The Role of Quantitative and Covariational Reasoning in Developing Precalculus Students' Images of Angle Measure and Central Concepts of Trigonometry

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The paper reports results from an investigation of a precalculus student's conceptions of angle measure, a radius as a unit of measurement, and trigonometric functions. This study consisted of a teaching experiment conducted with students from an undergraduate precalculus course. The findings reported here focus on one student and reveal the importance of the ideas of angle measure and the radian for developing coherent understandings of trigonometric functions. Specifically, these ideas provided a foundation for one student developing coherent meanings and effectively reasoning about the geometric objects of trigonometry (e.g., right triangles and the unit circle) in relation to trigonometric functions. This study also reveals the importance of quantitative and covariational reasoning in supporting the student's constructed understandings.

Introduction

Research on both students' and teachers' understandings of trigonometric functions reveals individuals having limited, fragmented, and deep-rooted understandings of trigonometry (Brown, 2005; Fi, 2003; Thompson, Carlson, & Silverman, 2007; Weber, 2005). Also, Thompson (2008) recently argued for the need to rethink the trigonometry curriculum such that coherence is promoted through leveraging common foundations (e.g., angle measure) between the various settings of trigonometry (e.g., the unit circle and right triangles). To further complicate the matter, research on students learning trigonometry is rather sparse. In an attempt to help fill this research gap, I designed a teaching experiment that was informed by research findings in trigonometry, curriculum suggestions, and research literature on reasoning I deemed critical to constructing trigonometric understandings (e.g., quantitative and covariational reasoning).

The teaching experiment outlined in this paper involved three undergraduate precalculus students. In addition to discussing the design of the instructional sequence used in the teaching experiment, this paper reports findings from analyzing one student's actions during the teaching experiment. The reported results focus on the student's actions as he engaged in various tasks whose design was informed by the available research literature. Analysis of the student's actions reveals the role of quantitative and covariational reasoning in his construction of flexible and coherent understandings of trigonometric functions. Also, the data reveals the foundational role of angle measure, and namely the radius as a unit of measurement, in constructing understandings of trigonometric functions.

Theoretical Perspective

Glasersfeld's radical constructivism (1995) and Piaget's theory of genetic epistemology (Chapman, 1988; Piaget, 2001) informed the design, implementation, and analysis of this study. Stemming from these perspectives, a researcher (or teacher) must consider another individual's knowledge as fundamentally unknowable. Knowledge is gained through individual experiences, where the experiences are entirely unique to the individual. Also, this knowledge is not *of* anything; there is no one-to-one correspondence between what knowledge is of and the knowledge itself. Rather, knowledge is what comes together through the processes of an individual altering his or her knowing (mental schema) in response to a cognitive perturbation or disequilibrium.

As defended by Jean Piaget, reflection is a major aspect of learning and possibly the most important aspect of building knowledge. Reflection attributes learning to the ability of the mind to “stand still” and attempt to make sense of an experience. In addition to the idea of reflection, Piaget and Glasersfeld identify the notion of abstraction, which is made possible through the comparison, separation, and connection of experiences. Through the mental activities of reflection and abstraction, the reorganization and construction of cognitive structures is achieved. Again, it is important to emphasize that these processes are completely dependent on the individual and are unseen by any other observer. Also, note that an individual’s experiences depend on the current model of knowing of the individual. This current model defines the experience.

An implication of considering an individual’s knowledge as fundamentally unknowable is that a researcher can only construct tentative models of a subject’s knowledge. Due to the autonomy of a subject’s knowledge, a researcher must construct, test, and refine models of a subject’s mathematics, where each iteration is intended to result in a more viable model than the previous model. This approach to researching student understandings forms a central aspect of a teaching experiment.

Background

The study discussed in this paper builds on a previous investigation into students’ conceptions of angle measure (Moore, 2009), where the study discussed here sought to further investigate the role of quantitative and covariational reasoning relative to trigonometric functions, as well as angle measure. Although the current research literature available in trigonometry is sparse, the studies that are available consistently report teachers and students having difficulty constructing coherent and flexible understandings of trigonometric functions and topics foundational to trigonometry.

Research commonly reveals both students and teachers having fragmented and under-developed conceptions of angle measure (Akkoc, 2008; Brown, 2005; Fi, 2003; Moore, 2009). These investigations report teachers lacking meaningful understandings of the radian as a unit of angle measure and that teachers are much more comfortable with degree measurements. For instance, Fi (2003, 2006) observed teachers trivially converting between radian and degree angle measures, but these teachers were unable to give a well-defined description of radian measure beyond this conversion. Also, multiple studies (Akkoc, 2008; Fi, 2003, 2006; Tall & Vinner, 1981; Topçu, Kertil, Akkoç, Kamil, & Osman, 2006) report teachers not viewing π as a real number when discussed in a trigonometric context. Rather, these teachers were observed graphing π radians as equal to 180 (the number, not degrees), where other teachers described π as the unit for radian measurements (e.g., a radian is so many multiples of π).

In response to the limited mathematical understandings often constructed by teachers, Thompson, Carlson, and Silverman (2007) focused on engaging teachers in tasks designed to necessitate their re-conception of the mathematics they teach. The authors (Thompson, et al., 2007) argue that the teachers involved in the study held strong commitments to the meanings they had attached to their high school curriculum. For instance, the teachers were attached to introducing trigonometry using right triangles, rather than angle measurement and the unit circle. In this case, the teachers’ previous understandings dominated what the teachers imagined themselves teaching even after the authors' brought to the teachers' attention the incoherence of their meanings.

In another valuable study, Weber identified that students were unable to discuss various properties of trigonometric functions or estimate the output values of trigonometric functions for various input values. The author argues that a limitation to this group of students was that they lacked the ability to construct the geometric objects needed to reason about trigonometric functions. For instance, the students could not approximate $\sin(\theta)$ for various values of θ. Instead, the students often stated that they were not given enough information to accomplish this task and that they needed an appropriately labeled triangle. As another example, when students were asked why $\sin(x)$ is a function, not one of the students from a lecture-based class was able to provide a meaningful answer. This is consistent with research literature that reveals the difficulty of students constructing a process conception of function (Harel & Dubinsky, 1992; Oehrtman, Carlson, & Thompson, 2008).

Weber's study (2005) also discusses the results of an experimental course in trigonometry. Weber claims that the improved performance in the experimental course was tied to the students' use of the unit circle. That is, students who showed improved performance often revealed reasoning that was context focused, which was enabled by their ability to construct these contexts. However, he is quick to note that not all approaches to trigonometry using the unit circle will result in improved student understandings. Rather, he stresses the importance of students understanding the process of creating the unit circle when constructing understandings of trigonometric functions.

