Strong Metaphors for the Concept of Convergence of Taylor Series

Jason H Martin
Arizona State University
Jason.H.Martin@asu.edu

Michael Oehrtman
Arizona State University
Michael.Oehrtman@asu.edu

We present results from interviews that were conducted with university calculus, real analysis, and numerical analysis students in an effort to characterize their conceptions of the convergence of Taylor series. During a detailed analysis of the interviews, we discovered that several students consistently relied on a single metaphor throughout several tasks. We were surprised by the students' commitment to these metaphors (emphasis) and the degree to which they influenced student responses (resonance). In this paper we describe some of the metaphors that students used and how they appeared to both enable and constrain the students' reasoning.

Introduction

Taylor series have been used throughout history by scientists and mathematicians, like James Gregory, Isaac Newton, Gottfried Leibniz, Leonard Euler, and Joseph Louis Lagrange, to approximate functions using polynomials. In response to Bishop Berkley's criticisms of Newton's lack of development of a rigorous calculus, Lagrange even attempted, although unsuccessfully, to avoid Newton's use of infinitesimals and make Taylor series the foundational building block upon which all of calculus was constructed (Burton, 2007; Grabiner, 1981). Today, Taylor series are frequently used in physics and engineering to simplify complicated equations and they play a foundational role in the theory of complex analysis. Even so, the topic of Taylor series is usually treated in four or fewer sections of a traditional calculus text (e.g., Hass, Weir, & Thomas, 2007; Larson, Edwards, & Hostetler, 2005; Stewart, 2008). Because of the many applications of Taylor series, students may revisit Taylor series in theoretical or applied classes such as differential equations, analysis classes, modern physics or physical chemistry, or a various engineering courses. We ask the question, what images guide students' reasoning about the convergence of Taylor series as they establish this initial conceptual foundation?

Background

Portions of this paper are part of an initial study conducted in partial fulfillment for a degree of Doctor of Philosophy in Mathematics (Martin, 2009). At that time, a review of the literature revealed that very little research had been conducted specific to student understanding of the concept of Taylor series. In most cases, any study that addressed convergence of Taylor series did so while considering a broader topic, such as limits or function approximation techniques. Two articles that address Taylor series in detail are Kidron and Zehavi (2002) and Kidron (2004) which both report on a qualitative study that enabled them to describe how animations from a CAS were used to enhance student comprehension of approximation techniques using Taylor series before the development of the formal theory. These reports showed that the CAS not only allowed the students to get a sense of the problem context, but the CAS spurred students to ask questions concerning the material and provided a tool for them to investigate their questions. Even so, Kidron and Zehavi (2002) reported that the dynamic graphs were a "source of trouble" in formulating the formal ε - N definition for limit of sequences (p. 226), and Kidron (2004) observed that students who had not done the actual analysis work (on paper) struggled with interpreting the graphical images (p. 328). Even though both articles contain examples of different types of student reasoning while engaging in tasks related to Taylor series, neither Kidron and Zehavi (2002) nor Kidron (2004) attempted to elaborate on the details of the different student conceptions.

Other works that included references to Taylor series while investigating broader issues are Oehrtman (2002, 2009), and Alcock and Simpson (2004, 2005). Utilizing conceptual metaphor from interaction theory (Black, 1962, 1977), Oehrtman (2002, 2009) characterized "students' spontaneous language and patterns of reasoning about limits as they emerged in the process of learning" (2002, p. 2) in an attempt to "capture global patterns in students' responses" (2002, p. 103). To achieve this goal, Oehrtman observed calculus students engaging in novel limit problems, including a Taylor series problem. When encountering this Taylor series problem, Oehrtman (2002, 2009) describes several metaphors he observed students using, for example, approximation, collapse, and proximity metaphors. Perhaps the most notable metaphor applied to Taylor series, is the approximation metaphor which is characterized by structures involving "estimates," "error," "accuracy," etc. which involve an unknown actual quantity and a known approximation. For each approximation there is an associated error and a bound on the error.

$$\text{error} = |\text{ actual value} - \text{approximation }|$$
$$\text{approximation} - \text{bound} < \text{actual value} < \text{approximation} + \text{bound}$$

The approximation is viewed as being accurate if the error is small and the accuracy of the approximation improves with each successive step. Oehrtman (2002) noted that students using this metaphor may describe Taylor polynomials as approximations to a generating function with a corresponding remainder that is the difference between the generating function and the approximating Taylor polynomials. Since the structure of the approximation metaphor reflects the structure of the ε - N definition for series, this metaphor can lend itself to allowing students to develop a more formal understanding of the limit concept.

Alcock and Simpson (2004, 2005) discussed student responses to several questions concerning sequences and series, and one of these questions was particular to Taylor series. In their exploration of responses of students who tended to produce visual representations verses those who tended not to produce visual representations, they found that students with an internal sense of authority and those who had an ability to link concepts together were more likely to produce correct responses. Of the few cases where Alcock and Simpson specifically referenced student responses to the Taylor series task, they mostly did so to give an example of an erroneous response.

Since none of the studies mentioned above specifically attempted to analyze and describe conceptualizations of the convergence of Taylor series by identifying and categorizing particular reasoning patterns, Martin (2009) sought to achieve this objective while accounting for different levels of exposure to series. To address this study goal, data were collected from 131 students in undergraduate mathematics classes, 10 graduate students, and 6 faculty from a mid-size four-year university and from a regional community college. All 131 students completed an in-depth questionnaire about their understanding of Taylor series, and eight of these students subsequently participated in nor more than two face-to-face, task-based, individual interviews. For a list of some of the interview tasks that will be mentioned in this paper, see Table 1.

Table 1
Interview Tasks

First Ten Interview Tasks for All Participants
1. What are Taylor series? 2. Why are Taylor series studied in calculus?
3. What is meant by the "=" in "$\cos x = 1 - x^2/2! + x^4/4! - x^6/6! + - \cdots$ when x is any real number?" 4. What is meant by the "$(-1,1)$" in, "$1/(1-x) = 1 + x + x^2 + x^3 + \cdots$ when x is in the interval $(-1,1)$?"
5. What is meant by the word "prove" if you were asked to, "Prove that sine is equal to its Taylor series." 6. What are the steps in proving that $\sin x = x - x^3/3! + x^5/5! - x^7/7! + - \cdots$?
7. How can we estimate sine by using its Taylor series?
8. What is meant by the "approximation" symbol in "$\sin(x) \approx x - x^3/3! =$ a Taylor polynomial for sine when x is near 0?" 9. What is meant by the "near" in "$\sin(x) \approx x - x^3/3! =$ a Taylor polynomial for sine when x is near 0?"
10. How can we get a better approximation for sine than using $x - x^3/3!$?

* Tasks appearing in the same row appeared on the same page of the interview handout.

To account for the effect of the amount of exposure of undergraduate participants with Taylor series, undergraduate participants were selected from calculus, real analysis, and numerical analysis classes after having prior exposure to Taylor series. Using the construct of concept images developed by Tall and Vinner (1981) and adapting questionnaire tasks from Williams (1991), Martin (2009) categorized some of the different ways in which experts and students conceptualized Taylor series. Some of the concept images utilized by experts and students included what Martin (2009) described as pointwise, sequence of partial sums, dynamic partial sum, remainder, and termwise (p. 145). Not only did Martin (2009) find that experts tended to utilize more conceptual images, but that they were more prone to utilize multiple conceptual images in close proximity and they could move between conceptual images more efficiently and effectively as needed. In addition, most students in Martin's study tended to lack a well-formed visual image of Taylor series convergence while experts did not. For the student who did not lack a well-formed image, he utilized multiple conceptual images throughout the interviews in ways similar to that of an expert. Therefore, Martin (2009) suggests that the part of the gap between expert and novice understanding of convergence of Taylor series may be bridged through pedagogical approaches to Taylor series incorporating more graphical images.

