Exploring the Learning of Mathematical Proof by Undergraduate Mathematics Majors through Discourse Analysis

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Abstract This study addressed the inherent and age-old quandary of learning mathematical proof. The aim of the study was to explore the nature of the learning of mathematical proof by undergraduate mathematics majors through the lens of discourse. Additionally, the study investigated mathematics majors' sense of a learning community in relation to their participation in a seminar on learning mathematical proof utilizing small-group discourse. A communicational approach to cognition—or commognition—provided the theoretical and research perspective for the study.

Background

Student difficulty with mathematical proof has been widely documented. Procedural, conceptual, and communicative issues reoccur in the literature as particularly problematic areas. Baker and Campbell (2004) observed, for example, that their students struggled with understanding the process of proof construction, the precision involved in writing in mathematics, and the application of rules of logic in proof construction. Moore (1994) found that conceptual understanding, mathematical language and notation, and getting started on proof were the three major sources of mathematics and mathematics education students' difficulties. Weber (2001) hypothesized types of strategic knowledge that undergraduates lacked in abstract algebra proofs, including knowledge of the domain's proof techniques, knowledge of which theorems are important and when they will be useful, and knowledge of when and when not to use syntactic strategies.

Understanding the processes involved in mathematical productivity is informative in a discussion on the the teaching and learning of mathematics. To gain insight into how mathematicians create mathematics, Sriraman (2004) interviewed five successful mathematicians from large Ph.D. granting universities. Analytic induction of the interview data revealed that, for these mathematicians, engaging in social interaction, imagery, heuristics and intuition very often preceded proof construction. Sriraman noted that the mathematicians' approach to proof is very different from the logical approach presented by most textbooks. In a study on proof validation, Weber (2008) found that mathematicians use a variety of strategies. These include formal reasoning, the construction of rigorous proofs, informal deductive reasoning and example-based reasoning. Additionally, the mathematicians' conceptual knowledge, the mathematical domain of the proof, and the status of the proof's author were important factors in validation.

Sfard's (2002, 2008) communicational approach to cognition—or commognition—provided the theoretical and research perspective for the study. Briefly, the approach is rooted in sociocultural psychology, which views learning as becoming a *participant* in distinct activities. Its focus is on the interactional and contextual aspects of learning, as opposed to the cross-cultural invariants of interest to cognitivists. Sfard argues that the cognitivist acquisition model of learning has proven inadequate in explaining tough issues in mathematics education, not the least of which is the persistent failure of some students to learn mathematics. On the other hand, the participationist framework provides a hopeful outlook on learning. Instead of focusing on intangibles, such as students' mental schemas, we can focus on something that is alterable—the social context. Commognition, then, is a term that stresses the fact that thinking (individual cognition) and interpersonal communication are different manifestations of the same phenomenon. The learning of mathematical proof continues to confound students of mathematics. To this point, however, research on learning mathematical proof has tended to focus on a daunting list of what students lack—procedural knowledge, strategic knowledge, conceptual understanding, a command of mathematical knowledge, a desire to understand why mathematical statements are true, and the list goes on. The primary significance of this study was its approach to examining the learning of mathematical proof through the lens of mathematical discourse. The careful analysis of the contextual data has the potential to begin to inform ideas about fostering effective mathematical communication and, in tandem, mathematical learning in relation to mathematical proof.

Methodology

The setting of the study was a zero-credit seminar focusing on mathematical proof for freshman and sophomore mathematics majors. The seminar was largely informal, with students working from the text *How to Read and Do Proofs* (Solow, 2005) in small groups to improve their knowledge of proof. The primarily qualitative study had nine participants. A multiple methods strategy of data collection was employed. First, audio recordings of small-group discourse on mathematical proof were collected along with participants' related work. Participants additionally completed the *Classroom Community Scale* survey (Rovai, 2002). Finally, interviews were conducted.