Although the research literature specific to trigonometry is limited, research in related areas of mathematics education offer further insights to the critical reasoning abilities necessary for constructing understandings of trigonometric functions. As previously mentioned, developing a process conception of function has been shown to be a difficult task for students (Harel & Dubinsky, 1992; Oehrtman, et al., 2008). Reasoning about a function as a process is critical in trigonometry, as the output of the functions cannot be trivially computed by hand.

Engaging in covariational reasoning has also been shown to be a difficult, but highly important way of reasoning (Carlson, Jacobs, Coe, Larsen, & Hsu, 2002; Oehrtman, et al., 2008). As trigonometric functions formalize covariational relationships, it is of the upmost importance that students develop the ability to reason covariationally in order to reason about the relationships referenced by trigonometric functions. Furthermore, if students are expected to reason covariationally about the relationships formalized by trigonometric functions, then it is inherently necessary that they conceive of the quantities composing these relationships.

Quantitative reasoning (Smith III & Thompson, 2008) provides a model of the mental actions of an individual conceiving of a situation, constructing attributes that admit a measurement process composing his or her conceived situation (e.g., quantities), and reasoning about relationships between these constructed quantities. The quantitative structures that result from these actions can form a foundation upon which a student can reflect and construct understandings that an observer would call mathematical. Although critical, this process has been shown to be non-trivial and highly complex for students (Moore, Carlson, & Oehrtman, 2009).

Lastly, Thompson (2008) argues for a rethinking of the trigonometry curriculum in order to promote coherence between the various settings of trigonometry. He suggests that this coherence can be gained through an appropriate approach to angle measure. In the case that students construct meaningful understandings of angle measure and the process of angle measurement, students can leverage these understandings to construct flexible and coherent understandings of trigonometric functions in the various settings they are introduced (e.g., the unit circle and right triangles). This outline by Thompson, in addition to the research findings and theories previously discussed, formed a foundation for the study presented in this paper.

Conceptual Analysis of Trigonometry

I imagine many would argue, and argue legitimately, that current approaches to trigonometry are already reliant on angle measure. However, this reliance does not necessarily imply that trigonometry is developed on foundations of angle measure, or that angle measure itself is fully developed such that it can be leveraged as a foundation to trigonometry. Thompson's (2008) suggestion for leveraging angle measure to create coherence goes beyond simply placing a lesson of angle measure previous to trigonometric functions. Rather, the manner in which angle measure is developed must be taken as a significant factor in students constructing understandings of trigonometric functions. The understandings of angle measure students construct must be immediately *and* developmentally beneficial for learning. As a counter-example to this approach to coherence, the current draft of the Common Core State Standards Initiative suggests, "limit[ing] angle measures to degrees" in right triangle trigonometry contexts (2009, p. 15). In practice this may reduce the cognitive load for students when constructing understandings of trigonometric functions in right triangle contexts. However, an approach grounded in this suggestion may not offer coherence between unit circle trigonometry and right triangle trigonometry, particularly in light of the research findings previously presented.

Conceptual analysis is a learner-centered tool to help achieve this difficult goal of coherence (Thompson, 2008). Two interrelated uses of conceptual analysis are 1) describing ways of knowing that are immediately *and* developmentally beneficial for learning and 2) analyzing ways of understanding a body of ideas based on describing the coherence between their meanings, where coherence refers to both compatibility and support of ideas. These two uses of conceptual analysis provide a tool for determining instructional goals and the development of curriculum, both of which are elements of the discussed study.

I now briefly present a system of ideas of angle measure and trigonometric functions in order to provide an example of a conceptual analysis of a topic. This conceptual analysis drove the design of this study, and namely the design of the instructional and interview tasks. I also must acknowledge that much of this conceptual analysis is based on the groundwork laid by Thompson (2008) and Thompson, Carlson, and Silverman (2007). I note that a similar conceptual analysis is presented in a previous study into student conceptions of angle measure (Moore, 2009). This conceptual analysis is rehashed here in order to situate the data analysis within the instructional goals driving the teaching experiment.

A central goal of the study was to support the subjects in constructing an understanding of the *process*, or derivation, of angle measure. This can be accomplished by approaching the process of angle measure in terms of measuring the length of an arc subtended by the angle as a fraction of the circumference of that circle. So, the degree is a unit of angle measurement that refers to an angle subtending $1/360^{\text{th}}$ of any circle's circumference, and the radian is a unit of angle measurement that refers to an angle subtending $1/(2\pi)^{\text{th}}$ of any circle's circumference. Both units of measurement have a fixed total number of units that rotate through any circle's circumference centered at the vertex, and measuring the openness of an angle involves determining the fraction of the total circumference of a circle and how many of the total units correspond to this fraction. Thus, if x degrees or r radians are subtended by an angle, both measurements correspond to the same fraction or percentage of a circle's circumference, yielding $x/360 = r/2\pi$.

In addition to an arc length's fraction of a circle's circumference, an angle measurement made in a number of radians describes a multiplicative comparison between a length (e.g., arc length) and the length of a radius. Thus, if an angle has a measurement of 2 radians, the length of the arc subtended by the angle is twice the length of a corresponding radius. Or if an angle has a measurement of π/4 radians, the length of the arc subtended by the angle is $\pi/4^{\text{ths}}$ the length of a corresponding radius. Note that the focus here remains on a quantitative relationship between the arc length subtended by the angle and a second quantity (in this case the radius), opposed to a position or label on the unit circle.

In order to leverage an image of angle measure as a subtended arc, as well as the radius as a unit of measurement, the sine and cosine functions can be introduced within the context of circular motion. The cosine and sine functions can be explored as *functions* that have an input of angle measure, in radians, and an output that is a fraction, or percentage (in decimal form), of one radius. The output of the cosine function is the abscissa of the terminus of the arc subtended by the angle and the output of the sine function is the ordinate of the terminus of the arc subtended by the angle, with both measured as a fraction of one radius. This definition, along with the ideas presented above, allows the development of the cosine and sine functions coherently in each context (Figure 1); the cosine and sine functions have an input of angle measure, measured in radians, and output length measured as a fraction of a radius (where the hypotenuse of a right triangle forms a radius). The outputs of the cosine and sine functions are measurements formed by a multiplicative comparison regardless of the context. Furthermore, if the radius (length of the hypotenuse) is held constant and the angle measure varies, then the output values of the cosine and sine functions vary accordingly.