For the purposes of this paper we focus our attention primarily on undergraduate student participants and attempt to identify patterns of metaphorical reasoning employed by individual students when addressing Taylor series convergence.

Metaphorical Reasoning

We analyzed interview data for student reasoning by employing a theoretical perspective of conceptual metaphor based on Max Black's (1962, 1977) interactionist theory and as used by Oehrtman (e.g., 2002, 2009). In general, metaphorical reasoning involves conceiving of unfamiliar aspects of a literal domain in terms of similar aspects of a more immediately understood metaphorical domain (*Figure 1*). Reciprocally, the selection of the metaphorical domain and attention to and interpretation of its important characteristics is influenced by the existing and emerging conception of the literal domain. Metaphorical attribution is achieved through the interaction within the resulting dynamic system. This dialectic allows for conceptual innovation that far exceeds what is possible by reasoning entirely within either domain. In many cases, metaphorical reasoning can serve to reduce the cognitive load entailed when reasoning with and about complicated mathematical structures involving the interaction of multiple concepts. Black distinguished emphasis and resonance as necessary characteristics of strong metaphors, those that have the potential to be ontologically creative for the user. Emphasis is the degree to which the user is committed to applying the chosen metaphorical domain and resonance is the degree to which the metaphor can support "elaborative implication," the development of additional inferences not contained within the original metaphor. Black (1970) also distinguished metaphorical reasoning using only proverbial knowledge and commonplace inferences from reasoning with theoretical models requiring systematic complexity and capacity for analogical development. Scientists and mathematicians may already possess a well-developed conceptual structure and use rigorous criteria for employing a theoretical model to help reason about a new concept. Students' metaphorical reasoning, however, is often highly idiosyncratic and based on the most salient images available at a given moment of conceptual development.

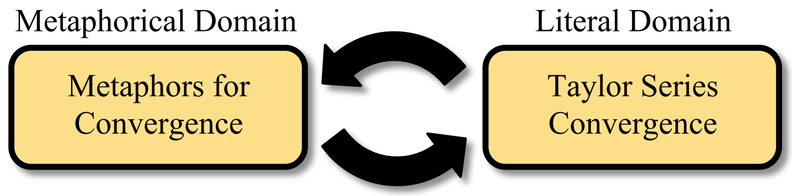


Figure 1. Dynamic Interaction Between Domains

Results

Some of the metaphors that emerged from our analysis of calculus and analysis students reasoning about convergence of Taylor series are similar to those previously described by Oehrtman (2002, 2009). After a close analysis of individual student interviews, additional metaphors based on part / whole relationships and operations preformed on cutable objects emerged from the data. We were surprised by the degree to which the individual students using each of these metaphors persisted in their usage of the metaphor throughout the interview tasks and the number of implications inferred from these metaphors concerning Taylor series. In this section we will highlight different aspects of each of these metaphors and the resulting implications for student reasoning.

For each metaphor we will focus on one student's, Brian for part / whole and Jordan for cutable object, usage of the metaphor in response to multiple tasks concerning Taylor series convergence. Brian and Jordan were not only different in their metaphorical implementation but also in their formal understanding of the literal domain, Taylor series convergence. From an epistemological perspective, one might conclude that Brian and Jordan are at different stages of understanding of Taylor series convergence. For instance, Brian was consistently imprecise in his responses to interview tasks as references particular to the mathematical structure of Taylor series were missing in most of his comments. Therefore, Brian should not be viewed as an expert reasoning scientifically about Taylor series in some systemized way, but as a novice who was still making sense of Taylor series and its corresponding implications. In contrast Jordan was more specific in his responses to Taylor series tasks and his responses better reflect the mathematical structure found in Taylor series. Even so, there are certain entailments of the metaphor utilized by Jordan that led him to erroneous conclusions. Thus, to better understand Brian's and Jordan's reasoning, we will focus on the underlying metaphorical structure and its implications, mathematically correct or not, on their emerging understandings of Taylor series convergence.

Part / Whole Metaphor. In the part / whole metaphor, there is a whole and there is a part and nothing else because the difference between the part and whole is not emphasized (*Figure 2*). In the context of determining some sought after convergent, the part is viewed as insufficient and perhaps potentially misleading, whereas the whole is able to orient the user and allow him to precisely identify the convergent. We will begin this section by looking at the orienting nature that the whole had for Brian and then look at the role of the part in relation to the whole.

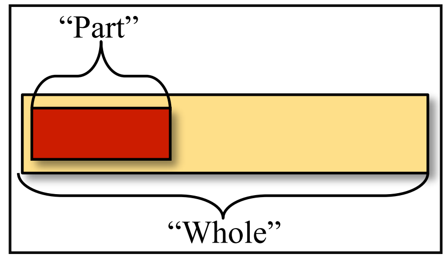


Figure 2. Part / Whole Metaphor

Consider the following excerpts:

Interview Task 3: What is meant by the "=" in
"$\cos x = 1 - x^2/2! + x^4/4! - x^6/6! + - \cdots$ when x is any real number?"

Brian: I'm thinking it has something to do with like a Riemann sum. That's just what comes to mind. Uh, if I add up, if I were to add up all of these it would give me a definite point.

In the above excerpt Brian appeared to cue off of his prior notion of Riemann sums, and later when attempting to describe what he meant by his reference to "Riemann sum", Brian added,

> I'm thinking that because these are all fractions [pointing to the terms of the series] of I guess cosine curve or function, it's gonna give me one single point. Um, kinda of a sum-, it's gonna, it's gonna give me a summary more or less is what I'm thinking since it's all these little points adding up to one point. I'm thinking it's gonna converge into something.

Even though it is unclear what "these" were for Brian (he did not explicitly say that "these" were terms of the Taylor series), or what "fractions" were for Brian (he did not state that they were coefficients for x or if he had imagined plugging in numbers for x to get "fractions" to be added up in the expanded Taylor series), there is a consistent structure in which an "all" yields a sought after single resultant through some limiting process. In the first excerpt, the "all" gave a "definite point" through an "add up" process. In the second excerpt, the "all" converged into "one single point" that was a "summary" of the "all". Subsequently in the interview, Brian was asked what he meant by his reference to "converge into something" found at the end of the excerpt, and Brian reiterated that he meant that "it's going to come to a point" and that this point was a specific number. Later in the same interview, when asked, "How would you go about estimating $\sin(103)$?" Brian's response included a reference to "all those numbers" that got him "one number that we're looking for" through a "summation" process.

In Interview Task 4, Brian was asked about the meaning behind the interval of convergence $(-1,1)$ for the geometric series $1/(1-x) = 1 + x + x^2 + x^3 + \cdots$. Twice Brian referred to the infinite amount of numbers in the interval $(-1,1)$, but his understanding of what the Taylor series was doing with any given number from the interval was not transparent. This prompted the interviewer to ask Brian to clarify what could be done with the infinite amount of numbers that he was referring to in relation to the given Taylor series. Brian responded,

> Well, like in number 3, I'm thinking in these infinite amount of numbers you're going to find some type of mass. I mean it's gonna be… If I were to add up all of them, it would somehow equal one finite number [holds both hand up as if holding something between] as oppose to all these infinite numbers [moves both hands away from each other]. I mean, it's gonna be just around this number consist-consistently.

Brian's use of "mass" appears to illustrate a limiting process in which an infinite amount of numbers are "around" some sought after number "consistently." Given all the numbers, a "massing" of numbers does not yield an infinite amount of numbers but instead oriented Brian to be able to identify the sought after "one finite number." Although mathematically imprecise, all the previous excerpts point to a metaphorical structure used to reason about Taylor series. The whole, whether it was composed of terms of a Riemann sum, fractions of a Taylor series, a collection of little points, or numbers, through some limiting process, like an "add up" process or "massing", oriented Brian to be able to precisely identify a sought after single resultant, usually in the form of one point or one number (*Figure 3*).

