Modeling the comprehension of proof in undergraduate mathematics

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Abstract

Although proof comprehension is fundamental in higher-level undergraduate mathematical courses, there has been no research on what exactly it means to understand a mathematical proof at this level and how such understanding can be assessed. In this paper we address these issues by presenting a multi-dimensional model of proof comprehension and illustrating how each of these dimensions can be assessed. Building on Yang and Lin's (2008) model of reading comprehension of proofs in school geometry, we contend that in undergraduate mathematics a proof is not only understood in terms of the meaning, operational status and logical chaining of its statements (as Yang and Lin delineate), but also in terms of its higher-level ideas, the methods it employs, or how it relates to specific examples. We illustrate how each of these types of understanding can be assessed in the case of a specific proof.

1. Introduction

In advanced mathematics courses, students spend a substantial amount of time reading proofs. Students read proofs from their mathematics textbooks, they read proofs in professors' lecture notes, and they read and listen to the proofs professors present in class. Presumably, one of the main goals of reading all these proofs is that students understand them and learn from them. Therefore, knowing what it means to comprehend a mathematical proof and being able to assess students' comprehension of a given proof are important issues in the learning and teaching of undergraduate mathematics.

However, exactly what it means for a proof to be understood, and how we can tell if students comprehend a given proof remain open questions in mathematics education. In a systematic study of the literature, Mejia-Ramos and Inglis (2009) found that out of a sample of 131 articles related to the notions of proof and argumentation in mathematics, only three articles focused on students' comprehension of given proofs. This finding is consistent with calls from other researchers (e.g. Selden & Selden, 2003; Mamona-Downs & Downs, 2005) who have suggested that more research on proof reading is needed. Furthermore, Conradie and Frith (2000), Rowland (2001), Schoenfeld (1988), and Weber (submitted) have argued that students' comprehension of a given proof is often measured at a superficial level, by asking them to reproduce it, or modify it slightly to prove an analogous theorem. These findings suggest that more sophisticated ways of assessing students' comprehension of a proof are needed. The objective of his paper is to present a model of what it means to comprehend a mathematical proof at the undergraduate level and to illustrate how this comprehension can be assessed.

Besides being of theoretical interest, a model of proof comprehension would also be of practical significance. On the one hand, mathematics professors could use a model to inform their own teaching methods and to create assessments that would more accurately measure student understanding. On the other hand, as Conradie and Frith (2000) note, better assessment measures could lead students to read and study proofs in more sophisticated ways. A more detailed consideration of implications and contributions of this research is presented in the last section of the paper.

2. Literature Review

Conradie and Frith (2000) contended that current methods of assessing students' proof comprehension are inadequate and described proof comprehension tests that they have developed to address this difficulty. In these tests, students were asked to read a given proof and to answer a series of questions related to that proof. However, given that their intention was mainly to propose an alternative way of assessing students' mathematical knowledge , they did not elaborate on a theoretical model of proof comprehension, or on the theoretical importance of the different types of questions included in their tests.

In a pioneering article, Yang and Lin (2008) made an important first step toward understanding proof comprehension by introducing what they called a *model of reading comprehension of geometry proof* (RCGP). Yang and Lin's model consists of four levels. At the first level, termed *surface*, students acquire basic knowledge regarding the meaning of statements and symbols in the proof. At the second level, which Yang and Lin called *recognizing the elements,* students identify the logical status of the statements that are used either explicitly or implicitly in the proof. In other words, in this second stage students identify whether these statements are employed in the proof as premises, conclusions, or known properties. At the third level, termed *chaining the elements*, students comprehend the way in which these different statements are connected in the proof, by identifying the logical relations between them. Finally, at the fourth level, which Yang and Lin referred to as *encapsulation,* students interiorize the proof as a whole by reflecting on its generality and application to other contexts.