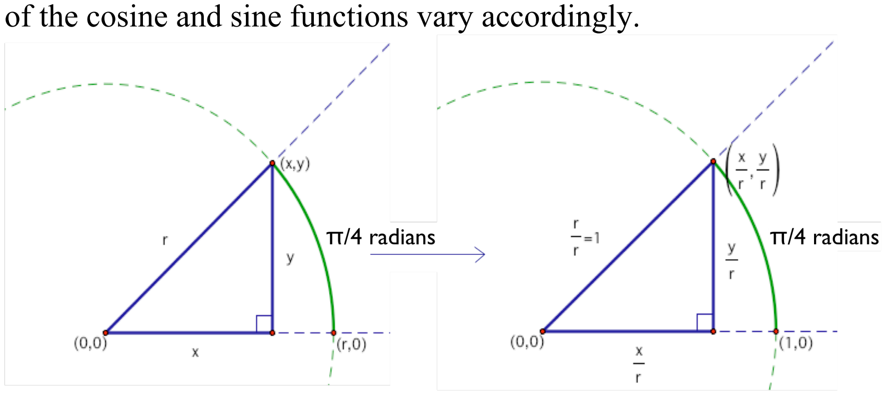


Figure 1

This approach additionally enables a student to leverage the radius as a unit of measurement relative to a circle of any radius length. By reasoning about a circle's radius as a unit of measurement for both positions and arcs, all circles can be conceived as having a radius of one unit and a circumference of 2π units (e.g., the unit circle).

Overall, this presented system of ideas, in addition to the research findings discussed above, formed a foundation for the design and analysis of this study. This study sought to determine the understandings constructed by students as they interacted with tasks designed to promote these ways of reasoning.

Methodology

The purpose of this study was to gain insights to the role of quantitative and covariational reasoning in a student's construction of understandings of trigonometric functions and topics foundational to trigonometry.

Subjects and Setting

The results presented in this paper focus on one student, Zac. At the time of the study, Zac was enrolled in an undergraduate precalculus course at a large public university in the southwest United States. The precalculus classroom was part of a design research study where the initial classroom intervention was informed by theory on the processes of covariational reasoning and select literature about mathematical discourse and problem-solving (Carlson & Bloom, 2005; Carlson, et al., 2002; Clark, Moore, & Carlson, 2008). The head of the precalculus curriculum design project was the instructor of the course. The subject was chosen on a volunteer basis and the subject was monetarily compensated for his time.

Zac was a full-time student at the time of the study. Zac was a Caucasian male in his early-twenties, an ethnomusicologist and audio technology major, and received a "B" as his final grade. Additionally, Zac did not plan on taking any math courses beyond his current precalculus course.

Data Collection Methods

The researcher (myself) withdrew Zac and two other volunteer students from the precalculus classroom in order to perform a teaching experiment (Steffe & Thompson, 2000) consisting of eight ninety-minute sessions. During the classroom sessions, the researcher acted as the subjects' instructor. Also, the teaching sessions occurred over a four-week period. Each teaching session included all three subjects, the researcher, and a fellow colleague as an observer. Immediately after each session, the researcher debriefed with the observer to discuss various observations during the teaching sessions and possible refinements to make to the future teaching sessions and interviews.

The design of the study also included a 60-minute pre-interview with Zac. The intention of this interview was to gain insights to Zac's understanding of angle measure upon entering the study. The pre-interview followed the design of a clinical interview (Clement, 2000) and Goldin's (2000) principles of structured, task-based interviews.

Exploratory teaching interviews totaling four hours were conducted with Zac throughout the study. Specifically, a two-hour interview occurred after the fourth teaching session and a second two-hour interview occurred after the final teaching session. The purpose of these interviews was to gain additional insights into the developing conceptions of Zac as he progressed through the instructional sequence. The interviews followed Goldin's (2000) principles of structured, task-based interviews with one significant addition: the interviews also involved instruction and the posing of additional questions based on the actions of the subject. This offered the researcher an opportunity to implement tasks "on the fly" as his model of Zac's understandings evolved. This enabled the researcher to gain additional insight to the possible limitations in Zac's current ways of thinking and the necessary reasoning abilities to overcome these limitations.

All interviews and teaching sessions were videotaped and digitized. Upon completion of the study, video data was first transcribed and analyzed following an open and axial coding approach (Strauss & Corbin, 1998). The researcher analyzed Zac's behaviors in an attempt to discern the mental actions behind his observable behaviors. The researcher then attempted to identify patterns or connections between these actions, and the role of quantitative and covariational reasoning in the subject's actions.

Results

Analysis of the interviews and teaching sessions offered insights to Zac's constructed understandings over the course of the study. In this section, I first describe his conceptions of angle measure upon entering the study. I then present data illustrating his constructed understandings of angle measure and trigonometric functions as a result of his interactions with the instructional and interview tasks. During this discussion, I also identify the role of quantitative and covariational reasoning in his constructed understandings.

Initial Conceptions of Angle Measure

Upon entering the study, Zac held a self-admitted limited understanding of angle measure. Although Zac had completed precalculus in high school and calculus I at a different university, he claimed, "I never really thought about [angle measurement]." Also, when presented with specific angle measures (e.g., one degree), Zac had difficulty explaining the meaning of the measurements. Zac's explanations ranged from referencing properties of various geometric objects to vaguely identifying arcs and areas between two lines. For instance, Zac discussed two perpendicular lines having ninety degrees and a straight line as one hundred and eighty degrees. Although he alluded to areas and arcs between the two lines, he was unable to provide a meaning of angle measure that included a systematic process that the measurement was based on.

In light of Zac's difficulties, he was given the supplies of a compass, waxed yarn, and a ruler and then asked to measure an angle during the pre-interview. Zac immediately responded that he could not accomplish the task, further illustrating his lack of a systematic process of measuring an angle. During Zac's engagement with this task, he also referenced the tool of a protractor. Zac described, "they have…a [protractor] that's already designed out, show's you where all the angles are." An important aspect of this description is Zac's reference to a protractor showing "*where* the angles are." Such a statement is consistent with an individual focusing on objects and the positions of lines without a systematic coordination of quantities (e.g., quantifying an arc length's fraction of a circle's circumference).

