in this study, we examine three case studies that serve to illuminate the notion of solution set, described above. This is a specific aspect of the addition and multiplication principle (and the interaction of the two principles) that was developed in the conceptual analysis, and that the data confirmed as being important for students as they solve counting problems.

Relevant Counting Problems. In order for the reader to best appreciate student responses in the case studies below, we first discuss in detail two counting problems (and their solutions) that will be featured in the following section.

<u>The Card Problem</u> (adapted from Tucker, 2002): How many ways are there to pick 2 different cards from a standard 52-card deck such that the first card is a <u>seven</u> and the second card is <u>not a spade</u>? The most common and efficient solution to this problem[1] employs a case breakdown. If (Case 1) the 7 of spades is the first card, then there is one choice for the first card,

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[1]Counting problems tend to have multiple solutions. An attempt has been made here to identify problems where judicious use of cases can greatly simplify the solution.

and there are 12 remaining spades in the deck. This leaves 51-12=39 choices for the second card, and so this Case can occur in 1*39=39 different ways. If (Case 2), the 7 of spades is *not* the first card, then there are three choices for the first card (the three non-spade 7's), and there are 13 remaining spades in the deck. This leaves 51-13=39 choices for the second card, and so this case can occur in 3*38=114 different ways. Therefore, the total number of ways of picking two cards where the first is a 7 and the second is not a space is given by the sum of the number of solutions of these two disjoint cases: 39+3*38=153.[2]

<u>The Apples/Oranges Problem</u> *(Adapted from Martin, 2001): How many different nonempty collections can be formed from five (identical) apples and eight (identical) oranges?* There are a number of ways of arriving at a solution to this problem. An efficient explanation is to note that we can choose 0 to 5 apples (6 possibilities) and 0 to 8 oranges (9 possibilities), yielding 6*9=54 possible combinations of 0 to 5 apples and 0 to 8 oranges. We cannot choose 0 of each (which can occur in one way), and there are thus 6*9-1=53 collections.

Results and Discussion

The results of the study detail three case studies related to the two problems above. The purpose of these case studies is to highlight the impact that the notion of a solution set has on students' understandings and uses of the addition and multiplication principles.

Case 1: Danny. Danny is a student in an introduction to statistics course, with little prior experience counting. Throughout the two interviews he displayed fair problem solving abilities.

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[2] A common student error, however, is to avoid a case breakdown and instead to (mis)apply the multiplication principle, arguing that there are four choices for the first card chosen (the four 7's) and there are 51-13=38 remaining choices for the second card chosen (39 non-spades, minus the first card that had already been chosen). Such reasoning leads to an (erroneous) answer of 4*38=144. However, the multiplication principle requires that the two steps in this solution method be independent. In this problem, because the first card could actually be a spade, the number of choices for the second card is dependent upon which card is chosen first. In the face of such dependent choices, a case breakdown is an appropriate way to proceed.

He showed some evidence of monitoring (e.g., re-reading the problem carefully, asking himself whether a solution made sense, and checking his work), although his monitoring did not catch all of his mistakes, and he failed to arrive at correct solutions in several of the problems. Furthermore, the monitoring he did exhibit tended to happen during the discussion of his work as the result of prompts by the interviewer and not of his own accord. Based on this, we characterize Danny not as being completely devoid of problem solving skills and strategies, but as not being particularly strong in this regard. Danny did not show any evidence that he thought of counting as enumerating a set of solutions.

The purpose of Danny's case is to highlight an unusual error that relates to the addition principle, and, more specifically, to the notion of a solution set. While solving the Cards problem, Danny correctly recognizes that the 7 of spades is an issue. He correctly breaks the problem up into two "scenarios," as he calls them, depending on whether the 7 of spades is chosen as the first card. He counts each case individually, arriving at correct numerical values for each, but he does not know how to combine the two values. He believes that the total number of ways of choosing cards "depends," and he does not combine them in any way to get a total number. As highlighted in the excerpt below, even after being asked more than once for a total answer, he does not see that adding the two together will yield the total number of possible solutions.

> D: So it's dependent, I would say.
> E: So is there a total answer, or would you say it depends?
> D: I think it depends… I would say that, depending upon whether the 7 of spades is the first card drawn or not, whether it's a different 7 that comes out, you're going to get different number of ways that you can make this combination, because, yeah, this way [the 7 of spades is not first] you're missing a non-spade from the deck already.
> E: Do you think it's possible given what you've found to determine a total number? Like, um…
> D: Hmm.

E: Can you use the information you have at all to decide the total number of ways of doing it, or do you think it depends?
D: Um, I would say…Well, no, I don't see a way to do that, actually. I would say that this [points to the case where the 7 of spades is first] is your total possible…however, should you get a seven that's not a spade, this [points to the case where the 7 of spades is not first] is your new total possible. I think it does hinge on that first card that comes out.