Yang and Lin (2008) focused predominantly on the first three levels (which seem to be crucial in the comprehension of high-school geometry), while leaving the fourth level noticeably unspecified. In particular, Yang and Lin indicated that their instrument for measuring students proof comprehension was not aimed at diagnosing if a student had reached this top level (p.71).

In spite of the apparent importance of proof comprehension, there is no other discussion of this subject in the mathematics education literature. Our model builds on Yang and Lin's model by adapting its first three levels to the context of undergraduate mathematics, and by expanding it to include other dimensions that are crucial in the comprehension of mathematical proofs at the undergraduate level.

3. Development of our model

To identify the ways in which a proof might be comprehended, we took into account a wide variety of sources. First, we considered theoretical articles from the mathematics education literature that discuss the nature of proof (e.g., Duval, 2007), the purposes of proof (e.g., Hanna, 1990; de Villiers, 1990), and different ways of presenting proofs (e.g., Leron, 1983; Rowland, 2001). From these theoretical discussions, we inferred different aspects of proof that mathematics educators regard as crucial in the learning and teaching of proof. Second, we read philosophical articles on how proofs are used by the mathematical community (e.g., Rav, 1999; Mancosu, 2008; Dawson, 2006). These articles provided us with a few aspects of proof that philosophers judge to be central in mathematical practice. Third, we examined the practice of mathematicians in three ways. We considered (i) introspective reports of mathematicians about their own experience with proof (Thurston, 1994), (ii) empirical studies investigating mathematicians' behaviors when reading proofs (Smith et al, 2009; Weber, 2008), and (iii) interviews with nine mathematicians about what they hoped to gain from reading the proofs of others and what they hoped their students would gain from the proofs that they presented

(Weber, submitted). Finally, we reviewed studies on general reading comprehension (e.g. McNamara et al. 1996; Lesley & Caldwell, 2009) looking to determine the different aspects of general written text that these models focus on.

In our investigations, we realized that there is not a single way a proof may be understood. For instance, some researchers have argued that proofs are understood in terms of the mathematical reasons justifying each assertion in the proof (e.g., Selden & Selden, 2003; Weber & Alcock, 2005), while others have said that proofs are understood in terms of their key ideas (e.g., Leron, 1985) or the proof method that is deployed (e.g., Rav, 1999). Similarly, some mathematicians have claimed to be able to understand a proof in terms of its logical details but not its method or key ideas (cf., Alibert & Thomas, 1991), while others have claimed to understand the larger idea of a proof while not grasping all of its logical details (e.g., Thurston, 1994; Rav, 1999). Hence, unlike Yang and Lin (2008), our model does not build on a single dimension or hierarchy of understanding, but rather has six different types of understanding that we believe are not dependent upon one another.

In the following section, we present the dimensions of our model. For each dimension, we describe the importance of this aspect of proof understanding and how one may design questions to assess it. We then illustrate what types of questions might be asked to assess each dimension by providing examples of multiple choice questions regarding the specific proof given below. We work with multiple choice questions because this type of item lends itself to large-scale studies on students' understandings of specific proofs (and proofs presented in specific formats), which is a goal of our research program. However, assessment questions need not be multiple choice: the type of item employed depends upon the goals of the researcher or teacher.

Claim: $4x^3 - x^4 + 2\sin x = 30$ has no solutions.

Proof: Consider the functions $f(x) = 4x^3 - x^4$ and $g(x) = 2\sin x$.

Since $f(x)$ is a polynomial of degree 4 whose leading coefficient is negative, $f(x) \to -\infty$ as $x \to \infty$ and $x \to -\infty$. Hence, $f(x)$ will have an absolute maximum. Taking the first derivative of $f(x)$ yields $f'(x) = 12x^2 - 4x^3$. Setting $f'(x)$ equal to zero and solving for x yields $x = 0$ and $x = 3$. These are the critical points of $f(x)$. The absolute maximum must occur at a critical point. $f(0) = 0$. $f(3) = 27$. Hence, $f(x) \leq 27$ for all x.

