Student Justifications for Relationships between Concepts in Linear Algebra: Analyzing Student Reasoning Using Adjacency Matrices

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Abstract

A central goal of most any linear algebra course is to ensure that students develop relational understanding of concepts, become proficient at various techniques, and develop personal and/or informal justifications for relationships between concepts. This report addresses linear algebra students' personal justifications for relationships through the analysis of videorecorded, end of the semester problem solving interviews. In particular, the interview question analyzed in this report prompted students to consider, given an invertible matrix A, whether 5 different claims are true or false. These claims are formally part of what many texts refer to as the Invertible Matrix Theorem (e.g., Lay, 2003). This report will address student responses by highlighting the informal or personal justifications students provided for relating invertibility, linear independence, determinant and span in R^3. *Furthermore, we develop an innovative method using adjacency matrices to analyze students' personal understandings of concepts and the connections students make between concepts.*

Purpose and Background

A central goal of most any Linear Algebra course is to ensure that students develop relational understanding of concepts (Skemp, 1987), become proficient at various techniques, and develop personal and/or formal justifications for relationships between concepts. For mathematicians, the rich relationships between concepts make linear algebra an especially beautiful and elegant discipline. For students, however, these rich relationships often result in significant difficulties (Dorier, 2000). Some of these interwoven concepts that students have difficulties with, central to any first course in linear algebra, include but are not limited to linear independence, determinant, span, and invertibility. Many of these concepts and their relationships are formalized in one of the keystone pieces of theory first introduced in an introductory linear algebra course, namely the Invertible Matrix Theorem (e.g., Lay, 2005). This report aims to sift out the understandings students have of these concepts and modes in which students relate these concepts to justify for themselves the validity of a claim.

A secondary goal that arose in the course of this analysis is how to explore the connections students make between concepts and their personal understandings of concepts. One such tool often used and explored is concept map (Williams, 1998). Though concept maps provide a powerful way to chart the connections that students make, this tool becomes unwieldy when a large number of concepts are invoked and connected. Thus, instead of using conceopt maps, we develop an innovative approach using adjacency matrices from graph theory. The methodology we develop resonates with the work of Schvaneveldt (1998), who studied understanding of proximity data (Schvaneveldt, 1989), and with the work of Harary, Norman and Cartwright (1965), who examined visualization of propositional logic through implication digraphs. Adjacency matrices have also been used as a framework to capture the emergence and intercoordination between conceptual and procedural knowledge in the ebb and flow of a 4th grade classroom conversation (Strom, Kemeny, Lehrer & Forman, 2001). However, this methodology has yet to be used for understanding students' individual, personal reasoning and understandings, particularly at the undergraduate level.

Methods

Data Collection Methodology

Data for this analysis comes from a semester long classroom teaching experiment in a linear algebra course at a large southwestern university. As part of this project, nine students participated in individual, semi-structured interviews (Bernard, 1988). Each interview was approximately 90 minutes long and consisted of several questions. The data sources for this analysis consist of video recordings of the interviews, as well as copies of all written work. Additionally, each interview was transcribed completely. The particular question from the interview that was analyzed for this report was the following:

> Suppose you have a 3x3 matrix A, and you know that A is invertible. Decide if each of the following statements is true or false, and explain your answer.
> a) The column vectors of A are linearly independent.
> b) The determinant of A is equal to zero.
> c) The column vectors of A span R^3.
> d) The null space of A contains only the zero vector.
> e) The row-reduced echelon form of A has three pivots.

The concepts involved in this question and their relationships had been formally summarized in the students' text as the Invertible Matrix Theorem (Lay, 2003). Furthermore, all claims are true except claim (b). In the interview, students were given part a) first, then part b), etc. This allowed us to focus more clearly on students' understanding of each connection.

Throughout the course of the interview, students were encouraged to think through their reasoning and answer aloud. Follow-up questions were frequently asked, aimed to unpack student's understandings of each concept individually and how the connections build from these understandings. For example, following claim a), if a student did not readily provide their understandings aloud, the interviewer asked the following:

- What does it mean for a set of vectors to be linearly independent/dependent?
- Do you have a geometric way of thinking about linear independence/dependence?
- What does it mean for *A* to be invertible?
- How does that relate to what you previously said about linear independence/dependence?