This example highlights two important aspects of the solution set as it relates to the addition principle. First, we see that even though Danny was able to break the problem into cases, he did not immediately see that he should add the cases together – the "addition" aspect of the addition principle was not clear to him. It is noteworthy that this same issue arose in more of Danny's work as well. The counting survey that he completed, as well as additional interview tasks, confirms a tendency not to add different cases. This is an important existence proof of a student who does not recognize the fact that when one breaks a problem into disjoint cases, he should add the results to arrive at the total. This example also suggests that Danny is not thinking of his total answer as a set of solutions. His failure to add the two cases betrays a lack of solution set thinking; there is no evidence that he was, even informally, envisioning a set of objects that he had to count. There was seemingly a disconnect between the case breakdown and its effect on what he was ultimately trying to count. We thus see from Danny a surprising error that highlights, generally, the kind of problems that beset students as they solve counting problems, and, more specifically, that offers one example of a lack of solution set thinking as it relates to the addition principle.

Case 2: Holly. Holly is a discrete mathematics student, and she was in the midst of a counting unit in this course while the interviews were taking place. She therefore had counting resources available to her, and she seemed comfortable with tree diagrams, choose notation, factorials, and certain prototypical problem types. For the most part, she displayed relatively weak problem solving skills throughout the two interviews. This was evidenced by a tendency to move quickly through problems, without much evidence of monitoring – she often did not read the problem carefully, check her work, or make sure her methodology aligned with what the question asked. Further, on occasion she would display an inclination to superficially match formulas or techniques to a given problem, even when they were not appropriate to use. In those instances the counting resources were arguably liabilities as she tried to solve the problems.

We highlight two examples from Holly's interviews. First, Holly was also given the Cards problem described above. Unlike Danny, she did not recognize that the choice for the second card was dependent upon whether the first card was a 7 of spades. She therefore proceeded to apply the multiplication principle, arriving at the following solution.

> H: …There are four ways to pick the first card, being a 7, and then the second one to not be a spade – let's see, you have 52 minus 1 again, and then you have to take out not a spade, the 13 spades. So 52 minus 14 is 38. So I would multiply those [the 4 and the 38].

At the end of this excerpt, she multiplied the numbers together and moved on to the next problem. She thus completed the problem rather quickly, and, as was somewhat typical, she did not go back to check her work. In addition to her lack of monitoring, in this example Holly gave no indication that she ever visualized an example of what she was trying to count (let alone the entire set of solutions). It is conceivable that she had one in mind, but her lack of a case breakdown, along with her failure to write down any solutions or verbalize what a solution could look like, point to an absence of solution set thinking. It appears that she was functioning on a superficial level, and her focus on surface features of the problem (it resembled other multiplication principle problems she had seen) prevented her from actually having to consider what she was trying to count. And, ultimately, we argue that it was both a lack of monitoring, and a lack of even an informal consideration of a set of solutions (or even one solution), that contributed to her missing this problem.

In contrast to her work above, we draw attention to Holly's work on the Apples/Oranges problem. This problem did not look familiar to her, and she therefore did not try to match her solution to a particular formula or problem type. Instead, she had to write out some visualization of what she was trying to count. She drew a diagram like the one below, representing the respective numbers of apples and oranges she could have, and she began to reason through the solution. Initially she drew eight branches from each row of A's – one branch each to one through eight O's. Similarly, she drew one branch from each row of O's to each of one through five A's. This initial work resulted in a solution of 5*8 for the A's connected to the O's, and 8*5 for the O's connected to the A's, plus the 13 solutions she had listed below. This resulted in 5*8 + 8*5 + 13 = 93 solutions.

A
AA
AAA
AAAA
AAAAA
O
OO
OOO
OOOO
OOOOO
OOOOOO
OOOOOOO
OOOOOOOO

Interestingly, Holly thought that this number was two high, and in a moment of monitoring she went back to check her work. Because she had the visualization before her, she was able to refer back to the solutions she had counted, and she came to the conclusion that some double counting had occurred. She realized that the 8*5 that came from the O's being branched to the A's was already accounted for in the 5*8, and she knew she had to take care of that double counting. She proceeded to cross out the double counting and adjust her answer accordingly. She went on to draw a 0A above the A in the diagram, and she drew a branch from this 0A to one to eight O's. She called this her final solution and got 6*8=48 as the final solution.

The reader may note that this solution is still not quite correct – she is missing the solutions that have zero oranges. The final visual representation that she came up with is not complete, in the sense that it does not accurately display the entire solution set. However, in spite of this, her answer is very nearly correct. She was able to engage more deeply with the problem and the quality of her effort is remarkably different from her work on the Cards problem. Most interesting is the fact that she was able to use the visual drawing (which, arguably, represents at least a partial solution set) to detect double counting. It was the act of reviewing the written solutions in front of her that allowed her to catch her error. Therefore, we see here that even a partial representation of a solution set enabled the student to have greater success in counting.

