A Study of the Role of Intuition in the Development of Students' Understanding of Span and Linear Independence in an Elementary Linear Algebra Class

Frieda Parker
University of Northern Colorado
Advisors: Hortensia Soto-Johnson and Cathleen Craviotto

In this paper, I report on preliminary results from my dissertation study on the role of intuition in students' learning of span and linear independence in an elementary linear algebra class. The purpose of my research is to examine the relationship between the quality of students' understanding of span and linear independence with respect to the role of intuition in their learning of these concepts. This qualitative study is based on the multiple case study tradition and employs the theoretical perspective of social constructivism. Methodological issues of importance in this study are how to assess the quality of students' understanding and how to evaluate the role of intuition in that understanding. Findings from this study might inform the development of more effective teaching practices for span and linear independence in linear algebra courses.

Sierpinska (2005), in a speech at the12th International Linear Algebra Society Conference, reflected upon her extensive research into the teaching and learning of linear algebra. In two teaching experiments, she and her colleagues attempted to improve students' understanding of theoretical concepts by having students do computational activities and computer labs that provided concrete illustrations of key ideas. A surprising result of these experiments was

> the gap between successful and less successful students widened instead of shrinking. The former learned a lot more than in the ordinary course and became even more enthusiastic about mathematics, and the latter became more confused or, worse, developed strong misconceptions while believing that they have finally understood the subject. (p. 1)

To explain these results, Sierpinska conjectured the less successful students understood the concrete representations of concepts in a practical rather than theoretical way. To test this hypothesis, she and her colleagues created definitions of practical and theoretical thinking. They distinguished between these two modes of thinking using five categories: reasons for thinking, objects of thinking, means of thought, main concerns, and products of thinking (Sierpinska, 2004). Next, they designed instructional practices aimed at improving students' theoretical mind-set. They found students' understanding of theory did improve with these new instructional practices, but the distinction between practical and theoretical thinking was an oversimplification of the patterns in successful students' thinking. Successful students were not only theoretical thinkers, they had a practical understanding of theory. Sierpinska believes a combination of practical and theoretical thinking characterizes the art of doing mathematics. Purely theoretical thinkers cannot find smart ways to solve problems and purely practical thinkers cannot develop deep conceptual understanding.

The research by Sierpinska (2004) and her colleagues demonstrated that while only the more successful students consistently engaged in theoretical thinking, all students engaged in practical thinking. Taken from another point of view, one could consider theoretical thinking as a reflection of formal mathematics and practical thinking as a "natural" way of thinking. Using this perspective, some researchers have found a misalignment between mathematical theory and the natural way students approach learning mathematics. For instance, Tall (1991), wrote in his epilogue of *Advanced Mathematical Thinking* that many of the book's contributors believed students' difficulties in learning advanced mathematics could be explained by the difference in the logic of formal mathematics, upon which most university instructors base their instruction, and the logic of students' cognition. More recently, in their discussion of advanced mathematical thinking, Mamona-Downs and Downs (2002) suggested traditional teaching of mathematics does not "connect with the students' need to develop their own intuitions and ways of thinking" (p. 170).

Mamona-Downs and Downs' (2002) observation of the disconnect between traditional teaching and students' way of thinking leads to another view of natural or practical thinking: intuitive thinking. Fischbein (1987, 1999), who has contributed significantly to the theoretical and research base about the role of intuition in the learning of mathematics, believes experimental and descriptive research provides evidence that formal knowledge of a mathematical topic is insufficient for people to produce mathematics. That is, knowledge of axioms, definitions, theorems, and proofs does not enable someone to solve problems or develop proofs; they also need intuition about the concepts (Fischbein, 1987). This suggestion aligns with Sierpinksa's belief that the art of doing mathematics results from a practical understanding of theory.

A didactical question follows from these observations: How can a mathematics teacher draw upon students' intuitive thinking to help them develop a practical understanding of the mathematical theory? Sierpinksa (2004) argues one cannot derive the answer to this question entirely from general epistemological theories or from psychological studies of cognition; research into specific mathematical content is also needed. Torff and Sternberg (2001), who reviewed the literature about the role of intuition in education, suggest teachers would benefit from a better understanding of how students' intuition coincides or conflicts with teaching practices, but more research is needed in this area. Taken together, the areas of study suggested by Sierpinska, Torff, and Sternberg underlie the purpose of my research. In my research I am studying how students' intuitions influence their learning of two particular mathematical concepts: linear independence and span. In order to gain some insight into the interaction of students' intuitions and teaching practices, I conducted the study in one linear algebra class. My research question is

What is the role of intuition in the development of students' understanding of span and linear independence in the context of the instruction environment of an elementary linear algebra class?