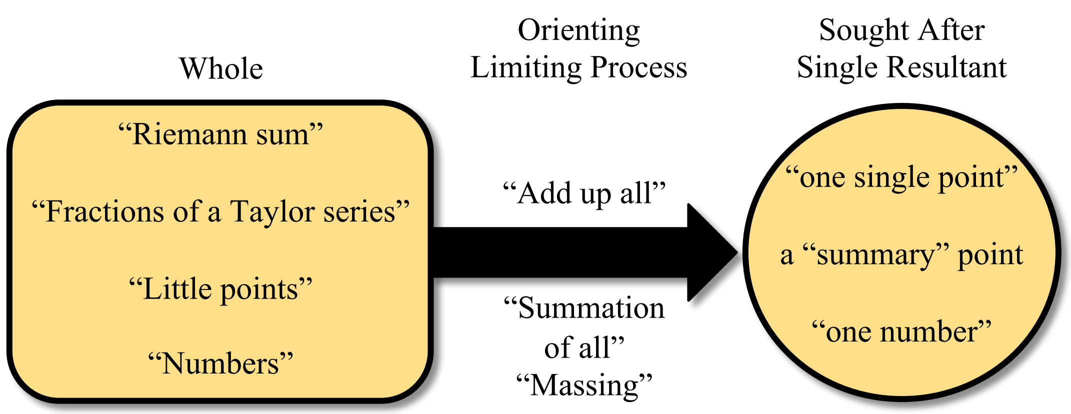


Figure 3. Brian's Orienting Nature of the Whole

Now we will look at the role of the part in relation to whole. In the following excerpts, the part is seen as insufficient when compared to the whole and lacks the orienting nature that allowed Brian to precisely identify the sought after convergent. As with previous excerpts, in the following excerpts Brian avoided particular references to the mathematical structure of Taylor series. Instead, Brian seemed to revert back to talking about something that he may have believed he had a well developed ability to discuss, convergence on a number line.

Interview Task 7: How can we estimate sine by using its Taylor series?
Brian: Well, it's called an estimate because it's not exactly that specific number that it's, uh, revolving around. It's just gonna be somewhere in the ball park of that specific number.

Following these comments, Brian alluded to plugging in numbers whose numerical representation contained several "9's" following the decimal, and then concluded

> If I were to think of a [holds both hands up with palms facing each other], just a number line, you know, I'm coming from the left hand side [moves left hand inward], I'm coming from the right hand side [moves right hand inward], and this is the number it's gonna stop at [moves both hands very close to each other]. It's an estimate, it's not exactly reaching it, but it's the best we can do.

In all the previous excerpts, not just the last two, Brian avoided specific details of Taylor series and talked about Taylor series using more global terminology. If fact, most of Brian's previous utterances easily translate to convergence on a number line. Furthermore, all of Brian's gestures are consistent with convergence on a number line since his gestures lacked any indication of a second dimension. It was as if all the points or numbers on a number line were adding up or massing to one single point or number on the number line (*Figure 4*). In the two previous excerpts, estimates played the role of the part and were seen as "not exact" and "not reaching" but in the "ball park" of what was sought after. Therefore, like the part, estimates were insufficient for precisely identifying a desired resultant but were "the best we [could] do" when trying to locate the resultant.

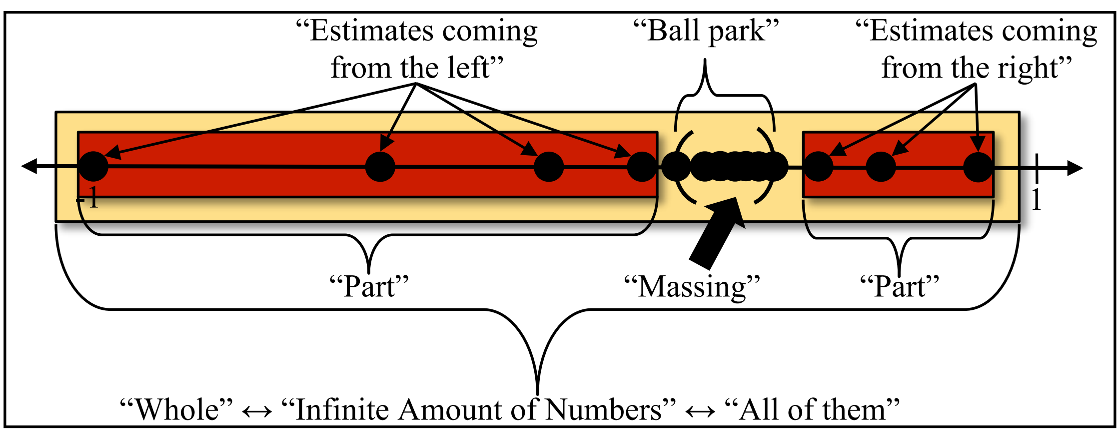


Figure 4. Brian's Part / Whole Metaphor on a Number Line

The aspects of the part / whole metaphor projected onto the Taylor series literal domain result in the infinite sum seen as the whole, whereas the part is the finite Taylor polynomial. In the next excerpt Brian specifically addressed the role that a Taylor polynomial plays in approximating the Taylor series.

Interview Task 8: What is meant by the "approximation" symbol in "$\sin x \approx x - x^3 / 3!$ = a Taylor polynomial for sine when x is near 0?"

Brian: Just because it says approximation and nothing else, I'm, I'm guessing it's gonna equal only a portion of what the whole Taylor series would equal. It's not gonna equal the whole answer, it's just gonna get me one little section of it [holds up left hand with thumb and index finger extended close together].

Here the third degree Taylor polynomial $x - x^3 / 3!$ was viewed as only a "portion" of the whole. Again, the lack of the orienting nature of the part was implied since the "portion" only gives "one little section" of the whole. It should be noted that there is no indication that Brian's reference to the "little section" and the corresponding gesture was related to some interval upon which Brian believed that the Taylor polynomial was an accurate approximation to the generating function. Instead, his references to "little sections" and the accompanying gestures seem to indicate a purely formulaic conception. That is, Brian appeared to compare and contrast a little section of the Taylor series expansion viewed as formula verses the formula for the whole Taylor series expansion (*Figure 5*). Immediately following the previous comments, Brian spontaneously proceeded to reason by analogy using Google Maps.

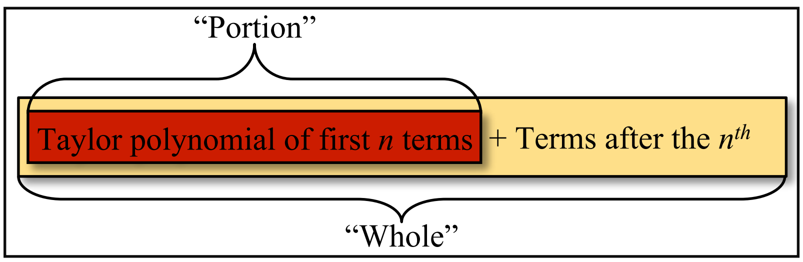


Figure 5. Brian's Part / Whole Metaphor Applied to Taylor Series

> It's almost like Google Maps. I'm gonna show, you know, [holds up left hand with thumb and index finger extended close together] this one little section, but if I, if I pan out or whatever, gonna show me [circular motion with right hand] exactly everything. I think the Taylor series is like the whole view [holds up both hands extended across body with palms facing each other]. And any time I show an approximation [pointing to the Taylor polynomial in Interview Task 8], it's just gonna give me that [holds up right hand with thumb and index finger extended close together] little piece.