Focal and preoccupational analyses were performed on the audio data to determine the object-level and meta-level features of the mathematical discourse/learning. The focal analysis and preoccupational analysis tools were recently developed by Sfard and Kieran in their work on cognition as communication (Kieran, 2002; Sfard, 2002; Sfard, 2008; Sfard & Kieran, 2001). Taken together, these tools enable an investigator to examine the effectiveness of communication (the unit of analysis), a precondition of a learning interaction. Focal analysis is a means for distilling the object-level "features of discourse that can count as indicators of its effectiveness or the lack thereof" (Sfard & Kiearn, 2001, p. 50). In focal analysis, a general comparison is made between what is said and what is done. This gives a detailed look at the mathematical content of the conversation. Preoccupational analysis is concerned with participants' engagement in conversation. It "deals with the question of how the participants of a conversation move between different channels of communication (private and interpersonal) and different levels (object-level and meta-level)" (Sfard & Kieran, 2001, p. 57). In this analysis, utterances are classified as reactive or proactive. That is, of interest is whether the interlocutor reacts to a prior comment or makes a response-inviting comment. Utterances can be further classified as personal or interpersonal. Descriptive statistics and typological analysis were used respectively to summarize the survey and interview data.

Findings

Small-group discourse appears to be a comfortable way for novice learners to approach a more expert discourse on proof. The study's participants had a sense of community in the seminar on mathematical proof utilizing small-group discourse. The discourse may also have contributed to the connectedness that they felt with their fellow math majors both inside and beyond the seminar walls. Moreover, the participants viewed being able to communicate about mathematical proof as the conduit to a universal math community.

There was evidence that while the paired-discourse of peers may serve to help learners organize and clarify thinking on familiar content, collaboration itself may do little in the way of helping them to penetrate a more expert discourse. However, the analysis of discourse in this study revealed that within small-group discourse on mathematical proof, there may arise *natural* opportunities to steer the discourse in the direction of increasing sophistication—what I have called discursive entry points. This finding potentially gets at the heart of what has classically been called the "learning paradox." Paired discourse between peers may provide learners an "un-artificial" way in which to "bump into," so to speak, those things that are unlearnable because they must be known before the process of learning can begin. It is hypothesized that, with the skilled intervention of an expert in the discourse, these entry points might be "ripe for ripening." A brief discussion of three types of discursive entry points is given next.

Discursive Entry Point 1: When the Interlocutor(s) is Aware that the Approach to Proof has Failed

In one sample of small-group discourse, Jacob and Sherri are attempting to prove that given a right isosceles triangle, its area equals one-fourth the square of the hypotenuse. From the beginning of the transcript Sheri seems anxious to "jump right in" to working with formulas. Despite some initial prompting from Jacob, the two devote very few utterances to planning the proof in the form of developing a key question and abstract answer (techniques from Solow, 2005). Knowing that the right triangle she is working with is isosceles, Sherri substitutes *y* for *x* in the Pythagorean Theorem ($y^2 + y^2 = z^2$) and then solves for y^2. Substituting the resulting expression, $z^2/2$, back in to the Pythagorean Theorem yields $z^2 = z^2$ and much laughter. After admitting to "going in a big circle" Sherri immediately takes up "playing" with a new formula, the area formula. Sherri gets quite a chuckle when she comes across the statement of reflexivity ($z^2 = z^2$) in her work. "Apparently that wasn't right", she says. There is no indication from her utterances, however, that she takes the time to determine *why* she got the result she did.

Sherri was looking for the *right* answer and it was apparent to her that she did not obtain it. Thus she tried something else. Revisiting the key question and its answer at this natural break in the discourse may have been particularly powerful, however, for moving Sherri toward a more expert discourse on two levels. First, it provided an opportunity for Sherri to begin to value adopting a more strategic approach in her work. Second, it served as an opportunity to move Sherri from a discourse on proof more or less discrete in nature (a routine or order of steps to go through) to one more holistic (proof as an endorsed narrative or objective truth). It is hypothesized that when the interlocutors are aware that the approach to proof has failed, intervention might include asking them to explain *why* the approach failed in relation to that which they are trying to demonstrate. Moreover, if a trial-and-error approach is used, it may be valuable to ask interlocutors to compare the success of various approaches in relation to the proof process.