The Intuitive Mind and Education

Wilson's (2002) theory of the "adaptive unconscious" supports the idea that intuition has a role in learning. Wilson defines "unconscious" as "mental processes that are inaccessible to consciousness but that influence judgments, feelings, or behavior" (p. 23). Wilson uses the adjective "adaptive" to indicate that nonconscious thinking is an evolutionary adaptation. The adaptive unconscious plays several roles in processing the extraordinary quantity of sensory information the brain receives every second. One role is implicit learning, which is learning without effort or awareness of what is learned. Such learning usually occurs through pattern recognition. The adaptive unconscious is capable of learning complex information, sometimes faster and better than the conscious mind. It is also responsible for "gut feelings" by communicating its implicit learning via emotions and feelings.

Another role of the adaptive unconscious is to select and interpret information, in effect serving as a gatekeeper to the conscious mind. Survival depends upon organisms accurately interpreting their world, therefore the filtering and interpreting mechanisms of the adaptive unconscious have evolved to generally provide a functional worldview. However, the adaptive unconscious' goal of maintaining a person's sense of well-being sometimes conflicts with its goal of accurately interpreting the world. Wilson (2002) believes this is "one of the major battlegrounds of the self, and how this battle is waged and how it is won are central determinants of who we are and how we feel about ourselves" (p. 39).

The adaptive unconscious and the conscious mental systems have different characteristics. Wilson (2002) summarizes the nature of these two systems as follows:

> The adaptive unconscious is an older system designed to scan the environment quickly and detect patterns, especially ones that might pose a danger to the organism. It learns patterns easily but does not unlearn them very well; it is a fairly rigid, inflexible inferencemaker. It develops early and continues to guide behavior into adulthood. Rather than playing the role of the CEO, the conscious self develops more slowly and never catches up in some respects, such as in the area of pattern detection. But it provides a check-and-balance to the speed and efficiency of nonconscious learning, allowing people to think about and plan more thoughtfully about the future. (p. 66)

Wilson claims the adaptive unconscious is a significant part of a person’s cognitive and affective personality.

Torff and Sternberg (2001) suggest intuition plays a role in all learning contexts. They define intuitive conceptions as “knowledge or knowledge structures that need not be available to conscious reflection but that act to facilitate or constrain task performance” (p. 7). “Intuition,” then, is “the process through which intuitive conceptions are acquired and used” (p. 7). Two other terms are “primary intuitive conceptions” and “secondary intuitive conceptions.” Primary intuitive conceptions are genetically encoded intuitions universal to the human species. These intuitions are developmental, not externally acquired. For example, when learning physics, students appear to have innate conceptions of force and agency, which are not acquired in school. Secondary intuitive conceptions are knowledge resulting from learner-environment interaction and so are particular to the sociocultural conditions inherent in the learning environment. A key question for educators is how intuitive conceptions help or hinder the learning of students.

Torff and Sternberg (2001) review the literature about the constructs of implicit learning and tacit knowledge. Implicit learning occurs outside conscious awareness both in terms of the process of learning and what has been learned. Empirical studies in a variety of settings have confirmed the existence of implicit learning, but there are still many controversies involving the conceptualization of this construct and the appropriate research methodologies. Open questions include whether implicit learning yields abstract knowledge or situation-specific knowledge and the nature of the relationship between implicit and explicit processes of learning. However, researchers generally see implicit learning as a positive process that supports learning in the classroom. The product of implicit learning is tacit knowledge, which is "procedural knowledge that guides behavior but is not readily available for introspection" (p. 10). Studies have shown that successful individuals not only have a large body of explicit knowledge, but also have extensive implicit knowledge. Sternberg and his colleagues believe tacit knowledge is a positive force in education and they recommend the explicit teaching of the valued tacit knowledge in each discipline.