The range of $\sin x$ is $[-1, 1]$. Hence, the range of $2\sin x$ is $[-2, 2]$. Therefore, $g(x) = 2\sin x \leq 2$ for all x.

Thus $4x^3 - x^4 + 2\sin x = f(x) + g(x) \leq \max(f(x)) + \max(g(x)) = 27 + 2 = 29 < 30$.

Therefore, $4x^3 - x^4 + 2\sin x = 30$ has no solutions.

4. Dimensions of model

4. 1. *Meaning of terms and statements*

One of the most fundamental ways of comprehending a text is to understand the meaning of individual terms and statements. In the case of proof, this amounts to understanding the meaning of statements, symbolic expressions, terms, and definitions. Yang and Lin (2008) feature this aspect of proof prominently at their *surface level*. Research suggests that students often fail to understand the meaning of key terms when reading a proof (Conradie & Frith, 2000), which hinders their ability to comprehend other aspects of the proof. Furthermore, researchers have argued that less successful proof readers do not try to understand the meaning of key terms and statements (e.g. Weber, Brophy, & Lin, 2008) .

Ways to assess a reader's comprehension of this aspect might include asking him or her to identify the definition of a key term in the proof, or to specify what is meant by some of its statements. For instance, with respect to the proof given above, one could ask the following question (correct answers are underlined):

> Which of the following are consequences of the statement "the range of $2\sin x$ is $[-2,2]$". **Check all that apply**:
>
> A. One cannot substitute an x value into $2\sin x$ unless $-2 \leq x \leq 2$.
>
> B. There is no a such that $2\sin a = 3$.
>
> C. There is an a such that $2\sin a = 0$.
>
> D. $2\sin x \geq -2$ for all x.

This question could be used to assess the extent to which a reader understands the concept of the *range* of a function or the way in which this term is used in this particular sentence of the proof. In general, there are at least three different features within this dimension of proof comprehension:

1. *Understanding the meaning of the theorem itself*. The proven claim is an important statement in any given proof. Assessing the extent to which readers understand the meaning of the proven statement may involve asking them to:
    a. *State the theorem in a different, but equivalent manner* (e.g. "write the theorem in your own words", "write the theorem in terms of [*a particular notion*]", "which of the following statements are equivalent to the theorem?")
    b. *Identify trivial implications of the theorem* (e.g., "which of the following statements are true based on the theorem?", "which of the following are consequences of the theorem?")
    c. *Identify examples that illustrate the theorem* (e.g., "which of the following cases verify the statement for a particular example?", "which of the following cases does the theorem rule out?")

2. *Understanding the meaning of individual statements in the proof.* This refers to a reader's comprehension of statements in the proof, other than the proven theorem itself. The multiple-choice item presented above illustrates one way of assessing this specific feature of the given proof. In general, the three types of questions used to assess readers' comprehension of the proven theorem may be employed to assess their comprehension of other statements in the proof.
3. *Understanding the meaning of terms in the proof.* This third feature refers to a reader's comprehension of specific terms or expressions appearing in the proof. Demonstrating an understanding of these terms and expressions may involve being able to:
    a. *State the definition of a given term in the proof* (e.g., "define [*term*] in your own words", "which of the following statements defines [*term*]?").
    b. *Identify examples that illustrate a given term in the proof* (e.g., "is [*specific example*] an example of a [*term*]?", "which of the following cases exemplifies (or does not exemplify) a [*term*]?").

In some respects, this is the most basic dimension of understanding in our model in the sense that one could conceivably answer these questions without ever having read the proof itself (although reading the proof may certainly help develop understanding of key terms and statements).

4. 2. *Justification of claims*

In a proof, new statements are deduced from previous ones by the application of accepted mathematical principles (e.g., theorems, logical rules, algebraic manipulations). However, as with all scientific texts, a proof would be impossibly long if all of its logical details were explicitly stated (see Chi et al, 1994). In many cases, the reader needs to infer what previous statements, and what mathematical principles, are used to deduce a new assertion within a proof (Weber & Alcock, 2005), and research has illustrated how undergraduates often fail to do this when reading a proof (e.g., Alcock & Weber, 2005; Weber, 2009).