The analysis reported here addresses the first three claims, a) – c), and the understandings each student expressed and the connections each student made regarding invertibility, linear independence, determinants, and span of R^3.

Analysis Methodology

Initially we thought of analyzing student understandings of each concept and connections between concepts using concept maps (Williams, 1998). However, as the number of different understandings and personal interpretations students expressed for invertibility, linear independence, determinants and span increased, concept maps quicky became an unwieldy tool. As a result, concept maps were not explicitly used but led to the use of similar tools and ideas in graph theory, specifically, the use of adjacency matrices.

To illustrate how we used adjaceny matrices, consider Figure 1, which represents a student's understanding of the relationships between concepts A – D. In the course of a student's interview, such a map would represent relating concept A to concept B on two separate occasions, using concept B to imply concept C, and connecting concept C to concept A.

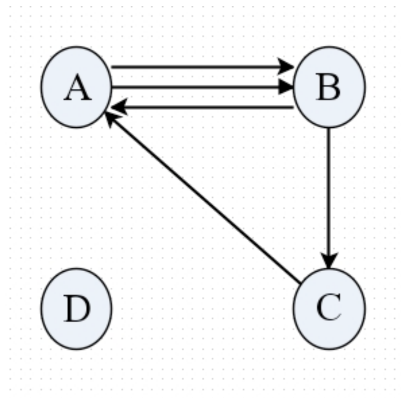


Figure 1. Sample Digraph

Figure 1 is an example of a digraph, which is defined in the following way. A *graph* consists of a finite set of points called vertices and a finite set of lines connecting these vertices called edges. A graph where direction is indicated for every edge is called a *directed graph* or *digraph* (Harary et al., 1965; Ore, 1990). Thus, in Figure 1, the vertices of the digraph indicate the concepts and the directed edges indicate connections.

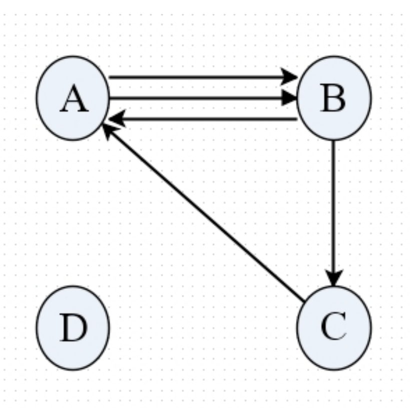
In pure mathematics, graphs are not always studied through their visual representation; instead, mathematicians can study a less visual, more systematic representation of a graph called the adjacency matrix. For a given digraph its *adjacency matrix* is defined as a square matrix with one row and one column for each vertex, where the entry $a_{ij}=k$ indicates k edges connect vertex v_i to vertex v_j, and entry $a_{ij}=0$ indicates there exists no edge connecting vertex v_i to vertex v_j (Chartrand & Lesniak, 2005; Harary et al., 1965). Note that in many contexts, multiple edges from vertex v_i to vertex v_j are prohibited. Since we made no such restriction, the entries of the adjacency matrix can be any nonnegative integer. Below is the adjacency matrix M for the digraph depicted in Figure 1.

	A	B	C	D
A	0	2	0	0
B	1	0	1	0
C	1	0	0	0
D	0	0	0	0

Figure 2. Adjacency Matrix *M*

The entry "2" in row A and column B indicates there are two edges, or connections, beginning with concept A going to concept B. Similarly the entry "1" in row B and column A denotes one edge beginning with concept B and connecting it to concept A. Observe that in this sample matrix concept A is connected in two different ways to concept B and concept B is connected by one edge to concept C. Furthermore, there is no directed edge beginning with concept A leading to concept C. This is not to say concept A could never be connected to concept C; instead, Figure 1 shows the only way to get from A to C is by going through B first. This idea is also defined in graph theory. A *walk* is a sequence of connected vertices, and the *length of a walk* is the number of edges in that walk. Adjacency matrices can be uses to clearly identify walks of lengths greater than 1 by examining powers of the adjacency matrix.