In this analogy he created a correspondence between Google Maps and Taylor series where the "little section" of Google Maps was related to Taylor polynomials and the "panned out" view of Google Maps was related to the "whole view" of Taylor series. The "panned out" view in Google Maps allowed Brian to orient himself to determine the exact location that he was looking for, whereas the little section was insufficient. For example, in *Figure 6*, if one focuses only on the "little section," because of the indication of water on the right, one might be led to believe that the "little section" is on the east coast of a continent or at least the east coast of some lake. Only after seeing the relationship of the "little section" to the "panned out" view is one no longer mislead and is able to conclude that the "little section" is in San Diego. Thus, determining the exact location of that "little section" would be an exercise in futility without the "panned out" view. For Brian, this is similar to the relationship between the Taylor polynomial and the "whole view" of Taylor series. Therefore, this prominent image available to Brian appears to have influenced his understanding of part / whole relationships and subsequently, his ability to elaborate on Taylor series convergence.

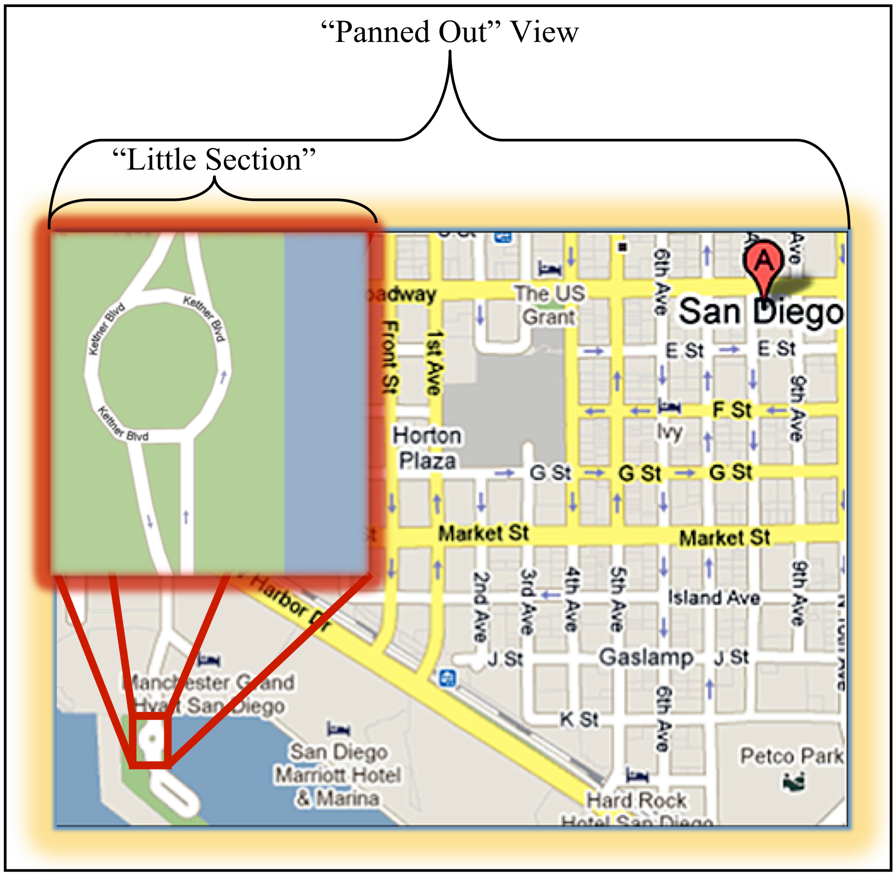


Figure 6. Brian's Google Maps Analogy

In another analogy, Brain likened the convergence of Taylor series to a grading scale (*Figure 7*). Brian claimed that if a student received an 'F' on one exam and an 'A' on another exam, then it would be difficult to tell which grade really represented the student's overall grade. Perhaps the first grade was a "fluke," and thus, a misleading piece of information. Brian claimed that considering more grades gives the teacher the ability to determine what grade the student actually should receive. Therefore, a few grades, which constitute a part of all the grades, are insufficient to determine the overall grade, but when all grades are taken as a whole, this orients the teacher to be able to correctly determine the overall grade.

It is also worth noting that when he was talking about the "little section" of Google Maps, the two grades, or about the "approximation" using Taylor polynomials, he used similar minimalist gestures, such as "thumb and index finger extended close together" or "moving both hands very close to each other with palms facing each other" as if easily grasping the "little section" or "approximation." In contrast to a small gesture, when "panning out" on Google Maps or discussing the whole view of the Taylor series his gestures embodied a larger scale, such as a

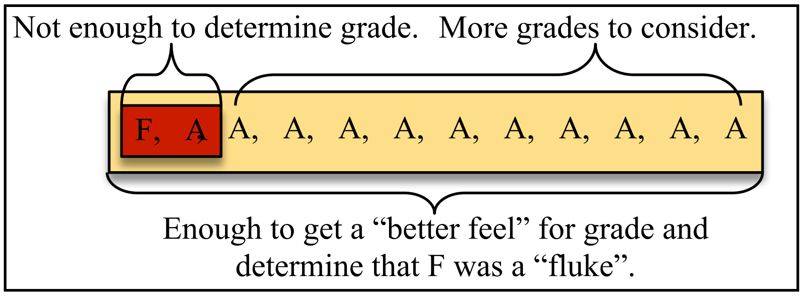


Figure 7. Brian's Grading Scale Analogy

"circular motion with left hand" or "moving both hands away from each other". These gestures taken together with his utterances unify the structure of the part / whole metaphor across seemingly very different contexts.

Immediately following his grading scale analogy Brian continued to apply the part / whole metaphor. Brian stated,

> I'm thinking if this one little piece of the Taylor series [pointing to $x - x^3 / 3!$] shows, just this one little piece [holds right hand up with thumb and index fingers extended close together], it's not going to give as much as opposed to maybe, you know, these other numbers are around that but it's going to zero in [points hands at each other with palms facing toward body] or home in on [points hands at each other with palms facing toward body] something more definite.

These comments further illustrate the insufficiency that Brian placed on the Taylor polynomial when compared to the Taylor series. The Taylor polynomial $x - x^3 / 3!$ simply did not "give as much" as the Taylor series. Once again, even though Brian began this excerpt by talking about Taylor series convergence in reference to the formula, Brian seemingly reverted back to talking about convergence as one would talk about convergence on a number line. Here the Taylor polynomial $x - x^3 / 3!$ not giving "much" was compared to all the "other numbers" that "zero in" and "home in on" on something "definite." Even though Brian made this switch from Taylor series formulas to points on a number line, the part / whole structure was still present.

When later asked what the Taylor polynomial approximation gave, Brian elaborated on his reference to it not giving "much" found in the previous excerpt. At first he simply restated that the Taylor polynomial gave an "approximation." Then he added that it gives what the Taylor series "might be" and "could be" in the form of an "estimate" and a "guess" that is "just one piece of the puzzle." Even though Brian alluded to the Taylor polynomial as being an "estimate" and a "guess" there is still an element of insufficiency when compared to the whole Taylor series. Therefore, for Brian, the part was one "little piece" that was insufficient and potentially misleading when compared to the whole that can "zero in" on something "definite."

Clearly, the extent to which Brian understood the role of Taylor polynomials in approximating Taylor series appears very minimal as indicated by his continued convention of discussing Taylor series convergence like one would discuss convergence on a number line. Thus, one could conclude that Brian never indicated a formal understanding of Taylor series convergence. Furthermore, one could argue that Brian never indicated a formal graphical understanding of Taylor series convergence. Even when directly presented with graphical tasks, he was unable to draw Taylor polynomials based on a given graph of a generating function and he was unsuccessful in stating any relationship to Taylor series when given graphs of Taylor polynomials. When asked to describe the graphical effect of adding more terms to a Taylor polynomial, Brian responded in his typical non-Taylor series specific language. Even so, one should not view Brian as having no understanding of Taylor series convergence, but instead, like many students encountering Taylor series for the first time, he has an emerging notion of Taylor series convergence that has not yet clearly distinguished itself from prior notions of convergence, such as convergence on a number line. Leveraging convergence on a number line and metaphor to reason about Taylor series convergence may have reduced Brian's cognitive load as he encountered Taylor series.