One way of comprehending proofs involves grasping how new assertions follow from previous ones. This dimension is analogous to Lin and Yang's (2007*)* third level of proof comprehension (*chaining the elements*). Assessing this dimension may involve asking a reader to make explicit a justification that is implicit in the proof, or to identify the specific statements within the proof that provide the basis for a given claim. The following question illustrates this type of assessment:

> In the proof, which justification best explains why $f(x) \leq 27$ for all x?
> **Check only one:**
>
> A. Because $f'(3) = 0$ and $f(3) = 27$.
>
> B. Because $f(x)$ is a polynomial of degree 4.
>
> C. Because $f(x)$ has a global maximum either at $x = 0$ or at $x = 3$, and $f(0) \leq f(3)$.
>
> D. None of the above justifications explains why $f(x) \leq 27$ for all x.

This question could be used to assess whether or not the reader understands the way in which the proof establishes the given inequality. In general, in order to assess the extent to which readers comprehend the justification of claims in the proof, one may ask them to:

1. *Make explicit an implicit warrant in the proof.* Generally, proofs include expressions of the form "Since A, then B", where the claim B is justified simply by citing statement A. In this type of expressions, the general rule according to which statement A is sufficient to conclude statement B is left implicit. In some cases (e.g. proofs appearing in specialized publications) this may be done under the assumption that this general rule is

obvious to the reader, and therefore it would be superfluous to mention it. In other cases (e.g. proofs appearing in undergraduate mathematics textbooks) this may be done with the expectation that readers work out for themselves what are these general rules. Assessing whether or not readers know these implicit, general rules, may involve asking the readers to make these rules explicit (e.g. "which one of the following general rules justifies [*claim of the form 'Since A, then B'*] in the proof?")

2. *Identifying the specific data supporting a given claim (or backward justification).* It is also common for proofs to include expressions of the form "Hence, C", in which the claim C is justified by some unspecified subset of all the previous statements in the proof. For instance, the proof presented above includes the expression "Hence $f(x) \le 27$ for all x", leaving unspecified exactly which statements in the proof allow us to conclude the inequality. Oftentimes these claims are justified by statements immediately preceding them in the proof, but it is not uncommon to find that these claims are justified by statements in different parts of the proof. Assessing the extent to which a reader understands how a claim presented in this manner is justified may involve asking him or her to identify the specific statements within the proof that provide the basis for the claim (e.g. "which of the following statements in the proof allow the conclusion of [*claim of the form 'Hence, C'*]?")
3. *Identifying the specific claim that is supported by a given statement (or forward justification).* Proofs often reach specific results, or state specific assumptions without specifying in an explicit manner how these results or assumptions are used in the proof. Therefore, another way of assessing this dimension of proof comprehension involves asking a reader to identify the exact place in the proof where a given piece of information

is employed as justification of new claims (e.g. “which of the following claims in the proof logically depend on the assertion that [*specific proposition in the proof*]?”).

4.3 *Logical Structure*

In a study of mathematicians’ validation of proofs, Weber (2008) found that mathematicians partitioned proofs, and understood the proof technique being used, by looking at what Selden and Selden (1995) referred to as a *proof framework*. Selden and Selden noted that a proof framework consists of the beginning and end of a particular proof (or in the case of a proof component, a sub-proof) that allow one to infer the statement being proven and the proof technique (e.g., direct proof, proof by cases) being used. For instance, in the proof given above, once it is established that its strategy is to analyze two components of the original function, the top-level logical structure of the proof could be described in terms of the following framework:

Proof: Consider the functions $f(x) =$_____ and $g(x) =$_____
Since $f(x)$ ______________________________, then ________________.
Since $g(x)$ ______________________________, then ________________.
Therefore, $f(x) + g(x) = 30$ has no solutions.