Although Zac did not provide an explanation of the process of measuring an angle relative to measureable attributes and relationships between these attributes, he was able to solve a problem that explicitly identified an arc length. The presented problem prompted Zac to determine an unknown angle measurement that corresponded to an individual traveling 32 feet around a Ferris wheel with a radius of 51 feet. When solving this problem, Zac explained that ninety degrees corresponded to one fourth of the circumference. Zac then computed one fourth of the circumference (80.1 feet) and constructed the equation $\frac{90}{80.1} = \frac{x}{32}$, subsequently solving for a correct angle measure of 35.95 degrees.

After the researcher prompted Zac to explain his solution, Zac claimed, "Well it's just, if you're given three variables and you just need one more…it gives you three of the four you need. It's a very easy equation to find a fourth." It light of Zac's actions, it appears that his solution method was mostly grounded in the "type" of problem and performing calculations. That is, Zac's constructed equation was a result of encountering a problem in which he was given three known values and one unknown value. Although on the surface his equation appears to imply some form of proportional reasoning, Zac's calculational focus implies that the equation was not a reflection of quantitative relationships. This was further emphasized when the researcher asked Zac to interpret one of the ratios of his constructed equations. In response, Zac immediately calculated the ratio and explained that he could multiply by the third given number to obtain the answer. As the researcher continued to probe Zac, Zac's descriptions of the two ratios did not include referencing the quantities of the situation.

In summary, Zac's actions during the pre-interview revealed an apparent lack of a systematic process of angle measurement. Although Zac alluded to various attributes (e.g., arcs and areas) and geometric objects (e.g., lines), he did not appear to reason explicitly about quantities (e.g., attributes admitting measurement processes) and relationships between quantities. Also, when describing a protractor, Zac referenced that the protractor showed "where the angles are," apparently reasoning about positions. Lastly, although Zac obtained a correct angle measurement when solving a problem, the reasoning driving his solution did not appear to be based on quantitative relationships. Rather, the "type" of problem led to his solution choice, and Zac did not explain a quantitative meaning behind his solution process even after the researcher specifically focused Zac on explaining the meaning of a ratio he constructed.

Constructed Conceptions of Angle Measure

In response to Zac's suggested use of a protractor and lack of a systematic process of angle measurement, the researcher designed and implemented an activity during the first classroom session that prompted Zac to construct a protractor. Additionally, the task required the construction of a protractor for multiple units of measurement in order to provide Zac the opportunity to reflect and identify a common structure of these various units.

Zac was first asked to construct a (half circle) protractor such that 8 units (named gips) rotated through a full circle[1]. Zac initially claimed dividing "it" in half would result in a measurement of two gips. After the researcher asked Zac to explain this division, Zac described, "the protractor, I just drew a line down the middle and that gives me two gips." During this interaction, Zac appears to focus on breaking up the object, rather than the perimeter, of the protractor into two equal pieces leading him to approximate a vertical line that went down the "middle" of the protractor.

Zac's focus on the entire object of the protractor presented difficulties when he attempted to construct a measurement of one gip. Although Zac identified a goal of creating two equal "pieces" of the protractor, he had difficultly accomplishing this goal. As he continued to have difficulty creating the protractor, the researcher decided to intervene with a question intended to refocus Zac's actions. Zac had correctly identified the mark for two gips by breaking the object of the protractor into two pieces, but his method was based on estimating the "best" line to create two equal pieces, or areas. Thus, the researcher asked Zac for a method to determine if the original line marking two gips was in the correct location. Zac responded that he could "measur[e] out the perimeter," while subsequently suggesting a calculation to determine the number of centimeters of arc length per one unit of angle measurement. As a result of the researcher's questioning, Zac's actions began to shift to identifying various measurable arc lengths and relating these to a number of units of angle measurement.

Although Zac correctly identified a measurement of arc length that corresponded to a unit of angle measurement, another instructional goal was to promote Zac reasoning about an angular

[1] The problem statement left the term "circle" undefined in order to promote a discussion that distinguished the measurement of the circumference of a circle from the measurement of the area of a circle.

unit of measurement in relation to the entire circumference of a circle. In order to foster this reasoning, the researcher posed multiple ratios (e.g., each ratio represented one-fifteenth of a circle's circumference) of arc lengths (measured in centimeters) relative to the corresponding circumferences of circles (measured in centimeters), where each arc length corresponded to the same unit of angle measurement. The researcher also posed the ratio of 1/15. After Zac calculated the numerical value of each ratio, the researcher asked Zac to interpret the results of these calculations, with the expectation that Zac would explain a quantitative meaning of the ratios.

Zac then described, "You are taking one-fifteenth of the full circumference, then dividing it by the full circumference." Thus, when interpreting the presented ratios, Zac conceived of each expression's numerator as a measurement of one-fifteenth of the corresponding circumference (the denominator). Thus, it appears that Zac conceived of the numerator and denominator of each ratio as representing measurements of quantities, while also conceiving of the ratio as a multiplicative comparison between these two measurements. After discussing this phenomenon relative to an angle measurement of one quip[2], the researcher returned Zac to an angle measurement of one degree. Zac immediately explained that one degree corresponds to an angle subtending "one-three hundred and sixtieth of a circle's circumference," adding that this was true for any circle centered at the vertex of the angle.

In summary, Zac's actions relative to constructing a protractor imply that Zac reconstructed of his image of angle measurement, and specifically degree measurement, such that angle measures implied a quantitative relationship. Zac reasoned about the process of measuring an angle as measuring a subtended arc length as a fraction of a circle's circumference. This understanding required that Zac conceive of an angle subtending an arc length that was measureable relative to the circumference of the circle. Through the action of constructing protractors in various units, with the researcher prompting Zac to reflect on this process and the quantitative meaning of various calculations, Zac began consistently describing angle measure as quantifying a fraction, or percentage, of a circle's circumference cut off by the angle.

After Zac constructed an understanding of degree measurement that appeared consistent with the instructional goals (e.g., measuring a fraction of a circle's circumference subtended by the angle), Zac engaged in an activity prompting him to i) construct a circle using a piece of waxed yarn as the radius of the circle and ii) measure the circumference of the circle in a number of yarn lengths. Zac's engagement in this activity resulted in him conceiving of 2π radius lengths rotating through a circle's circumference. Following this exploration, the researcher asked Zac to discuss the validity of the radius as a unit of angle measure. Zac responded, "it simplifies a circle…the circumference of a circle is equal to two pi r, where the radius is the unit, not inches, or anything like that. So it simplifies it…using it as an actual unit… one radius, and then the circumference is six point two eight radius." Zac's actions imply that he conceived of the radius "as an actual" unit for measuring an arc, which enabled Zac to conceive of 2π radii rotating through any circle's circumference. Additionally, Zac identified a circle as having a radius of "one radius." Thus, Zac's ability to construct and measure various quantities in a number of radii enabled him to conceive of all circles having a radius of "one radius" and a circumference of "six point two eight radius" (e.g., the unit circle).