Even though Brian's understanding of Taylor series may be less than desirable, all of these excerpts for Brian point to a similar structure used to reason about convergence across various contexts. We call this structure the part / whole metaphor. Depending on the context, the part may be composed of some points on a number line, the first few terms of a Taylor series, a "little section" zoomed in on a map, or a couple of grades. The whole is all the points on the number line, all the terms of the Taylor series, a "panned out" view of a map, or all the grades. When used to determine a sought after convergent, the "part" is an insufficient and potentially misleading portion of the "whole." Depending on the context, the "part" may be an approximation that can give one a "better feel" for the sought after specific convergent but this is only an "estimate" and a "guess." Only the "whole" can orient one to be able to precisely identify the sought after convergent.

Cutable Object Metaphor. In the cutable object metaphor an operation of "cutting" is preformed on one object to produce two new objects each of which can then be operated on by being "picked up" and used as needed (*Figure 8*). We will begin this section by looking at the object that will eventually be "cut" and then look at the two objects that are produced from the cut. Like the previous section, we will use one student's responses to illustrate these objects and the operations performed on these objects.

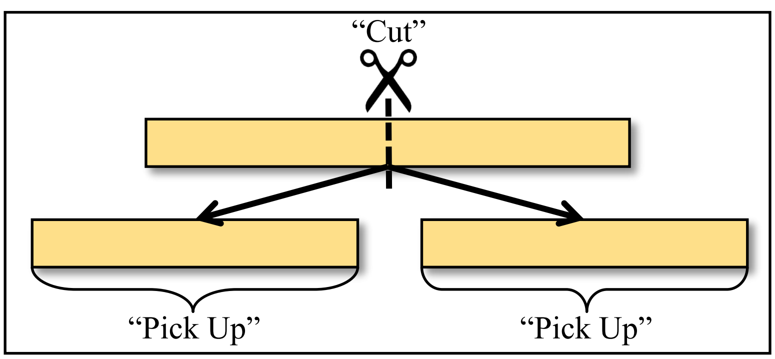


Figure 8. Cutable Object Metaphor

Consider Jordan's response the very first interview task found on the next page.

> Interview Task 1: What are Taylor series?
> Jordan: It's a sum of an infinite [moves right hand left to right touching the table three times] like polynomial, like from *n* equals whatever to infinity [waves right hand from left to right], or index, or whatever. And um, the summation is equal to the value of a function [right hand from left to right making frequent stops that touch the table].

Later in the interview, Jordan referred to the Maclaurin series for e^x as "just a representation of uh, e^x as a polynomial." When responding to a task involving the geometric series $1+x+x^2+x^3+\cdots$, Jordan failed to take into account the interval of convergence and stated that the domain of the series was all real numbers because "the domain of a polynomial is all real numbers." Jordan's persistent relating of Taylor series to infinite polynomials made a different points throughout the interview, and the immediacy of Jordan to make this relation found in the previous excerpt indicate the high level of influence that this image of Taylor series had on Jordan's reasoning. This image of Taylor series as an infinite polynomial is not new. Historical evidence suggests that mathematicians like James Gregory, Isaac Newton, and Brook Taylor all viewed Taylor series as infinite polynomials (Grabiner, 1981; Jahnke, 2003). Experts today reason using this image but they are also very aware of the image's potential pitfalls (Martin, 2009). Even though this image led Jordan into one of the image's pitfalls when he overlooked the role of the interval of convergence, we will see that it provided Jordan with an avenue of reasoning that allowed him to correctly answer several questions concerning Taylor series convergence.

In the first excerpt, Jordan not only indicated that he viewed a Taylor series as an infinite polynomial, but that the infinite polynomial was "equal" to the value of the generating function. On another occasion, Jordan noted that "for any given x" the Maclaurin series for cosine "would equal cos x." During his response to Interview Task 8, Jordan noted that the Taylor polynomial $x-x^3/3!$ is not the exact value for sin x but to get the exact value one has to use "every single term of the infinite, of the infinite, uh, polynomial." Not only did Jordan view a Taylor series as identical to the generating function formulaically, he viewed them as being identical graphically. In Questionnaire Task 5, Jordan was given the graph of sin x from -4π to 4π and asked to graph two Taylor polynomials. During the interview, Jordan brought up what the graph of the "entirety" of Taylor series would look like compared to just the graphs of the Taylor polynomials.

> Like the entirety of [Taylor series for sine] [sweeping motion with right hand from left to right across paper], the graph of the entirety of it, if that would be, okay, so it does equal the same thing, do that [traces over the curve in the air] and then do that [traces over an imaginary extension of the curve off of the paper]. I-I, or if it was a Taylor approximation where one is linear [hand motion in the air above the graph consistent with $y=x$], and then one is quadratic [hand motion consistent with a concave down parabola], one is, you know, cubic.

When asked if he would still trace out the curve for the generating function if the x-axis was extended to "100π, 1000π, 1000000π?" Jordan reiterated,

> For the Taylor series sure… Yeah, you can do that [tracing in the air with pen over an imaginary extension of the curve off of the paper]. Because uh, if *the* Taylor series [up and down movements from left to right touching table with right hand cusped down], if that's what we mean by that, is equal to sine [same left to right movement with right hand], then sure [tracing in the air with pen over the curve and beyond the paper], sine goes on forever periodically, same-same-same. [emphasis in original]

Throughout the interview Jordan persisted in affirming that the entirety of the Taylor series was identical to the generating function. Therefore, for Jordan nothing was missing from the Taylor series expansion to cause the expansion to be any different from the generating function. The Taylor series expansion was as much “there” as the generating function was “there” both formulaically and graphically. Hence, for Jordan this image of Taylor series expansions as infinite polynomials in their entirety contributed to each expansion’s encapsulation as an object.

One of the artifacts of viewing Taylor series as objects was that Jordan could now perform the operations of “cutting off” and “picking up” on the objects. As the next excerpt will demonstrate, Jordan referred to Taylor polynomial approximations as what you get when you “cut off” the formula of a Taylor series after a particular n^{th} term. Following the “cut” of the formula, the first n terms can be “picked up” to yield an approximation and the remainder is what is not “picked up.” The cut off / pick up operations suggest an idea of manipulating that which is already there. For example, one cannot “cut off” a series unless terms remain on each side of the cut after the operation of cutting has been preformed, nor can one “pick up” something unless it is there to be picked up. That is, if all the terms of a Taylor series are already present in their entirety in the form of an infinite polynomial, then a Taylor polynomial is what you get by cutting the infinite polynomial after a certain degree and picking up only those terms that are needed to achieve a desired accuracy. In contrast, if one viewed a Taylor polynomial through some add up process, a desired approximation is achieved by adding terms to a dynamically created polynomial. In the add up process, individual terms do not exist until prior terms have been added. Thus, the operation of “cutting” would not make much sense for someone who held this view.

