Which Path to Take?
Students' Proof Method Preferences

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Abstract: Students in their first proof course often lack flexibility in writing original proofs. Learning to utilize different proof schemes is a crucial factor in their development as mathematicians. In this paper, we will consider several student interviews in which students both think through existing proofs and attempt to generate original proofs. These interviews of beginning abstract algebra students highlight their capacity to explore multiple ways to prove the same statement.

Background

There is a transition time in university mathematics from computational mathematics to formal mathematics (Tall 2008). It is well documented that this shift is problematic for undergraduates (Larsen & Zandieh, 2008; Grassl & Mingus, 2007). Some universities have a "bridge to proof" course to attempt to help students in this transition.

Many studies have been done to investigate this difficulty. In one study, undergraduates' proof writing abilities were compared to doctoral students' abilities. This study found that even though an undergraduate understands mathematical proof and knows the relevant information, they may lack strategic knowledge to be able to successfully construct a proof (Weber, 2003).

Studies of particular interest are ones that gave students several different proofs of the same statement and asked them to analyze the proofs. Selden & Selden (2003) investigated how undergraduates read and verified several different student-produced "proofs" of a number theoretic statement. They found that students tended to focus on logical/detailed errors rather than global/structural errors. They also noticed that students improved throughout the interview process, suggesting that validating proofs could be taught.

Two studies incorporated questionnaires in which students chose a valid proof from a sample of possible “proofs.” In a large scale study of secondary students (age 14-15), Healey and Hoyles (2000) gave the students questionnaires that had a mathematical statement and several possible proofs. The students were asked to choose which proof method they would most likely use, as well as the proof that they thought would receive the highest marks from the teacher. They found that the students seemed to prefer to use empirical evidence as proof, though they recognized that this was not the expectation of their instructor. Almeida (2000) distributed a questionnaire to undergraduate math majors. This questionnaire had several statements and possible proofs, and the students were asked to choose the proof that they thought was correct. He found that though students profess to understand the nature of mathematical proof, their actual practices deviate from the formal position.

In these studies, the majority of the possible proofs were incorrect, and students were attempting to choose the proof that was correct. We were interested in what would happen if undergraduates were given several correct proofs that used different proof methods. Would they be able to recognize more than one correct way to prove the same statement?

Methodology

We were interested in investigating whether or not undergraduates can recognize different proof methods when looking at several examples of “proofs.” The research questions are:

Can students recognize different proof methods? How well do students choose a useful method to prove a statement, and how flexible are they if their first attempt fails?

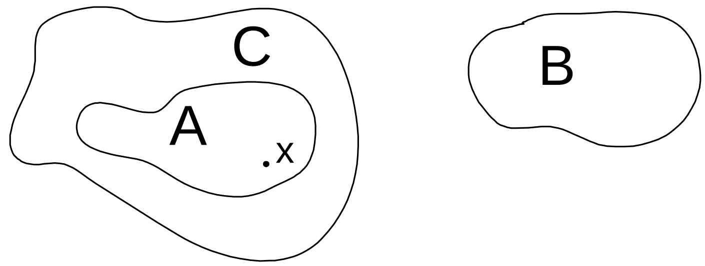
This study was a case study of students enrolled in the introduction to modern algebra course during the fall 2009 semester. This course is the first proof-based course in the undergraduate math program at the mid-sized research university where the study took place. Classroom observations were conducted twice a week during the Fall 2009 semester. Four volunteers from the course participated in semi-structured 45 minute interviews which were audio recorded. Two students were interviewed half way through the fall semester, and two were interviewed at the beginning of the Spring 2010 semester. Two of the students were male, two were female. They were all juniors or seniors majoring in mathematics.

In the interview, the participants were given two statements. The first statement was a set-theoretic statement. The students were given some time to think about the statement, given the freedom to draw or write, and asked whether or not they thought the statement was true. Then, they were given five possible "proofs" of the statement. Three were correct proofs, using different methods. Two were incorrect proofs, according to the standards that had been established in class. The students were asked to think aloud as they decided whether or not each proof was a valid proof of the statement. The statement and possible proofs are given below.

Statement 1: Suppose that $A \subseteq C$, and that B and C are disjoint. Prove that if $x \in A$, then $x \notin B$.

Proof A: Assume that $A \subseteq C$ and B and C are disjoint. Also assume that $x \in B$. Then, since $B \cap C$ is empty, we can deduce that $x \notin C$. Since $A \subseteq C$, we can conclude that $x \notin A$.

Proof B:



Proof C: Assume that $A \subseteq C$ and B and C are disjoint. Also assume that $x \in A$ and $x \in B$. Then, we have that $x \in C$, since $A \subseteq C$. Since $x \in B$ and $x \in C$, we know that $x \in B \cap C$. But, $B \cap C$ is empty by our assumption. This is a contradiction. Thus, it must be the case that $x \notin B$. Therefore, we have proved the statement.