[2] A measurement of one quip corresponds to an angle that subtends one-fifteenth of a circle's circumference.

As Zac continued through the study, he consistently leveraged his ability to construct subtended arc lengths that were measurable relative to the circumference and radius of a circle. For instance, when presented with a measurement of 2.2 radians, Zac first constructed a diagram (Figure 2) and *then* explained, "it means that the angle length…right here (*tracing arc length*) is equal to two point two radius lengths."

Figure 2

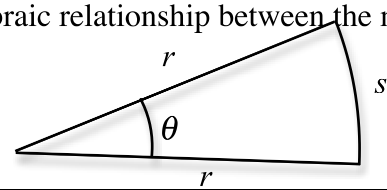
In this case, Zac's ability to reason about an arc length as measurable in a number of radius lengths enabled him to interpret the given measurement *without* converting to a number of degrees while constructing an angle of the corresponding openness. In a similar manner, Zac explained that a measurement of 0.5π radians corresponded to "pi halves radius lengths," while subsequently approximating this as 1.57 radius lengths.

Zac's ability to reason about angle measurements reflecting a fraction of a circle's circumference subtended by an angle also enabled him to flexibly convert between units of angle measurement. As an example, during the interview conducted after four teaching sessions, Zac constructed the equation $\frac{35}{360} = \frac{x}{2\pi}$ to convert 35 degrees to a number of radians. Recall that during the pre-interview, Zac constructed a similar equation (e.g., three known values, one unknown value) while providing a procedural explanation. Contrary to his explanation during the pre-interview, Zac supported his angle conversion by explaining, "Well what you're doing is just technically finding a percentage. Like thirty-five over three sixty is…nine point seven percent of the full circumference…so thirty-five of those degrees equals nine point seven percent…so all I have to do is multiply nine point seven percent by two pi." In this case, Zac's solution appears to be driven by his ability to reason about angle measure as conveying the fraction of a circle's circumference subtended by the angle. This enabled Zac to reason that the same percentage of 360 degrees and 2π radians were subtended by an angle, with his constructed ratios reflecting this quantitative relationship.

Lastly, Zac's ability to conceive of measuring an arc length as a fraction of the radius appears to have entailed reasoning indeterminately about this quantitative relationship. As an example, consider Zac's actions on The Arc Problem, which prompted him to construct an algebraic relationship that was *not* formalized during the classroom sessions.

The Arc Problem

Using the following diagram, determine an algebraic relationship between the measurements r, θ, and s.



After reading the problem statement, Zac claimed that the angle measurement was in radians and proceeded to construct an algebraic representation (Excerpt 1).

Excerpt 1

Zac: Alright. We'll say theta equals radians (*writing* $\theta = rad$), very very simple then. r theta is equal to s (*writing* $r\theta = s$). 'Cause theta is in radians, that means a percentage of the radius. Which would then be equal to this length (*tracing arc length*). So you multiply the percentage of the radius by the radius, you'll get the arc length.

During this explanation, Zac appears to leverage his ability to reason indeterminately about measuring an arc length as a fraction of the radius. This reasoning resulted in Zac constructing an algebraic representation reflecting his constructed quantitative relationship corresponding to measurements in radians. Zac offered further insights to this quantitative relationship when the researcher asked him to explain the formula $\theta = s / r$. Zac explained, "Well this is…a ratio, that'll give you a percentage of *r*." This explanation reveals Zac describing a presented formula relative to the quantities of the situation, rather than a procedural calculation (e.g., dividing both sides of the previous formula by *r*). Thus, it appears that his ability to reason about measuring an arc length as a fraction of the radius enabled him to construct meaningful algebraic representations between indeterminate measurements of these quantities.

In summary, over the course of the study Zac appears to have constructed a very flexible understanding of angle measure. Zac's understandings also appear to have stemmed from him conceiving of the process of measuring subtended arc lengths relative to the circumference or radius of a circle. This enabled him to construct understandings of various units of measurement that were rooted in quantitative relationships based on indeterminate values. As he attempted various tasks, Zac was able to leverage these reasoning abilities to reason about various angle measurements, convert between various angle measurement units, and construct solutions and algebraic relationships rooted in quantitative relationships.

Constructed Conceptions of Trigonometric Functions

In order to leverage Zac's ability to reason about a measurable arc length and the radius as a unit of measurement, the researcher designed an exploration of circular motion to promote Zac constructing the sine (and cosine) function. The context consisted of a bug traveling counter-clockwise on the tip of a 3.1-foot fan blade, and was thus named The Fan Problem. Also, the bug began from the standard position (e.g., 3 o'clock).

After the path of the bug was identified as a circle, Zac described that for a specific arc measurement in radians, the linear measurement of the arc would increase for an increasing radius, but the arc length's percentage of the radius would remain constant. Thus, relative to the context of circular motion, it appears that Zac's ability to reason about measuring an arc as a fraction of the radius promoted him conceiving of measuring the bug's distance traveled as a fraction of the radius of the fan.

Next, the researcher asked Zac to generate a graph relating the vertical distance of the bug above the horizontal diameter (referred to as *vertical distance* from this point forward, unless stated otherwise) with the distance traveled by the bug. Initially, Zac generated a graph with the output measured in feet and the input measured in radians. Also, Zac's graph perceptually resembled a graph of the sine function.

In order to further investigate the quantitative relationships conveyed by the graphs, the researcher asked Zac to explain the "shape" of his graph. Also, previous to the Zac's descriptions, the researcher added a graph composed of three linear segments conveying the same directional covariation (Figure 3) as his constructed graph. The researcher intended that this would promote Zac contrasting his graphical relationship with a linear relationship. Relative to the researcher's constructed graph, Zac responded that a constant rate of change implied that for an equal change of total distance traveled, the vertical distance changes by a constant amount. He subsequently claimed that the relationship between the vertical distance and distance traveled was not linear.