In Interview Task 6, Jordan was asked to give the steps in proving that sin x equals its Maclaurin series. During his initial response, Jordan brought up the role that the remainder plays in this proof. This prompted the interviewer to have Jordan “elaborate on the remainder.” Jordan replied,

> Um. So if you cut it off at this point, uh [marks two vertical lines on the handout separating the first four terms of the Taylor series from the rest of the equation], if you cut it off and use just the terms available to you, the first four terms, this is a number and uh, anything after that [circles the ellipses], which would include plus one over uh er, nine, yeah, that's right, [while writing “$+x^9/9! - x^{11}/11!$”] nine factorial, x to the ninth, uh, over eleven factorial, minus, uh, x to the eleventh, these are all trailing terms [circles the terms that he had just written] that also have a value if you add them together [holds both hands to the right as if holding something between them], and the remainder is what you didn't [waves right hand to the right] pick up.

Note that in the above excerpt, Jordan "cuts off" the first four terms of the Taylor series from the rest of the series. He indicated his understanding that more terms of the series exist following x^7 / 7! by relating the ellipses to the terms "after that" which he later exemplified by both verbalizing and transcribing the next two terms. In addition, he attributed the remainder, as something that one gets when the infinite polynomial is "cut off" where one side of the "cut" yields an approximation while the other side yields the remainder which is what is not "picked up" because it is not the approximation. Therefore, the cutable object metaphor, illustrated in *Figure 8*, when projected onto the Taylor series literal domain now resembles *Figure 9* for Jordan.

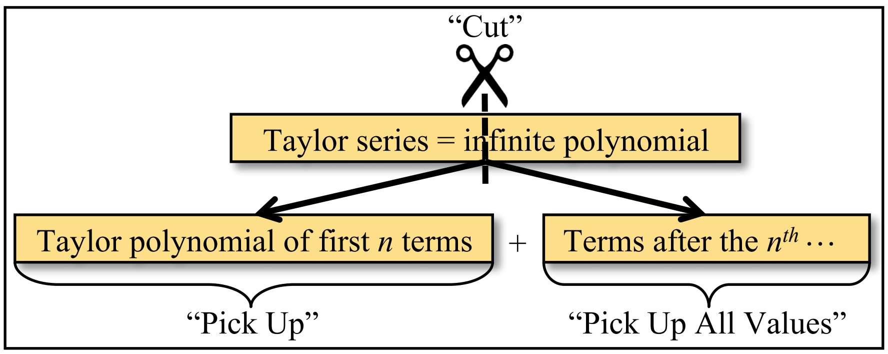


Figure 9. Jordan's Cutable Object Metaphor Applied to Taylor Series

This view appears to have led Jordan to an image of remainder, not as a difference, but as "trailing terms". In this task he equated the remainder to the "trailing terms" that start with x^9 / 9! and continue on with terms of higher degree as additionally indicated by the wave of his right hand to his right. It should be noted that Jordan's responses to other tasks clearly indicate that he additionally conceptualized remainder as a difference graphically (the length of a vertical line drawn between the approximating polynomial and the generating function) and algebraically (the absolute value of the difference between the approximating polynomial and the generating function for given values of *x*), but when strictly viewed through the lens of the cutable object metaphor he depicted the remainder as "trailing terms" of an expanded series.

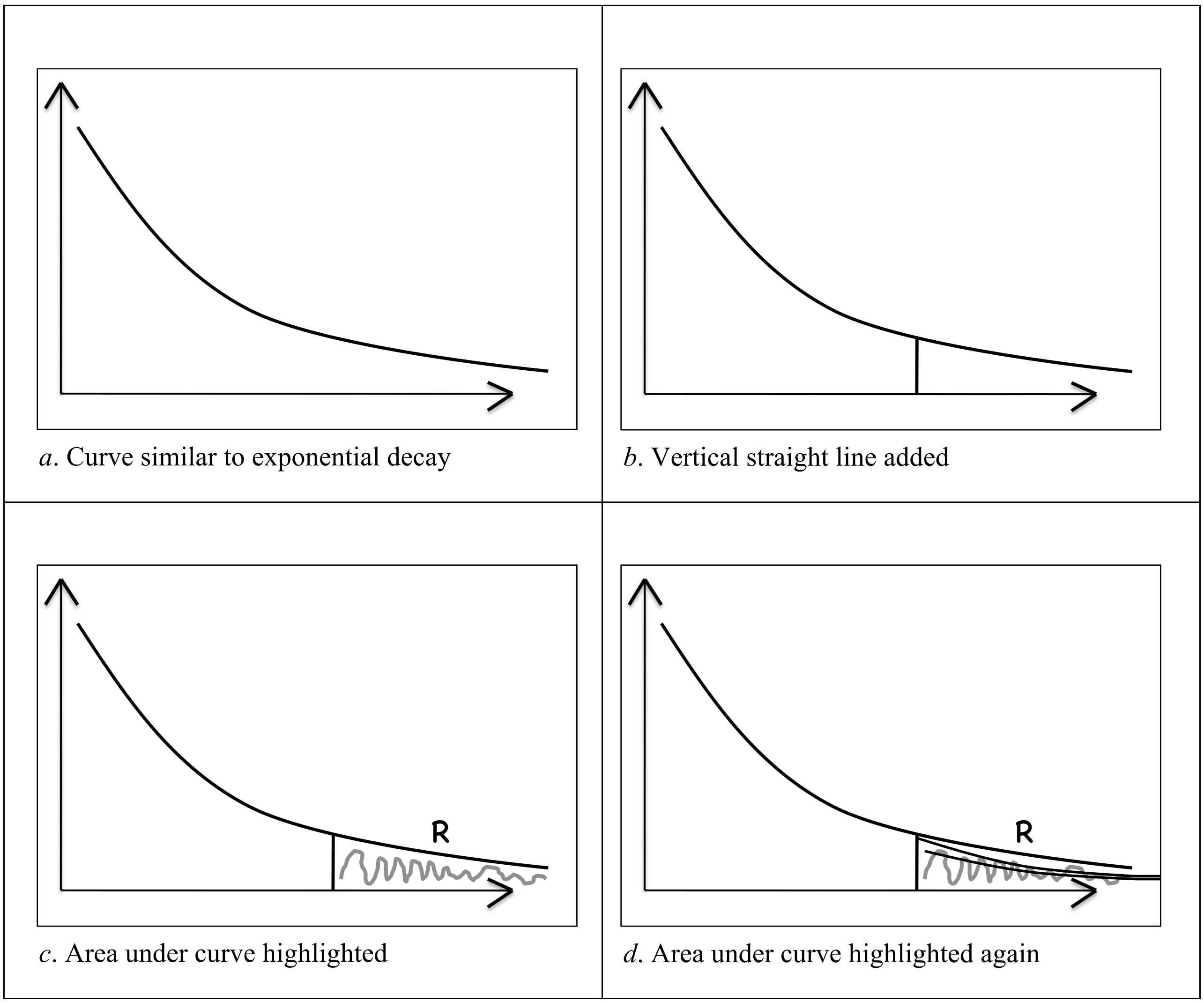
Immediately following Jordan's comments in the previous excerpt, Jordan spontaneously produced a graphical image.

Jordan: So, say if we have a graph [begins graphing] of this as it goes to infinity and you only add up, um [mumbles "y, x, whatever" and produced *Figure 10a*]. Um, if you only cut off this amount of term [makes the vertical line in *Figure 10b*] and this goes to infinity [highlights the right portion of the graph under the curve in *Figure 10c*], this has value and this is your remainder [writes the "R" found in *Figure 10c*].

I: Okay.

Jordan: So, if the remainder goes to zero, it means that you're [draws first nearly straight line directly below curve found in *Figure 10d*] picking up every single value [draws 2^{nd} nearly straight line below curve found in *Figure 10d*], um, and yeah.