Discursive Entry Point 2: When the Interlocutor(s) Raises Questions Specifically Related to Managing the Process of their Proof

In a different sample of small-group discourse, Lisa and Sara are attempting to prove that if n is an odd integer then n^2 is an odd integer. The conversation is characterized by interpersonal utterances by both interlocutors. How do the utterances of each shape its direction and success? Focal analysis reveals that the women's contributions are very different in their nature. Sara appears to take the lead, so to speak, from early on and throughout much of the conversation. She offers the key question, abstract answer, and the hypothesis and conclusion. She also is the first to articulate the logical arguments for the body of the proof and their justifications. Indeed, Sara might be described as managing the mathematical *content* of this discussion. Lisa's contributions, on the other hand, are mainly in the form of affirmations and questions. Some of Lisa's questions serve the purpose of clarification. Many other questions relate directly to the necessary *approach* for successful completion of the proof (which the women did not achieve in the allotted time.)

The preoccupational analysis reveals that Lisa questions the pair's approach/process nine different times within the discourse, but that these opportunities to move the proof forward may not have been fully exploited. Largely, Lisa's questions were left unattended by the pair. Moreover, it was not uncommon for Lisa to pose a question and dismiss it as unimportant all in one breath. Some of Lisa's questions included:

- Are we working from the bottom or the top first? I don't think it matters so which one?
- Should we prove n is even first and then if you add a one it's odd? But that would be backtracking that really doesn't make any sense.
- Maybe we should have did this from down here. You know what I mean, we should of said this is from the bottom up, yea, I think we should have.

When an interlocutors raise questions specifically related to managing the process of the proof, the questions likely arose for good reason. As such, each question serves as a discursive entry point—an opportunity for expert intervention. Experts might encourage the interlocutors to consider, rather than dismiss, their naturally arising questions and articulate specific mathematical answers to them.

Discursive Entry Point 3: When the Interlocutor(s) is in a Cyclical Ad-hoc Routine Course of Action

Communication is a patterned activity. We are able to participate in various forms of communication because we are familiar with the routines associated with them. Take the basic communication of greeting. The act of greeting follows a routine consisting of the elements of *when* and *how*. We must understand the context (when it is appropriate to greet someone or respond to another's greeting) and we must also understand how to do it (what words to use or gestures to make). In our everyday lives, greeting is a more or less automated routine. When a situation is less than familiar, however, one is often left searching for a routine. Sfard (2008) states:

> In situations that do not automatically evoke standard routines, an ad hoc pattern would often settle in from the very first exchange. This is particularly true of educational settings. There is a salient rhythm to interactions involving newcomers to a discourse, trying to become its full-fledged participants. (p. 197)

Although ad hoc, the patterns of newcomers "are made possible by certain standard discursive patterns, already known" to them (p. 199).

In the same sample as just discussed (discourse on proving that if *n* is an odd integer then n^2 is an odd integer), Sara and Lisa rely on an algebraic routine, with which they are no doubt familiar, to keep their discourse alive. The cycle repeats itself three times throughout the discourse. They know both when and how to square [that is they let n=2k+1 and find $n^2 = 4k^2 + 4k + 1$] and factor the resulting algebraic expressions [writing $4k^2 = 2(2k^2)(even)$ and$4k = 2(2k)(even)$]. They also recognize in their resulting algebraic expressions the form of an odd number [writing on their papers $4k^2 + 4k = even$ and *even +1 (odd)=odd*]. But it is here that the familiar algebraic routine no longer serves them in advancing the proof. They get stuck each time they go through the algebraic routine because they are not well-rehearsed in the next routine. They realize neither that it is a good time (when), nor how to rename $2k^2 + 2k$ as a new integer. Yet while the structure of their routine remains fundamentally the same, each cycle brings about modifications. In the second cycle, for example, the women support each of their algebraic maneuvers with justification. In the third cycle, they consider for the first time how working from a different direction might be advantageous. We see then, a change in what the women are saying and doing. Sfard (2008) notes that "this kind of change is typical of interactions involving participants who are in the process of individualizing a discourse" (p. 199). In this example, it is almost as if when the old and familiar routine comes up short, the learners tweak it and try it again. To learn is to become ever more fluent in a discourse. We see here a handful of successive approximations of the perhaps thousands needed to become fluent.