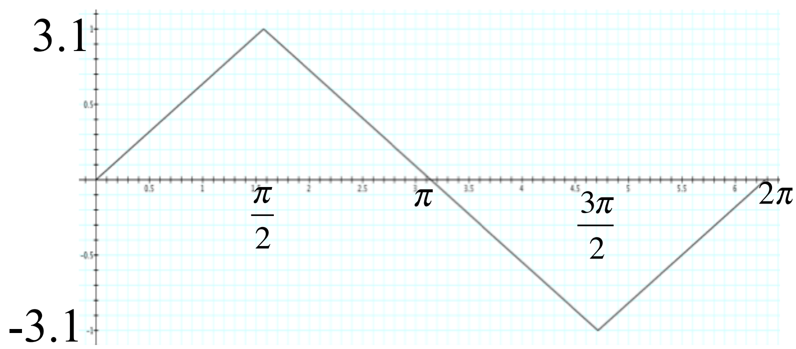


Figure 3

In order to verify his conjectured graph, Zac suggested considering equal changes of distance traveled by the bug while comparing the changes in vertical distance. This suggestion resulted in the researcher using the diagram of the situation, along with prompting from Zac, to identify that the change in vertical distance was decreasing for equal changes of total distance over the first quarter of a revolution.

As the discussion continued, Zac also distinguished between the vertical distance and a *change* of vertical distance, identifying that the vertical distance approached zero while the change of vertical distance increased over the second quarter of a revolution. Zac continued his description by stating that the vertical distance decreased at a decreasing rate for the third quarter of a revolution, while supporting this description by identifying that for equal changes of arc length over the third quarter of a revolution, "the change in vertical distance is going to get smaller."

Zac's actions imply that he conceived of both the vertical distance and the *change* in vertical distance as two separate, but measureable and related quantities. It also appears that Zac's ability to reason about a traversed arc as a measureable attribute enabled him to imagine an object traveling on a circular path while covarying a vertical distance with the total distance traveled around this path. This imagery enabled Zac to reason about amounts of change of vertical distance and how these amounts of change were changing relative to equal changes of arc length. Additionally, as Zac gave these descriptions relative to the situation, the researcher verified this covariation on Zac's constructed graph.

Next, the researcher chose to pursue the implications of Zac's chosen measurement units relative to the constructed graphical representation. The researcher presented a graph that was identical in its perceptual features except that the graph had maximum and minimum output values of one and negative one, respectively (e.g., the graph of the sine function). After constructing this graph and asking Zac to explain the output units of the graph given that it conveyed the same covariational relationship as the previous graph, Zac immediately suggested "the radian" as the output unit. Zac also supported this conjecture by identifying the locations where the bug was zero radius lengths and one radius length vertically from the center of the circle.

Subsequently, the researcher asked Zac how the outputs of the two graphs were related. Zac immediately explained, "multiply the percentage of the radius by the radius length." Thus, it appears that Zac's ability to reason about measuring a length relative to the radius enabled him to quickly identify the radius length as a possible unit of measurement for various measureable attributes of circular motion. In this case, Zac conceived of measuring both a vertical distance and a traversed arc length relative to the radius.

In summary, Zac's engagement on The Fan Problem revealed that his ability to conceive of measuring an arc length in a number of radius lengths formed a foundation for the construction of the sine function. To say more, Zac was able to reason about equal changes of arc length while comparing the corresponding changes of vertical distance. This promoted the construction of a graph that was rooted in a quantitative relationship. Zac also conceived of measuring these quantities in multiple units, while his described relationships between these units appears to have stemmed from reasoning about a multiplicative comparison between a length and the radius.

Zac's ability to reason about measuring a length as a fraction of the radius also led to him converting to a linear unit of measurement as needed. For instance, after the previous interactions, the researcher formalized the graph of the sine function as $f(\theta)=\sin(\theta)$. Zac then suggested that the graph he produced corresponded to the algebraic representation of $g(\theta)=3.1\sin(\theta)$, explaining that multiplying the output of f by the radius length in feet would give an output in feet. Also, Zac was able to give this description without calculating a specific value of $\sin(\theta)$ and multiplying this value by the radius. Thus, Zac's constructed quantitative relationship may have promoted Zac constructing a process conception of the sine function. As a result, Zac was able to reason that the measurement of the output quantity in a linear unit should be the same percentage of the radius length measured in that unit.

As The Fan Problem was conducted in a setting with all three subjects, Zac was prompted to engage in a similar task during an interview session. The researcher designed The Ferris Wheel Problem such that it would offer additional insights to the mental actions driving Zac's behaviors on The Fan Problem.

The Ferris Wheel Problem

Consider a Ferris wheel with a radius of 36 feet that takes 1.2 minutes to complete a full counter-clockwise rotation. April boards the Ferris wheel at the bottom and begins a continuous ride on the Ferris wheel. Construct a graphical and algebraic relationship that relates the total distance traveled by April and her vertical distance from the ground.

After reading the task statement, Zac described, "her vertical distance from the ground," paused, and traced a portion of the circle beginning at April's starting position. He then identified April's furthest distance from the ground during her ride. These actions by Zac imply he was constructing an image of the relationship between the quantities that he was attempting to relate. Zac then drew a larger diagram of the Ferris wheel, which he subsequently used to describe the covariation relationship between the two quantities (Excerpt 2).

Excerpt 2

1	Zac:	Ok. So a really easy way to do this is divide it up into four quadrants
2		(*divides the circle into four quadrants using a vertical and horizontal*
3		*diameter*). 'Cause we're here (*pointing to starting position*), for every
4		unit the total distance goes (*tracing successive equal arc lengths*), the
5		vertical distance is increasing at an increasing rate (*writing i.i.*)…Then,
6		uh, once she hits thirty-six feet, halfway up, it's still increasing but at a
7		decreasing rate (*tracing successive equal arc lengths, writing i.d.*)…Uh,
8		then when she hits the top, at seventy-two, it's decreasing at an
9		increasing rate (*tracing successive equal arc lengths, writing d.i.*)…And
10		then when she hits thirty-six feet again it's still decreasing (*making one*
11		*long trace along the arc length*), but at a decreasing rate (*tracing*
12		*successive equal arc lengths*, *writing d.d.*).
14	Int:	Ok, so in terms of this one, this quadrant (*pointing to the bottom right*
15		*quadrant*), could show me on there how you know it's increasing at an
16		increasing rate? Just show using the diagram...
17	Zac:	So like, a, she moves that much there (*tracing an arc length beginning at*
18		*April's starting position*), that much here (*tracing an arc of equal length*
19		*over the last portion of April's path in that quadrant*), uh, the vertical
20		distance there changes by that much (*tracing vertical segment on the*
21		*vertical diameter*), which is really hard to see with this fat marker. And
22		then, uh, the vertical distance here changes by that much (*tracing*
23		*vertical segment from the starting position of the second arc length*),
24		which is a much bigger change.
25	Int:	Ok.
26	Zac:	And then, you know, you can do the same for all of them.