Even though this image constructed in *Figure 10* is seemingly unrelated to Taylor series, Jordan likened it to Taylor series utilizing the cutable object metaphor and thus demonstrated the saliency of this image to the metaphor. In this case, the “cut”, instead of separating an expanded Taylor series into two objects, the cut separated the area under a curve into two pieces. He then appeared to note that the corresponding *x*-values on the right of the “cut” went to infinity, but regardless they still “have value” as the remainder. Implicit, is that the area to the left of the “cut” “has value” and can be “picked up” as the approximation. The implication of the remainder “having value” and his ability to “pick up every single value” are indicative of the encapsulation of the two pieces resulting from the “cut” as objects. Therefore, these seemingly unrelated contexts are unified by the structure of the cutable object metaphor (*Figure 11*).



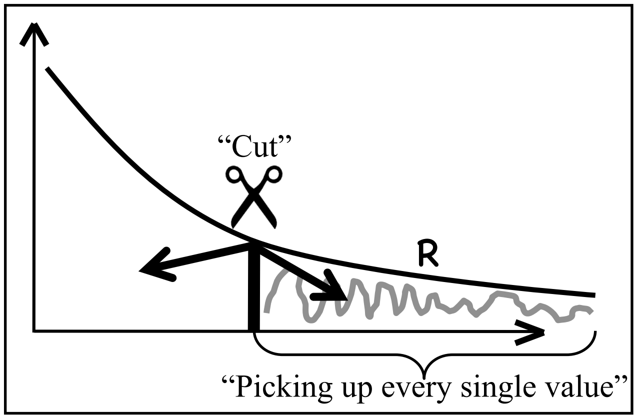
a. Curve similar to exponential decay

b. Vertical straight line added

c. Area under curve highlighted

d. Area under curve highlighted again

Figure 10. Reproduction* of the Progression of Jordan's Graph of a Cutable Object
*All images in this figure are reproduced based on the individual’s finished graph. The order of the steps were determined based on the audio and video evidence.



F*igure 11.* Cutable Object Metaphor Applied to Jordan's Graph

In the previous excerpts, Jordan has mainly focused on elaborating on the remainder properties of one of the objects that was produced from a "cut" in the context of expanded Taylor series. Most implications about the approximation properties from the second object produced from the cut were inferred from tacit clues. The next excerpt better confirms these implications of the non-reminder object produced from the "cut" as an object that is "picked up" for approximation purposes. In this case, Jordan will not specifically utter the words "pick up" but will instead use language suggestive of a structural equivalence to the "pick up" operation.

Interview Task 7: How can we estimate sine by using its Taylor series?
Jordan: Ah, perfect. You just, depending on the degree of accuracy you desire, you [waves right hand from left to right] take the terms of the Taylor series. Um, if you only want it to a certain decimal point, [turns the page back] going back to this [points to sine's Taylor series expansion in Interview Task 6], uh, we have this Taylor series representation and um if you want it within a specific degree of accuracy, you can take [highlights the first two terms in sine's Taylor series expansion in Task 6] the first two terms and that will give you some, some estimation, obviously not perfect.

In this excerpt, Jordan "takes" terms from a Taylor series to achieve a desired "degree of accuracy." In context of all the other excerpts, there is no indication to suggest that Jordan was currently viewing the Taylor series as anything but an infinite polynomial in its entirety. Later in the interview Jordan revealed that he viewed the "degree of accuracy" as a measurement of "how close … the approximation is to um, what you're approximating." The "taking" of terms illustrates the "pick up" operation of the cutable object metaphor in that it entails the same implications the "pick up" operation. For example, like with the "pick up" operation, terms cannot be "taken" unless they are already present. Furthermore, the "taking" of terms from an infinite polynomial in its entirety necessarily entails a separation (i.e. a "cut") from those terms not taken. Jordan concludes by additionally clarifying what he meant by "taking" terms of the Taylor series to achieve a desired accuracy by giving an example of where the first two terms of the series were "taken" to yield an estimation, albeit a not so "perfect" estimation.

Moreover, while responding to a questionnaire task, Jordan concluded that the geometric series $1+x+x^2+x^3+\cdots$ is an approximation to $1/(1-x)$ provided that he can "cut it off [one chopping motion with right hand]" and use a finite number of terms. But when using "infinitely [sweeping motion with right hand left to right]" many terms, the geometric series $1+x+x^2+x^3+\cdots$ is "identical" to the generating function $1/(1-x)$.

All of these excerpts for Jordan point to a similar structure used to reason about Taylor series convergence when responding to a range of tasks. We call this structure the cutable object metaphor. For Jordan, an image of Taylor series as an entire infinite polynomial appeared to contribute to its encapsulation as an object and subsequently, the encapsulation of the two objects following the "cut." The operation of "cutting" was preformed on the infinite polynomial object to produce two new objects, an approximation object and a remainder object. Each of these could then be "picked up" by Jordan and used as needed when encountering to a variety of tasks.

Discussion and Implications

The implications of part / whole and the cutable object metaphors elaborated above gave Brian and Jordan the facility to make sense of both approximation and proof tasks concerning Taylor series convergence across multiple contexts, and thus, demonstrated the high degree of resonance for each within the user. The immediacy with which the metaphors sometimes appeared, the respective commitment to these metaphors by Brian and Jordan throughout the interview tasks, and their frequent omission of other metaphors indicate the high degree of emphasis that they placed on the metaphor that they utilized. Therefore, for Brian and Jordan, the part / whole and cutable object metaphors, respectively, were strong metaphors that allowed them to creatively reason about Taylor series convergence.

When the approximation, part / whole, and cutable object metaphors point to the same mathematical object, they can appear to be nearly the identical because they all to various degrees reflect some of the structure of the mathematical object. These metaphors are not the same. For example, consider the remainder in each metaphor. For the part / whole metaphor, there is no remainder because there is only the part and the whole and nothing else. In the cutable object metaphor, the remainder is viewed as the tail of the series. Although this is the case for analytic functions on their respective intervals of convergence, this is not the case for those values of *x* outside the interval of convergence. Plus, the remainder as tail image may lead someone to incorrectly conclude that a Taylor series equals its generating function because the limit of the tail is zero as the initial degree goes to infinity. Only in the approximation metaphor is the remainder correctly viewed as the difference between the given Taylor polynomial and the generating function. Even though all metaphors contain an element that is used for approximation purposes, the approximation in the approximation metaphor, the part in the part / whole metaphor, and the approximation object produced from the cut in the cutable object metaphor, the approximation is either achieved through different means or has different entailments attached. In the cutable object metaphor, the approximation is achieved through the operation of "cutting" an object that is already in existence (the Taylor series viewed as an infinite polynomial in its entirety), but this operation is not necessary for the approximation metaphor because Taylor polynomials may be dynamically constructed through an add up process in which terms do not exist until prior terms have been successively added. The part from the part / whole metaphor is always viewed as insufficient and potentially misleading when compared to the whole, but this insufficiency and deceptive property is never attached to the approximation object in the cutable object metaphor. Equally, the orienting nature of the whole from the part / whole metaphor is missing from the approximation metaphor because in the approximation metaphor a good approximation can be achieved without knowing the entire

Taylor series. The approximation metaphor usually arises in response to an approximation task, but the part / whole and cutable object metaphors need not be limited to such tasks. Therefore, even though these metaphors can point to the same mathematical structure, their metaphorical structures are different.