Note that, as specified by Selden and Selden’s (1995) definition, this specific proof framework does not address detailed knowledge of the mathematical ideas in the proof (which would be needed to come up with the strategy in the first place and to fill in the blank spaces to complete the proof), but rather general knowledge about the overarching logical structure of the given strategy.

As Selden and Selden (2003) subsequently illustrated and Weber (2009) replicated, students often do not consider the proof framework of a proof they are reading, leading students to be unsure of what is being proven and unable to determine if a proof is correct.

Assessing a reader's comprehension of this top-level logical structure of a proof may involve identifying the role of a sentence or part of the proof framework, or the logical relation between two or more of these parts. For instance, in order to assess whether or not a reader of the proof given above understands the logical relation between the two main parts of its framework, one could ask:

> Which of the following best describes the logical relation between the second paragraph of the proof (beginning with "Since $f(x)$ is a polynomial of degree 4") and the third paragraph in the proof (beginning with "The range of $\sin x$ is $[-1, 1]$"). **Check only one:**
>
> A. The two paragraphs are logically independent.
>
> B. The third paragraph logically depends on statements in the second paragraph.
>
> C. The second paragraph logically depends on statements in the third paragraph.
>
> D. None of the above.

In general, assessing a reader's comprehension of a proof's top-level logical structure may involve:

1. *Identifying the purpose of a sentence within a proof framework.* Certain sentences in proofs can be associated with general proof frameworks, or strategies. For instance, when reading a proof of a conditional statement that begins by assuming a denial of the consequent, one may suspect that the rest of the proof will follow by contraposition or contradiction. Or, when reading a proof of an existentially quantified statement that begins by defining a non-arbitrary mathematical object, one would expect this object to be the candidate that establishes the required existence. Therefore, one way of assessing the extent to which a reader comprehends the logical structure of a proof could involve asking him or her to identify the role played by this type of statement in the proof framework (e.g. "which of the following best describes the way in which [*sentence of the*

proof framework] was used in the proof?" "which of the following best explains why the author of the proof states [*sentence of the proof framework*]"?).

2. *Identifying the purpose of a component of a proof framework.* Another way of assessing a reader's comprehension of a given proof's framework may involve asking him or her to identify the role of a given part of the proof within its framework. (e.g. "which of the following statements best justifies the consideration of [*specific case*] in the proof?")
3. *Identifying the logical relation of two components of a proof framework.* Finally, like in the item given above, one could assess this dimension of proof comprehension by asking a reader to identify the logical relation between two or more components of a proof's framework (e.g. "which of the following best describes the logical relation between [*two or more parts of the proof framework*]"?

4. 4. *Higher-level ideas*

Another dimension of proof comprehension involves compartmentalizing the proof into modules and chaining these modules to produce a proof summary that focuses on the higher-level ideas of the proof. This dimension is related to Leron's (1983) notion of *structured proofs*, in which the ideas of a proof are arranged in levels of autonomous modules, where the top level "gives in very general (but precise) terms, the main line of the proof" (p.174). To Leron, identifying the main idea of a proof was critical to understanding it. Thurston (1994) emphasized the importance of this dimension in his description of proof comprehension as the process of unraveling its key ideas from the rigmarole employed to carry them out. The mathematicians interviewed by Weber (submitted) also stressed the importance of the big idea of the proof. As an illustration, the proof given above could be structured in the following modules:

Level 1: The proof shows that $4x^3 - x^4 + 2\sin x \leq 29$, for all values of x, which clearly implies that $4x^3 - x^4 + 2\sin x = 30$ has no solutions.

Level 2: The function $4x^3 - x^4 + 2\sin x$ is split into two functions $f(x) = 4x^3 - x^4$ and $g(x) = 2\sin x$, where clearly $f(x)+g(x) = 4x^3 - x^4 + 2\sin x$. Then, by showing that $f(x) \leq 27$ and $g(x) \leq 2$, it concludes that $4x^3 - x^4 + 2\sin x \leq 29$.