Similar to Zac's actions during The Fan Problem, Excerpt 2 reveals him reasoning about a traversed arc length while describing variation in the vertical distance of April from the ground. Also, although Zac was asked to create a graph, Zac's descriptions *first* focused on the context of the problem. For instance, in order to support his rate of change description (lines 1-12), Zac identified equal changes of arc length on his diagram while comparing corresponding changes of vertical distance (lines 17-24). Finally, Zac described that the method he utilized in this first quadrant *could* be used for the other quadrants (line 26), implying he constructed a relationship between the quantities such that he was able to anticipate covarying the two quantities over the course of a revolution.

Immediately after these actions, Zac constructed a graphical representation (Figure 4). *During* his construction of the graph, Zac described the directional change and rate of change of the vertical distance (from the ground). In light of these actions, it appears that the graph constructed by Zac emerged from his quantitative relationships revealed in Excerpt 2. Additionally, this approach by Zac may have been promoted by The Fan Problem exploration.

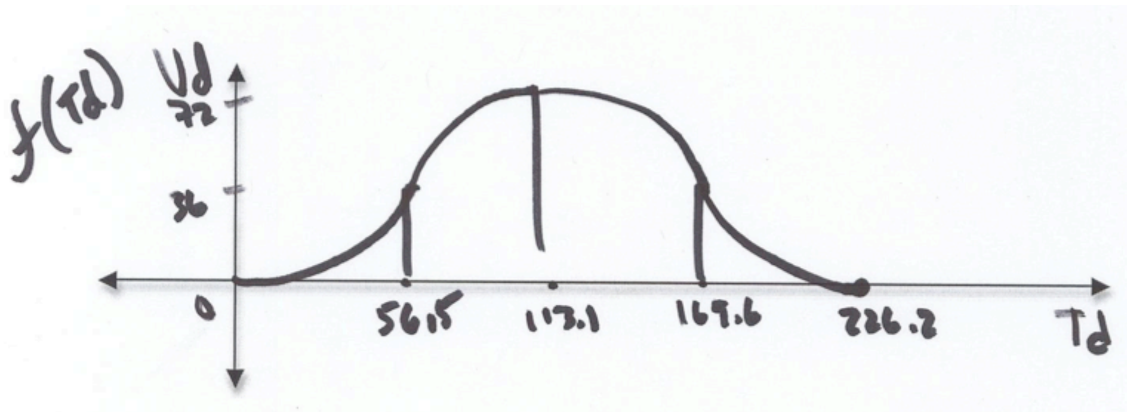


Figure 4

Zac then began constructing the corresponding algebraic representation to his graph. He first described, "…the total distance is the input to get the vertical distance," appearing to identify an input quantity and an output quantity. At this time, Zac rotated his diagram of the situation counter-clockwise by ninety degrees. As he rotated his diagram, he explained, "'cause then I can actually make sine work." Zac then paused for an extended amount of time, eventually stating, "I can still make sine work this way (*referring to the initial orientation of the diagram*)." This was followed by Zac explaining how he could "make sine work" (Excerpt 3).

Excerpt 3

Zac:	But I can still technically make it work here just by taking, by making it, uh, the starting point (*pointing to the bottom of the circle*) sixty-nine point six, um, feet around the circle. Or one sixty-nine point six feet around the circle. Or negative fifty-six point five feet around the circle. So it's going backwards (*tracing arc clockwise from the standard starting position to the bottom of the circle*).
Int:	Ok.
Zac:	I can still get the vertical distance that way. Um, so ya (*pause*). So (*long pause*), that means, since I'm doing that, that means whatever the vert-, or total distance is, I have to subtract fifty-six point five from it.
Int:	Ok.
Zac:	So let's see, vertical distance (*pause*) is equal to f of total distance (*writing*), which is equal to total distance minus fifty-six point five (*writing*), which will get me there (*pointing to the bottom of the circle*). (*pause*) And how (*inaudible*), divided by thirty-six feet to get me radians. (*pause*) And then I take the sine of that. So sine, so that will give me this one (*referring to first constructed graph, task two*).

By the completion of Excerpt 3, Zac constructed the algebraic representation of $vd = f(Td) = \sin\left(\frac{(Td - 56.5)}{36\,ft}\right)$. Zac's initial action of rotating his diagram appears to stem from his desire for the starting position of the individual to be at the standard position. However, after considering this action, he identified the position of the rider as measureable from the standard position along the circumference of the Ferris wheel (lines 1-6). This identified quantity enabled

Zac to explain that this counter-clockwise measurement would always be 56.5 feet less than the individual's distance traveled. As he continued, Zac converted the total distance to a number of radians in order to use this value as an input to the sine function (lines 16-18), an action likely stemming from his conception of the sine function taking an input measured in radians.

At this time in the interview, Zac's constructed algebraic representation was not mathematically consistent with his graph. Although he identified the correct argument for the sine function, Zac was yet to explicitly identify the output of the generated expression relative to the unit of measurement and the vertical distance being quantified. In order to gain additional insights to Zac's conception of his constructed algebraic expression, the researcher prompted Zac to explain his expression an additional time.

After Zac stated that the output of the sine is "the vertical distance," Zac explained, "[sine] will give me a percentage of the radius length, which I then need to multiply by thirty six." Thus, after the researcher asked Zac to further explain his constructed representation, Zac reflected on the output units of the sine function relative to the desired units of measurement, while reasoning about the output of the sine function as a fraction of the radius to alter his algebraic representation to $vd = f(Td) = 36\sin\left(\dfrac{(Td - 56.5)}{36\,ft}\right)$.

Next the researcher asked Zac to explain the distances related by the two representations. Zac claimed the distance traveled and "vertical distance," and in response the researcher asked, "Vertical distance from where?" Consistent with Zac's algebraic representations, he identified the vertical distance from the horizontal diameter of the Ferris wheel. Thus, it appears that Zac's algebraic representation was a mathematically correct reflection of the two quantities he was relating at *this* time in the interview. Although the problem statement prompted him for an output measured from the bottom of the Ferris wheel, which he used as the output value of his graphical representations, Zac conceived of the vertical distance above the center of the Ferris wheel when constructing his algebraic representations.

After the researcher prompted Zac to identify a "vertical distance" on his diagram, Zac claimed, "I was thinking about that actually…you're gonna have to change the formula." Immediately following this statement Zac altered his algebraic expression to $vd = f(Td) = 36\sin\left(\dfrac{Td - 56.5}{36}\right) + 36$. He justified this algebraic representation by comparing the vertical distance measured by his previous representation to the vertical distance requested in the problem statement (e.g., a difference of 36).