Fischbein (1987) has provided the most comprehensive discussion of the construct of intuition in mathematics learning. Whereas Wilson (2002) attributes the existence of the "adaptive unconscious" to the brain's need for a system to filter and process large quantities of incoming perceptual information unconsciously, Fischbein provides another perspective on the existence of intuition. Conscious cognition is often unable to infer and interpret from the given information with certainty, yet humans must have some sort of credible reality in order to act in meaningful ways. This creates an underlying need for certitude. Therefore,

> Intuition summarizes experience, offers a compact, global representation of a group of data, helps overcome the insufficiency of information, introduces behaviorally meaningful interpretations in a reasoning process, and thus confers on the mental activity, the qualities of flexible continuity, of firmness and efficiency which characterize an active, adaptive behavior. (p. 12)

Fischbein defines intuition by describing its key characteristics. The first is that "intuitions refer to self-evident statements which exceed observable facts" (p. 14). In other words, intuitions act like theories in that they extrapolate beyond the immediate information. Globality is a consequence of the self-evident nature of intuitions and describes how intuitions are applied generally often without regard to contextual considerations. Intuitions are also coercive; they assert themselves in such a way that one does not consider alternative explanations and theories.

The literature about the role of intuition in advanced mathematics and in education overall provide evidence that students' intuitions do play a role in their learning. In particular, this role can help or hinder learning. Studies support exploring the role of intuition in the learning of specific mathematical concepts, and paying particular attention to students' use and development of intuition in relation to the nature of the instruction environment.

Theoretical Perspective and Methodology

The emergent perspective (Cobb, 1995; Cobb & Yackel, 1996; Yackel & Cobb, 1996), which is a coordination of social and psychological learning theories, provides the foundation of my theoretical perspective. The *interactionist* view of classroom processes (Bauersfeld, Krummheuer, & Voigt, 1988) represents the social perspective, while a constructivist view of individuals' (both students and teacher) activity (von Glasersfeld, 1984, 1987) represents the psychological perspective. In the emergent perspective neither the social nor the psychological view dominates. Instead, they have a reflexive relationship in which the students and teacher influence the development of classroom social norms and, in turn, these social norms influence student learning. Intuition theory coordinates well with the emergent perspective and its underlying constructivist and social theories. Both perspectives suggest implicit learning is a consequence of students' participation in the learning environment. Students' implicit learning involves finding viable regularities in their experiences, which then stabilize into unconscious schemas. The resulting tacit knowledge constitutes essential elements of mathematical knowledge, such as when and why to invoke specific problem solving strategies. In addition, this tacit knowledge influences future learning.

The setting for this research was an elementary linear algebra course at a mid-sized public liberal arts university in the western United States. There were 32 students in the class. Students' majors included secondary mathematics teaching, physics, meteorology, and other science majors. For most students, this class is their first exposure to elements of advanced mathematics, such as proving and learning abstract structures. The class instructor was experienced in teaching this course at more than one university. The textbook for the course was David C. Lay's (2003) *Linear Algebra and Its Applications*. The course content began with matrix operations involving linear systems of equations. Subsequent topics included matrix invertibility, determinants, vector spaces, and eigenvalues and eigenvectors. The class met 3 days a week for 50 minutes. The prerequisite for the course is Calculus II. For most of the students, this will probably be their only class in linear algebra.

This research is a psychological case study. I selected 7 students from the linear algebra class with each individual student comprising a case. The case context is the learning environment of the students, which includes the physical setting, the instructional practices, the classroom culture, and the textbook. I selected individual cases to represent a maximum variation (Merriam, 1998) in students' understanding of span and linear independence. To do this, I collected data for all students in the class up through the first test and then used this set of data to select students. In particular, I chose the cases from the pool of volunteers and from those students who consistently attended class and turned in completed work.

I collected data related to both the learning context and the individual learners. Data related to individuals included written work resulting from homework problems, exams, and journals. I also conducted two interviews with each of the seven students. I conducted the first interview around the fifth week of the semester and the second interview about two weeks before the end of the semester. The interviews were semi-structured (Merriam, 1998) and lasted between an hour and an hour and a half. Interview questions included asking students about their understanding of the meaning of span and linear independence, their understanding of the formal definition of these concepts, and their understanding of procedures that involved these concepts. The instruction environment data consisted of classroom videotapes, classroom field observations, classroom and student artifacts, and the course textbook.

Preliminary Data Analysis

I am currently in the process of data analysis. The guiding principle of the analysis is to develop profiles of students' intuition and understanding and determine the nature of any associations that exist between intuition and understanding. My analysis of understanding is premised on Harel's (1997) assertion that the essential indicator of a student's understanding of a concept is when the student has the ability to solve problems related to that concept. This necessitates a descriptive approach to capturing student understanding that includes the connections students make between ideas (Zandieh). I ascertain the quality of students' understanding by determining the degree to which the student met the instructor's expectations of understanding.