Proof D: Assume that $A \subseteq C$ and B and C are disjoint. Also assume that $x \in C$ and $x \in B$. Then, we have that $x \in C \cap B$. Since B and C are disjoint, we have that $B \cap C$ is empty. This is a contradiction. Thus, we can conclude that $x \in A$.

Proof E: Assume that $A \subseteq C$ and B and C are disjoint. Also assume that $x \in A$. Then, we know that $x \in C$, since $A \subseteq C$. Since B and C are disjoint, we can conclude that $x \notin B$. This proves the statement.

These possible proofs were generated by the investigator. The two incorrect proofs were chosen to investigate both how students felt about pictures as proof, and whether or not students were able to recognize a correct proof by contradiction.

When the students had finished the first exercise, they were given a second statement.

Statement 2: Assume that n is an integer. If n^2 is even, then n is even.

The students were then asked to think about and try to construct a proof of the statement. Their different attempts to prove the statement were recorded and analyzed. This was done to see which proof methods that the students would choose for themselves. This statement was chosen because it could be proven directly, by contradiction, or by proving the contrapositive.

The interview data were transcribed, and themes were identified. These themes were analyzed in light of the classroom observations.

Results

All of the students were able to recognize that the first statement was true. Two drew a picture that was very similar to proof B to help them decide. The students then read the possible proofs and decided whether or not each proof was valid. The results are presented in the following table:

Proofs of Statement 1	Yes	Unsure	No
Contrapositive	0	1	3
Diagram	3	0	1
Contradiction	4	0	0
Incorrect Proof	2	1	1
Direct	3	0	1

The students were not able to recognize the proof of the contrapositive as a valid proof method. They all recognized the proof by contradiction, though two of the interviewees thought that an incorrect proof was correct. They both said it was a proof by contradiction, possibly because it said the word “contradiction” in it. Most of the students recognized the direct proof, though two mentioned that they thought the proof was just re-stating the statement. This could be because the proof was fairly short, so if the statement were more complicated, this result may not have occurred. Three of the four thought that a diagram was a valid proof. In fact, two of the students mentioned that the diagram was the same as what they drew to convince themselves that the statement was true. As the students talked aloud, they seemed to focus on checking the proofs line by line, which is consistent with Selden and Selden's (2003) findings.

When they were constructing original proofs, they employed a variety of methods. Two of the students focused primarily on generating empirical examples by plugging in numbers, though they admitted that this would not serve as a formal proof. Three attempted to prove the converse, but quickly realized that it was not a proof of the statement. Two of the students gave a correct proof of the contrapositive, but one of these did not think it was a proof of the statement. One student attempted to prove the statement by induction, and admitted that the reason was that she was learning induction in class at the time.

The students seemed to be distracted by what they were learning in class at the time. Two of the students, when asked to define subset, began to give the definition of subgroup. The observations confirmed that they were learning about subgroups in class that week. Another student, when asked about what he meant by "disjoint sets," began to talk about disjoint cycles in the symmetric group, which was what they had discussed in class the previous week.

In the first part of the interview, none of the students were able to recognize the proof of the contrapositive as a valid proof, however, two of the four used that method in proving the second statement. They all recognized the proof by contradiction in the first part, but none of the students used proof by contradiction to prove the second statement. The students seemed to be inconsistent in their understanding of the contrapositive and proof by contradiction.

Students also seemed to be unclear about the role of empirical examples and diagrams in proof. They seemed to recognize that a counterexample can be powerful, but empirical evidence was not the same as proof. Even though many of the students mentioned this, they still spent a considerable amount of time plugging in values for the variables in Statement 2. They also seemed to think that as long as a diagram represented the situation, it was considered a proof. One participant said, “It’s not a very good proof, but it’s still a proof.” This student may have recognized that it would not earn good marks from the instructor, but it still convinced her. This seems consistent with Healey and Hoyles’ (2000) findings.

Overall, the students seemed to be able to change methods during the interview process, but that may be because the interviewer was asking them if there were any other methods they could try. Most of the students didn’t begin with a method that would lead to a correct proof, but during the course of the interviews, three out of the four had either outlined or completed a valid proof of the statement. Only one of these, however, realized that the statement had been proved.

Discussion

The findings suggest that students who are in their first proof course are still developing their understanding of different proof methods. They may be able to recognize different methods in certain contexts, but not in others. Students may be able to read a proof and recognize the methods used, but are not able to construct a proof using that method, or vice versa.

Throughout the interview process, the participants were able to change methods fairly well. This may be because the interviewer asked if they could think of another way to attempt the proof. This suggests that students could be taught to be flexible in using different methods of proof.

This investigation used only two statements, so the data may depend heavily on the particular statements that were chosen. Future research could use a similar method with different statements. Perhaps a more complex statement would have different results. It would also be interesting to investigate how to teach students to be more flexible in their proof writing abilities. If we can help students to gain these skills, it will help them tremendously as they as they enter into the classroom as secondary teachers, or as they continue into more advanced courses in mathematics.

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