Overall, Zac's actions on The Ferris Wheel Problem were consistent with his actions on The Fan Problem. Throughout his solution process, Zac utilized the context of the situation to construct and (indeterminately) relate various quantities. Also, Zac's ability to reason about a varying arc length while coordinating changes in a vertical distance enabled him to construct a graphical representation that was a reflection of this quantitative relationship. This reasoning further enabled Zac to support rate of change descriptions by focusing on amounts of change between two quantities, which has shown to be a difficult line of reasoning (Carlson, et al., 2002). Finally, although Zac's initial algebraic representation was inconsistent with his graphical representation, he was able to reflect on his image of the situation to continually refine this representation. As Zac reflected on his representations relative to the quantities they related and their output units, he was observed refining the quantities he was attempting to relate. Zac's ability to reason about the sine function as formalizing an input-output process between two quantities also enabled him to alter his algebraic representation based on his constructed quantitative relationships.

Trigonometric Functions and Right Triangles

To conclude this results section, I will briefly describe the reasoning abilities exhibited by Zac during a right triangle context. During the interview following the first four classroom sessions, he was presented with a problem prompting him to identify the height of a building given that he was standing 1000 feet from the base of the building and looking at the top of the building with a line of site of 56 degrees with respect to the ground. Also, at this time in the study, a right triangle had not explicitly been identified during the classroom sessions.

After constructing a right triangle to represent the context of the problem, Zac explained, "the hypotenuse is the radius," and *constructed* a circle with a radius length equal to the hypotenuse. As a result of this conception, Zac explained the output values of both the sine and cosine functions as representing measurements relative to the hypotenuse (e.g., the trigonometric ratios), while explaining that the hypotenuse of the right triangle formed the radius of the circle. Thus, Zac's actions on a problem presenting a right triangle context reveal him leveraging his ability to construct a circle using a length (the hypotenuse) as a radius and subsequently reason about measuring quantities relative to this length. This also led to Zac reasoning about the trigonometric functions as formalizing input-output relationships between various quantities of a right triangle context.

In addition to Zac correctly solving this problem by leveraging understandings constructed during the previous classroom sessions, he verbally expressed an awareness of the reasoning driving his actions (Excerpt 4).

Excerpt 4

Zac: I always just thought hypotenuse was, you know, just that side of a triangle. You know, you could use Pythagorean's Theorem to find out what it was very easily. And now that we've figured out, you know, now I'm looking at it and seeing it's the radius, it makes a lot more sense to be able to find, you know, the horizontal and vertical distance according to the radius (*waving tip of pen across the radius*).

Although his feeling of things making "a lot more sense" could be argued to be a result of having multiple experiences with trigonometric functions, Zac's justification for this feeling was very specific. Zac explained that he had not previously considered the hypotenuse to be anything other than a side of a triangle (lines 1-2), but now he understands the radius, or hypotenuse, as a unit of measurement (lines 4-6). Thus, it appears that an outcome of the instructional sequence was Zac finding coherence in reasoning about measuring quantities relative to the radius, or hypotenuse.

Discussion and Conclusions

Over the course of the study, Zac appears to have constructed an understanding of angle measure that he was able to leverage when reasoning about trigonometric functions. Zac's ability to reason about the process of angle measure was central to this foundational understanding. Although Zac began with a self-admitted lack of understanding of angle measure, as Zac engaged in various tasks he conceived of quantities and relationships between quantities that enabled him to attach quantitative relationships to various units of angle measurement. These relationships between quantities (e.g., an arc length and the circumference or radius of a circle) also necessitated the construction of a circle centered at the vertex of the angle. The attachment of quantitative relationships to the various units of angle measure offered him much flexibility between units of angle measure. This resulted in Zac constructing meaningful angle conversions and formulas rooted in quantitative relationships. Moreover, Zac's understanding of the radius as a unit of measurement appears to have resulted in Zac's ability to spontaneously construct the unit circle. To say more, as Zac encountered various situations that included a circle, he conceived of measuring quantities relative to the radius, which resulted in him conceiving of any circle as a circle of a radius of one unit and a circumference of 2π radii.

Zac's ability to reason about a measurable arc length also enabled him to develop images of circular motion that included the covariational relationship of two quantities. An implication of this reasoning was Zac constructing various mathematical representations that were rooted in quantitative relationships. Opposed to simply recalling perceptual shapes from memory to construct a graph, Zac engaged in covariational reasoning within the context of the problem in such a way that graphs emerged from this reasoning. Zac's ability to reason about circular motion also appears to have resulted in him constructing a process conception of the sine function. To say more, Zac often reasoned indeterminately about the relationship between the two quantities formalized by the sine function. Rather than having to compute specific values and find differences between these values, Zac was able to leverage his image of circular motion to engage in covariational reasoning that was not dependent on numerical values. Thus, a possible product of approaching the construction of the sine function within circular motion was offering Zac a situation in which it was beneficial to construct a process conception of the sine function.

The instructional focus on conceiving of measurement units as quantitative relationships also appears to have created a foundation for Zac to construct coherence between right triangle contexts and the unit circle. As Zac encountered right triangle contexts, he was observed constructing circles using the hypotenuse of the right triangle, while subsequently measuring quantities relative to the hypotenuse. This construction process of using the hypotenuse of a right triangle as the radius of a circle appears to have enabled Zac to apply trigonometric functions to right triangle contexts in ways that were consistent with his use of the functions in circular contexts.

Zac's understanding of the unit circle and measuring quantities as a fraction of the radius also offered him a foundation to reflect upon during his solution processes. Although Zac's initial algebraic representations on The Ferris Wheel Problem would be considered incorrect relative to the written problem statement, these representations were consistent with the situation he was attempting to model. Then, as the researcher questioned Zac, he reflected on the context of the situation and identified various quantities and relationships between these quantities. This reflection resulted in him altering his algebraic representation such that it was consistent with the written goal of the problem.

This process of a subject continually refining her or his image of a situation, and the possible inconsistencies that naturally arise during this process, have been recently noted as a common occurrence in undergraduate students' problem solving processes (Moore, 2009; Moore, et al., 2009). Thus, if contextual situations are to be used as a foundation for mathematical reasoning and abstraction, this process of a subject continually modifying their image of a situation must be taken into consideration. In order to promote the construction of mathematical knowledge through quantitative reasoning, a researcher or teacher must be attentive to the contextual situations subjects construct while attempting to promote the construction of situations consistent with the instructional goals.

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