Case 3: Liam. Liam is a math major and was also taking discrete mathematics at the time of the interview. Like Holly, the interviews took place while he was in the midst of a counting unit for that class. He also noted that he had learned some counting while helping his wife in a community college mathematics course. He therefore had counting resources available to him, and he seemed comfortable with tree diagrams, factorials, choose notation, and certain prototypical problem types. In comparison with the other interviewees, Liam displayed relatively strong problem solving abilities throughout both interviews. This was evidenced by a tendency to take his time during the problems, and his work was marked by regular monitoring – he tended to read (and re-read) the problems carefully, frequently ask whether his work made sense, check his work for reasonability, and make sure his methodology was appropriate.

We highlight one example from Liam’s interviews, in which he worked through the Apples/Oranges. We will see that Liam was able to create a correct and complete representation of the solution set, which he referred back to often. Despite some initial missteps in solving the problem, having this visual referent allowed him to detect solutions he had missed and ultimately arrive at the correct solution.

After asking some clarifying questions about what the problem was asking, Liam properly articulated that he was trying to count non-empty collections of fruit. He asked himself aloud, “How can I represent that?” and he proceeded to write down a picture of a Cartesian plane. In this picture, he drew an 8x5 rectangle oriented on a set of axes, where the x-axis represented oranges and the y-axis represented apples, saying that “each point is a possibility.” Liam thus properly mapped the set of solution as points on a Cartesian plane. However, although he recognized the points as being the possibilities, he proceeded to compute the area of the rectangle, which misses the points along each axis. He arrived at an initial solution of 5*8-1=39 (the minus 1 was for the (0,0) point).

Because of his strong problem solving skills, however, Liam went on to check his work by counting out some of the solutions. He wrote a column of 5 4 3 2 1 0, (see below) representing the zero to five apples, and wrote an 8 next to the first four numbers (for the one to eight oranges) and a 7 next to the last number (he knew this had to be one less than the other cases because he wanted to avoid drawing zero of each). He added up the right column and got 47, which was in contrast with his previous answer of 39. After some thought, though, and after referring back to his diagram, he said, “Okay it should be 47, because I’m starting at zero not starting at 1.”

5	8
4	8
3	8
2	8
1	8
0	7
	47

At this point, Liam paused to address the discrepancy he found. After reviewing his solution some more, he stated the following, “Where did that go wrong? I over-counted by 8. Oh, is that why, because it’s inclusive of this line (the top of the box), which has 8 on it? Okay, so it should be 47.” Subsequent to this, Liam proceeded to explain his work back to the interviewer, and in doing so he again noticed a problem with his method, and he referred to his diagram of the Cartesian plane to do so. He continued to work through the problem and check his work, and he eventually arrived at the correct answer of 53. It is significant that Liam had a correct, complete, concrete representation before him of what he was trying to count. We believe that, due in large part to his correct visual representation of the solution set, along with his solid problem solving abilities, Liam was able to successfully navigate this problem.

Liam’s problem solving skills were significant in this problem, and his work was marked with monitoring such as checking for reasonability, carefully analyzing his work, and writing down a visual representation of what he was trying to count. What is most noteworthy, though, is that Liam’s visualization of the problem led to a model of a complete solution set (the points on the Cartesian plane). Although he did not immediately recognize this (he started with the area of the rectangle instead of the points along its perimeter), he was eventually able to recognize the solution in the model. Indeed, his model enabled him to refer back to his solution set and consider what he had counted, and it allowed him to identify solutions he had missed and ultimately to arrive at the correct solution.

Conclusion

We see from the case studies above that having a notion of a solution set appears to be particularly important in counting. Without such a notion, students are prone to hunt for surface-features of counting problems, and they may be unable to make sense of and utilize their counting procedures. But, with such a notion, students are better equipped to detect errors and arrive at a correct solution. While students in the study overwhelmingly did not show evidence of thinking in terms of a set of solutions, there is an indication that even a partial representation of a solution set helped students to be more successful in their counting activity.

This is a preliminary report, and its primary purpose is to inform and situate further research on the teaching and learning of combinatorics. Thus, even more than standing on their own, the results and conclusions here serve a purpose of guiding and directing subsequent work. In particular, the data seemed to corroborate the conjecture in the conceptual analysis about the importance of solution set; directions for further research therefore include an increased emphasis on solution set thinking. Perhaps more data could be gathered in order to find effective ways to uncover students' notions of solution sets. Also interesting would be to look for more evidence to help discover whether more successful counters tend to display stronger solution set thinking. Additionally, given the importance that a notion of solution sets seems to play in counting, there is a potential instructional design component about solution set thinking. It could be the case that the instruction of counting and counting principles (such as the addition and multiplication principles) might be framed in terms of solution sets, and that this could foster in students a deeper, more consistent approach to counting.

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