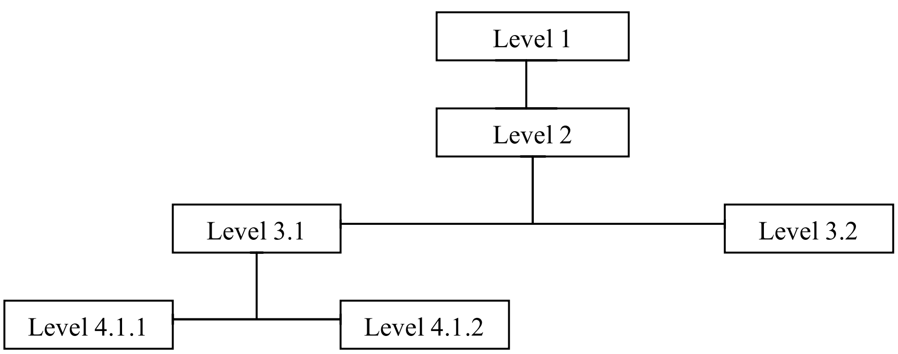
Level 3.1: $f(x) = 4x^3 - x^4$ has an absolute maximum. Furthermore, this absolute maximum occurs at a critical point. Since the critical points of $f(x) = 4x^3 - x^4$ occur at $x = 0$ and $x = 3$, and $0 = f(0) < f(3) = 27$, then 27 must be the absolute maximum. This means that $4x^3 - x^4 \leq 27$.

Level 3.2: Since $\sin x \leq 1$, for all values of x, then $g(x) = 2\sin x \leq 2$, for all values of x.

Level 4.1.1: Since $f(x)$ is a polynomial of degree 4 whose leading coefficient is negative, $f(x) \to -\infty$ as $x \to \infty$ and $x \to -\infty$. Hence, $f(x)$ will have an absolute maximum.

Level 4.1.2: In order to find the critical points of $f(x) = 4x^3 - x^4 + 2$, the proof finds the points in which the derivative of $f(x)$ is 0, or undefined. Since $f'(x) = 12x^2 - 4x^3$, and $f'(x) = 0$ whenever $x = 0$ and $x = 3$, these are the critical points of $f(x)$.

The structure of these modules can be represented by the following diagram:



The assessment of a reader's comprehension of the higher level ideas of a given proof may involve identifying summaries and sentences that capture these ideas. For instance, in the context of the proof given above, this assessment may include the following type of questions:

Which of the following summaries <u>best</u> recaps the ideas in the proof?
Check only one.

<u>A</u>. To show $4x^3 - x^4 + 2\sin x = 30$ has no solutions, the proof shows that $4x^3 - x^4 + 2\sin x \leq 29$. This is done by showing that $4x^3 - x^4 \leq 27$ and $2\sin x \leq 2$.

B. If $f(x) = 4x^3 - x^4$, $f'(x) = 12x^2 - 4x^3$. $f'(x) = 0$ when $x = 0$ or $x = 3$. $f(0) \leq f(3) = 27$. Also, $\sin x \leq 1$, so $2\sin x \leq 2$. Hence $4x^3 - x^4 + 2\sin x \leq 29 < 30$.

C. Polynomials of degree 4 have absolute maximums. Since $4x^3 - x^4 + 2\sin x$ is a polynomial of degree 4, it must have an absolute maximum. The proof shows that the absolute maximum of $4x^3 - x^4 + 2\sin x$ is 29, so it cannot equal 30.

D. None of the summaries capture the ideas of the proof.

In general, assessing a reader's comprehension of a proof's higher-level ideas may involve:

- *Identifying a good summary of the proof.* In this case, like in the item above, a reader is asked to provide or identify a summary of the proof that contains the ideas in the higher levels of the structured proof (e.g. "which of the following is the best summary of the proof?").
- *Identifying a good summary of a key module in the proof.* In this type of assessment the reader is asked provide or identify a summary of a key module of the proof (e.g. "which of the following best summarizes why [*significant result within the proof*]?"). In terms of structured proofs, this could be interpreted as summarizing a given node in a given level. For instance, in the proof given above, one could ask a reader to identify a sentence that best summarizes the way the proof justifies that $4x^3 - x^4 \leq 27$ (level 3.1).