Ad-hoc routine courses of action adopted by novice discursants have the potential, generally, to lead in a mathematically productive direction. However, the incremental and slow nature of each revision may not lend itself to the very real and practical constraints of classroom time. As such, the occurrence of a cyclical routine course of action can be considered as a natural opening in the conversation for expert intervention—a discursive entry point. It is hypothesized here that when learners fall into a cyclical ad-hoc routine course of action, teachers might determine how to assist the interlocutors in modifying those routines in mathematically productive ways.

Discussion

Sfard (2008) argues: "scaffolded individualization is the only way for a 'newcomer' to enter a discourse governed by rules different from those that regulated her communicational activity so far. Individualization, by definition, requires proactive participation—and help—of this discourse's 'oldtimers'" (p. 282). College teachers of mathematics will need practical models for facilitating small-group discourse in the classroom if it is to be used effectively in introducing students to mathematical proof and moving them toward a more expert discourse. In naming discursive entry points and identifying six factors that affect small-group discourse on mathematical proof, this study has laid the groundwork for further research and development of such models.

A main finding of this study was to reveal the potential for expert intervention in naturally occurring openings in beginner discourse, called discursive entry points. The conception of discursive entry points seems especially promising in the current era of mathematics education reform, which calls for greater student engagement/conversation in the classroom. Greater student engagement/conversation in the classroom, however, requires that the educator strike a very fine balance. The educator must work to foster a learning environment that allows for the entire range of mathematical discovery (from the trials and tribulations of working on a proof to the elation of seeing it to completion). He or she must also adeptly manage the problem of the learning paradox. Thus, the task of the future discursive researcher will be two-fold. First, researchers will need to identify and classify points in student discourse that are pregnant with potential. Where in students' small-group discourse will expert intervention bear the greatest fruit? What opportunities present experts the greatest chance of guiding learners to increased proficiency in the discourse of mathematical proof? Second, discursive researchers will no doubt be interested in the question of *how*. How do experts intervene when presented with discursive entry points in small-group discourse and how do learners respond? The findings of future exploration into naturally occurring discursive entry points in student discourse could potentially transform the teaching and learning of mathematical proof.

Solow (2005) opens his introductory proof text with a "preface to the instructor." Here, he captures the all too frequent general milieu of the teaching and learning of mathematical proof.

> The inability to communicate proofs in an understandable manner has plagued students and teachers in all branches of mathematics. The result has been frustrated students, frustrated teachers, and, oftentimes, a watered-down course to enable the students to follow at least some of the material, or a test that protects students from the consequences of this deficiency in their mathematical understanding. (p. xiii)

This state of affairs is at best unfortunate and at worst tragic. Mathematical proof, after all, is the backbone of all of mathematics. Moreover, what the above-described scenario does not portray is a learning situation in which students taste the "experience and joy of mathematical discovery" (Benson, 1999). Benson argues that the joy of discovery should be just as intense for the learner of proof as it was for the first mathematician who made the discovery.

The present study, I believe, presents a complex, but nonetheless hopeful, outlook on the teaching and learning of mathematical proof. Classroom methods that are dialogic in nature, such as small-group discourse, may hold the promise of promoting mathematical experiences for learners that are more authentic in nature—where the joy of mathematical discovery, if even on a small scale, is not uncommon. At the same time that dialogic methods of teaching mathematics gain increasing traction, we have in Sfard's (2008) *commognition* a complementary, not to mention revolutionary, foundation for researching mathematical thinking. This study has shown that the high-resolution analysis of student discourse reveals far more about student thinking on mathematical proof than could ever be found by looking, for example, at student work alone. This study represents the tip of the iceberg when it comes to using discourse analysis to better our understanding of how college level students learn mathematical proof and consequently how to teach mathematical proof.

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