Conceptual metaphors are not mutually exclusive, and multiple metaphors may be mixed by a single individual. A close examination of the transcripts revealed that both Brian and Jordan, even though their strong metaphors may be part / whole and cutable object, utilized elements of the approximation metaphor to various extents. For example, following the excerpt in which Jordan referred to "taking" terms of the Taylor series to obtain an approximation, Jordan used an analogy incorporating a computer that went "through a loop function" to estimate the value of *e* using a Taylor polynomial that was "added to" during each iteration of the loop. This analogy imposes the approximation metaphor by suggesting that a Taylor polynomial does not exist until the previous Taylor polynomial has been created by the computer and that the approximation gets successively better with each iteration. Following these comments, Jordan continued using the approximation metaphor in response to Interview Task 8, a task specifically asking participants to elaborate on the approximation properties of a given Taylor polynomial. In response to this task he mixed the approximation metaphor and with the cutable object metaphor when he noted that the "exact value" of the Taylor series, viewed as an "infinite polynomial," is obtained by "adding up" all the terms in the series. In this case, even though he referred to the "exact value" of the Taylor series as an "infinite polynomial," the approximation context of both the current and previous tasks appeared to influence his usage of the approximation metaphor. In summary, it appears that when Jordan cued off of his image of Taylor series as an infinite polynomial he was much more likely to use the cutable object metaphor before the approximation metaphor but when he cued off of the approximation properties of Taylor series, he may still use the cutable object metaphor but was more likely to use an approximation metaphor than when cueing off of Taylor series as an infinite polynomial. Therefore conceptual metaphors are not mutually exclusive but complement each other in a dialectic of metaphors for Taylor series convergence and their usage depends on various influencing factors idiosyncratic to each individual.

For a calculus student first encountering power series, Taylor series presents a complex mathematical structure that brings together many different concepts previously studied in calculus. Some of these concepts include the notion of variable, function, limit, sequences, and series which all interact together in the concept of Taylor series convergence. Not to mention the idea of pointwise and uniform convergence which play crucial roles when tackling questions concerning Taylor series convergence but for valid pedagogical reasons are many times in the background in a calculus classroom. Because of all these interactions between different concepts, a student encountering Taylor series may be overwhelmed by the high cognitive load necessitated to reason rigorously within the multiple concepts. Whether the topic be Taylor series or something else, conceptual metaphor provides a vehicle to help reduce the high cognitive load and make reasoning concerning a complicated topic more manageable. As we have seen in this study, students employ conceptual metaphors to reason about unfamiliar mathematical concepts spontaneously rather than systematically (Vygotsky, 1987) and thus such reasoning is often idiosyncratic, bound to concrete application, and lacks conscious awareness and volitional control. Instruction which attempts to develop systematic ways of reasoning about Taylor series necessarily interacts with these spontaneous concepts in what Vygotsky referred to as the zone of proximal development. It is in this zone where the creative aspect of learning takes place.

Both Brian and Jordan demonstrated the spontaneity of their reasoning using part / whole and the cutable object metaphors, respectively. Brian unprompted analogies with Google Maps and grades were grounded in concrete applications, but he lacked awareness that the part / whole metaphor disregarded a remainder element even when faced with approximation tasks. Similarly, Jordan had not fully systematized his reasoning concerning Taylor series to account for the implications of viewing the remainder as the tail of the series when using the cutable object metaphor.

Additionally, to help reduce the cognitive load encountered by a complex mathematical object, students may revert back to talking about that mathematical object in terms of something that he or she already knows how to discuss. For Brian, this appeared to be embodied in his idiosyncratic reasoning for Taylor series convergence as if it was some type of proverbial convergence on a number line (*Figure 4*). His excerpts suggest that in an attempt to understand Taylor series, Brian appears to have assimilated Taylor series into a previous established limit scheme, convergence on a number line. The part / whole metaphor may have served Brian well for convergence on a number line, and thus, Brian continued to reason about Taylor series using this same metaphor. From the data it is unclear if Jordan was specifically producing an image of an improper integral found in *Figure 10* and *Figure 11,* but it seems highly likely. One possibility seems that the proof of the integral test done in his calculus class may have influenced this salient imagery. If Jordan was indeed relating Taylor series convergence to an improper integral, then this is another example of how a student may attempt to assimilate Taylor series convergence into a previously established limit schema. A study that details previously established schemas in which Taylor series are assimilated and the extent to which these schemas are accommodated is worthy of further investigation.

Oehrtman (2008) described design research currently being pursued to develop an entire calculus and differential equations sequence using such research insights and pursuing the parallel design goals to i) reflect the structure of formal definitions of limits, ii) be based on natural language and ideas directly accessible to students, iii) be coherent in its application to all concepts defined in terms of limits, iv) have coherent meaning and structure across multiple representations, and v) be amenable to instructional techniques based on a constructivist theory of abstraction. Therefore, in this work we strive to interact with students' spontaneous concepts and assist them in developing their metaphorical reasoning in more systematic and mathematically fruitful ways. The insights gained by understanding the metaphors from this study can help instructors recognize students utilizing elements of these metaphors and engage them in a productive discourse that develops their scientific reasoning.

References

Alcock, L. J., & Simpson, A. P. (2004). Convergence of sequences and series: Interactions between visual reasoning and the learner's beliefs about their own role. *Educational Studies in Mathematics, 57*(1), 1-32.

Alcock, L. J., & Simpson, A. P. (2005). Convergence of sequences and series 2: Interactions between nonvisual reasoning and the learners beliefs about their own role. *Educational Studies in Mathematics, 58*(1), 77-100.

Black, M. (1962). Metaphor. In M. Black (Ed.), *Models and Metaphors* (pp. 25-47). Ithaca, NY: Cornell University Press.

Black, M. (1977). More about metaphor. *Dialectica*, 31, 433-57.

Burton, D. (2007). *The History of Mathematics: An Introduction* (6 ed.). St. Louis: McGraw-Hill.

Grabiner, J. (1981). *The Origins of Cauchy's Calculus*. Cambridge, Massachusetts: MIT Press.
Hass, J., Weir, M., & Thomas, G. (2007). *University Calculus*. Cambridge, Massachusetts: MIT Press.
Jahnke, H. N. (2003). Algebraic Analysis in the 18th Century. In H. N. Jahnke (Ed.), *A History of Analysis*. Providence, RI: American Mathematical Society.
Kidron, I. (2004). Polynomial approximation of functions: Historical perspective and new tools. *International Journal of Computers for Mathematical Learning, 8*, 299-331.
Kidron, I., & Zehavi, N. (2002). The role of animation in teaching the limit concept. *International Journal of Computer Algebra in Mathematics Education, 9*(3), 205-227.
Larson, R., Edwards, B., & Hostetler, R. (2005). *Calculus with Analytic Geometry* (8 ed.). Florence, KY: Cengage Learning.
Martin (2009). *Expert Conceptualizations of the Convergence of Taylor Series Yesterday, Today, and Tomorrow*. Unpublished Dissertation, The University of Oklahoma, Norman, OK.
Oehrtman, M. (2002). *Collapsing Dimensions, Physical Limitation, and other Student Metaphors for Limit Concepts: An Instrumentalist Investigation into Calculus Students' Spontaneous Reasoning.* Unpublished Dissertation, The University of Texas, Austin, TX.
Oehrtman, M. (2008). Layers of Abstraction: Theory and Design for the Instruction of Limit Concepts. In M. Carlson & C. Rasmussen (Eds.), *Making the Connection: Research and Practice in Undergraduate Mathematics, MAA Notes Volume, 73,* 65-80. Washington, DC: Mathematical Association of America.
Oehrtman, M. (2009). Collapsing dimensions, physical limitations, and other student metaphors for limit concepts. *Journal for Research in Mathematics Education*, 40(4), 396-426.
Stewart, J. (2008). *Calculus* (6 ed.). Belmont, CA: Brooks/Cole.
Tall, D., & Vinner, S. (1981). Concept image and concept definition in mathematics with particular reference to limits and continuity. *Educational Studies in Mathematics, 12*(2), 151-169.
Vygotsky, L. (1987). The Development of Scientific Concepts in Childhood. In R. W. Rieber and A. S. Carton (eds.) *The Collected Works of L.S. Vygotsky*. Vol. 1, 167-241.
Williams, S. R. (1991). Models of limit held by college calculus students. *Journal for Research in Mathematics Education, 22*(3), 219-236.