4. 5. *General method*

An important aspect of comprehending a proof involves identifying the procedures used in the proof and the ways in which these procedures can be applied (or re-interpreted) to solve other proving tasks. Rav (1999) argued that illustrating new methods was a primary reason for why mathematicians published proofs and read the proofs of others. Several other philosophers (e.g., Dawson, 2006; Avigad, 2006) have made similar claims. For instance, Bressoud (1999) argues that “[t]he value of a proof of a challenging conjecture should be judged, not by its cleverness or elegance, or even its ‘explanatory power’, but by the extent to which it enlarges our toolbox” (p. 190). The mathematicians interviewed by Weber (submitted) all claimed that finding new techniques was one of the most important reasons for why they read the proofs of their colleagues. Hanna and Barbreau (2008) argued that illustrating methods is a very important reason for presenting proofs in mathematics classrooms. Weber (submitted) documented how illustrating techniques was a common reason that mathematicians presented proofs in their lectures.

This dimension of proof comprehension involves understanding the general line of attack used to prove the theorem (or in sub-proofs within the theorem), knowing when this method is applicable, and being able to apply it in other settings. Therefore, one way to assess this type of comprehension is to ask the reader to identify the general manner in which this method can be applied in other settings. For instance, in order to assess a reader’s comprehension of the general method employed in the proof given above, one could ask:

Suppose that a student, Erica, wanted to use the ideas in the proof to show that $4\sin x + e^{-x} = 6$ has no positive solutions. Which of the following approaches would best suggest to Erica which approach to use? **Chech only one:**

A. Erica should find the x values of the critical points of $f(x) = 4\sin x + e^{-x}$, plug these values into $f(x)$, and show that these resulting values are all less that 6.

B. Erica should sketch the graph of $4\sin x + e^{-x}$ and show that this graph never intersects the line $y = 6$.

C. Erica should find the maximum value of $4\sin x$ and e^{-x} for positive values of x and show that the sum of those values is less than 6.

D. None of the preceding approaches are based on the ideas from the given proof.

In general, there are at least two ways in which this dimension can be assessed:

1. *Transferring the proof method:* This involves being able to successfully apply the method in the solution of a different argumentative task (e.g. "(Using what you have learned in the previous proof) prove that [*similar theorem*]")
2. *Reinterpreting the proof method:* This involves being able to identify the *general manner* in which the method of the original proof can be applied in a different proving task (e.g. Given a theorem T that can be proven using the method displayed in the original proof, ask "which of the following general procedures would you follow to prove T?", or "how would you start proving T?", or "which of the following lines in the proof would you have to modify to prove T?")

4. 6. *Application to examples*

Comprehending a proof often involves understanding how the proof relates to specific examples—that is, being able to follow a sequence of inferences in the proof in terms of a specific example. Many of the mathematicians Weber (submitted) interviewed emphasized that this was an indispensable tool that they used to gain an understanding of a proof that they were reading. One mathematician in this study said: "Commonly, if I'm really befuddled and if it's appropriate, I will keep a two-column set of notes: one in which I'm trying to understand the proof, and the other in which I'm trying to apply that technique to proving a special case of the general theorem." This is also the idea behind the generic proof presentation that Rowland (2001) advocates, and the idea of linking formal representations employed in a written proof to a reader's informal understandings of the proof in terms of examples or pictures (Raman, 2003). In the context of the proof presented above, one could assess this type of comprehension using the following question:

> Using the logic of the proof, which best exemplifies *why* $x = 5$ is not a solution to $4x^3 - x^4 + 2\sin x = 30$. **Check only one:**
>
> A. If $x = 5$, $4(5)^3 - 5^4 + 2\sin(5) = 4(125) - 625 + 2\sin(5) = 500 - 625 + 2(-0.95) = -125 - 1.92 = -126.92 \neq 30$.
>
> B. Since $4x^3 - x^4 \leq 27$, then $4(5)^3 - (5)^4 \leq 27$. Since $2\sin x \leq 2$, then $2\sin(5) \leq 2$. Thus, $4(5)^3 - 5^4 + 2\sin(5) \leq 27 + 2 = 29 < 30$. Therefore, 5 cannot be a solution to $4x^3 - x^4 + 2\sin x = 30$.
>
> C. We know that $4x^3 - x^4 + 2\sin x = 30$ has no solutions, so we know that plugging 5 into this equation cannot be 30.
>
> D. None of the above uses the logic of the proof.

In general, one could assess readers' understanding of this dimension by asking them to identify the way in which a sequence of inferences in the proof applies to a given specific example (e.g. "Using the ideas in the proof, which of the following statements best explains why [statement about a particular example]"?).

5. Discussion

5. 1. *Limitations of the model*

The model in this paper contains six different dimensions along which one can assess an individual's understanding of a proof. We do not believe that these dimensions are necessarily exhaustive. Indeed, some of the mathematicians interviewed suggested other ways that a proof might be understood, such as seeing if the proof could be salvaged if a hypothesis of the theorem was removed. We chose the current six dimensions either because they appeared prominently in the literature, because they were emphasized by many of the mathematicians interviewed, or because they were appropriate for undergraduate mathematics classes.

5. 2. *Contributions of this paper*

This paper builds upon the work of Yang and Lin (2008) and Conradie and Frith (2000) in an important way. Conradie and Frith (2000) proposed specific questions that they used to assess students' understanding of proof in their own classrooms. However, they did not provide a theoretical model that described the purposes of their questions, nor did they provide a method for teachers to generate their own questions to different proofs.

Yang and Lin made an important contribution toward delineating students' comprehension of a proof by proposing a four-level hierarchy of understanding. Their focus was on the first three levels, which dealt with the meaning of the statements within a proof and the logical structure of the proof that they presented. This is quite appropriate for proofs in a high school geometry course, which is the scope that Yang and Lin assigned to their model. However, in undergraduate mathematics courses, where the proofs become longer and more nuanced and more proof techniques are applied (proof by cases, proof by induction, and indirect proofs are common in undergraduate mathematics courses but rare in high school geometry), there are more sophisticated ways in which a proof can be understood. We incorporate Yang and Lin's contributions to our model, but elaborate on other aspects of understanding a proof that are only mentioned in passing or ignored by Yang and Lin, including understanding how components of a proof relate to one another, understanding the higher level ideas of a proof, being able to apply the method of the proof elsewhere, and applying the proof to a specific example.

5. 3. *Implications*

In this paper we have presented a multi-dimensional model of proof comprehension at the undergraduate level, and have illustrated ways in which each of these dimensions can be assessed. Delineating the ways in which proof can be understood has both practical and theoretical significance. At a practical level, a professor who can articulate specific learning goals when presenting a proof can emphasize its important components, and carefully targeted assessments can more accurately measure the extent to which his or her goals were reached. Furthermore, as Conradie and Frith (2000) point out, this type of assessment gives students an incentive to understand proofs at a deep level and focuses their attention on important aspects of the proof. Also, as Conradie and Frith (2000) note, professors can better assess the quality of their lectures if they can understand what aspects of a proof students successfully understood and what aspects they did not.

For researchers, a means to assess students' comprehension of proof can be essential for evaluating the effectiveness of mathematical instruction. For instance, structured and generic proofs are widely claimed to aid in students' comprehension of proofs (e.g., Leron, 1983; Rowland, 2001), but these claims have not been empirically tested. Our model can be used to test whether (and in which ways) these types of proofs improve comprehension.

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