My preliminary analysis indicates that students' understanding of span and linear independence is significantly influenced by their understanding of linear combination and solution. In turn, students' understanding of linear combination and solution appears to be dependent upon their skill and understanding of working with the multiple roles of variable that appear in the definitions and problems involving span and linear independence. Students with weak understanding of variable ignored variables in expressions, confused when variables were parameters or unknowns, and inappropriately substituted numbers for unknowns. The finding that understanding is a result of layers of concepts aligns with Zandieh's framework of understanding in which she asserts that the understanding of concepts, such as derivative, are predicated on layers of process-object pairs (Sfard, 1992). The other dimension of her understanding framework is *multiple representations*. Aspects of students' understanding can then be captured by looking at how students know each layer (process-object pair) when working with different representations (graphical, verbal, symbolic, etc.). I am in the process of determining how well Zandieh's understanding framework works with my data.

My work in creating students' intuition profiles has expanded to my considering a more general construct I am currently calling *thinking profiles*. These thinking profiles consist of indicators of students' intuitive thinking, problem solving patterns, and language use. Drawing from the literature, I developed a set of indicators of students' intuitive thinking with respect to mathematical thinking shown below.

1. Indicators that can appear when a student holds incorrect conceptions.
    1.1. The conception makes sense from an "everyday" perspective (Stavy & Tirosh, 2000; Vosnaidou, 2008).
    1.2. The conception relates to initial learning experiences (Fischbein, 1987; Wilson, 2002).
    1.3. The conception persists in the face of opportunities to change these conceptions (Fischbein; Vosnaidou; Wilson).
    1.4. The conception builds upon on a prototypical example, rather than definitions (Sierpinska, 2000).
    1.5. The conception is an overgeneralization of an example or concept (Ben-Zeev & Star, 2001; Fischbein).
    1.6. The conception derives from a surface feature of a problem or representation, rather than the problem structure (Fischbein; Harel, 1999; Stavy & Tirsoh).
2. Indicators that can appear when a student holds incorrect or correct conceptions.
    2.1. The quality of students' written examples.
    2.2. The metaphors and analogies students use to construct their understanding.
    2.3. The relative ease with which students learn different concepts.

My preliminary analysis does suggest that students with weaker understandings do have more indicators of intuition that interfere with understanding, such persistent incorrect conceptions and overgeneralization.

I developed a rubric for analyzing students' problem solving thinking. The inspiration for this rubric was The National Resource Council's (NRC) (2001) framework of mathematical proficiency, which includes five strands: strategic competence, procedural fluency, conceptual understanding, adaptive reasoning, and productive disposition. The four elements of my problem solving rubric are: strategic competence, procedural fluency, work interpretation, and conceptual understanding. Strategic competence is an indicator of the appropriateness of a student's choice of strategy or heuristic to solve a problem. Procedural fluency is the student's skill in executing a selected procedure. Work interpretation indicates how well the student interpreted the results of any procedural work with respect to the problem context. Conceptual understanding indicates the degree of thinking grounded in concepts rather than procedures. I am further deconstructing problem solving thinking by analyzing patterns of problem solving that occur for different problem types. Problem types vary from being primarily procedural to being significantly conceptual. Currently, I do not have any preliminary results for problem solving thinking.

Language use may be associated with students' understanding. Data from a pilot study I did suggested this possibility and Sierpinska (2000) found students who engaged in practical thinking used symbols and language inaccurately, unclearly, or nonsensically. My current data also has indications of this. Below is a sample of the variation of student explanations that often occurred in response to a problem. While the three students answered correctly that the statement was true, their explanations were very different.

Problem: True or False – If z is in Span{x,y} then {x,y,z} is a linearly dependent set.

Student 1: True. x,y would be a linear combination of z.

Student 2: True. If z is in the span{x,y} then z fits into the span. It is telling us that z can be dependent on x,y.

Student 3: True. Saying that z is in the span{x,y} can be translated to z is a linear combination of the vectors x and y. A set is linearly dependent if one vector can be expressed as a linear combination of the other vectors. Therefore, the set {x,y,z} is linearly dependent.

To date, I have been unable to find in the literature a framework or method for analyzing students' language use that is appropriate for my data and am currently in the process of developing an appropriate analysis strategy. I believe one part of this strategy will include analyzing students' use of vocabulary. Another aspect that may be important, but I am having difficulty in defining, is the sentence structure in students' writing.

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