If M is the adjacency matrix of a graph with vertices v_i and v_j, then the number of walks of length k is denoted by the entry a_{ij} of the matrix M^k (Chartrant & Lesniak, 2005; Harary et al., 1965). To illustrate this statement, consider again Figure 1 and the matrix M^2 in Figure 3.



	A	B	C	D
A	2	0	2	0
B	1	2	0	0
C	0	2	0	0
D	0	0	0	0

Figure 3. Matrix M^2

Notice that in row A and column C there is an entry "2", indicating there are two walks of length 2 through which concept A is connected to concept C.

One possible deficiency in using adjacency matrices to depict student reasoning is the loss of order of student's connections between concepts and understandings. To account for the loss of order in which each student made these connections, transcripts were coded following each connection described by the student with the letters denoting originating and terminating concepts. An explicit example of this will be given in the context of our results.

Results

Within the course of the interviews, students expressed a multitude of understandings for the concepts involved in the Invertible Matrix Theorem. Figure 4 contains a list of these concepts (coded as items A-I) and the subsequent understandings expressed by the students (coded numerically underneath the main concepts). These understandings were those given by the students and not a list of concepts generated by the course instructor or interviewer intended for the students to discover.

A. A is invertible
    1. *Calculator inverse* – Can use calculator to produce inverse
    2. *Row-reduce to Identity* – Augments the matrix with the identity; can row-reduce to find inverse
    3. *Formula: Inverse Matrix: no "divide by 0 errors"*– Can calculate inverse with a formula, using determinant; uses the 2x2 inverse matrix formula
    4. *Input-output: "don't lose information"* – Given output, can use inverse matrix to find input
    5. *Can reproduce A: "reproduce matrix A"* – Misconception, possibly confusing invertibility with another activity completed in class

B. A is noninvertible/singular
    1. *Calculator error* – Can use calculator to see if there is inverse
    2. *Cannot row-reduce to Identity* – Augments matrix with identity; cannot row-reduce to find inverse
    3. *Formula*: *Inverse Matrix: "divide by 0 errors"* – Cannot calculate inverse with formula using determinant; uses the 2x2 inverse matrix formula
    4. *Input-output: "lose information"* – Given output, cannot find original input
    5. *Cannot reproduce A: "cannot reproduce matrix A"* – Based on activity completed in class

C. Column vectors of A are linearly independent
    1. *No free variables* – No variable undefined, usually in the correlating system of linear equations
    2. *Unique solution to matrix equation/system of linear equation*
    3. *Geometric – No "Losing a degree of freedom";* magic carpet ride problem
    4. *Geometric* – Vectors are *not collinear or coplanar*
    5. *Proportional-algebraic* – No vector is a *scalar multiple* of another vector
    6. *Linear combination - algebraic* – No vector is a *linear combination* of another vector

D. Column vectors of A are linearly dependent
    1. *Free variables* – Some variable undefined, usually in the correlating system of linear equations
    2. *No unique solution to matrix equation/system of linear equation*
    3. *Geometric – "Lose a degree of freedom";* magic carpet ride problem
    4. *Geometric* – Vectors are *collinear/coplanar*
    5. *Proportional-algebraic* – One vector is *a scalar multiple* to another vector
    6. *Proportional-algebraic* –One vector is a *linear combination* of other vectors

E. $\det(A) \neq 0$: Determinant of A is nonzero
    1. *Procedure used to calculate $\det(A)\neq 0$:* Can use Rule of Sarrus/diagonals or Leibniz/cofactor formula to calculate nonzero determinant of *A*
    2. *Geometric – 2-d parallelogram with nonzero volume*
    3. *Geometric – 3-d parallelpiped with no/zero volume*

F. $\det(A) = 0$: Determinant of A is equal to 0
    1. *Procedure used to calculate $\det(A)=0$:* Can use Rule of Sarrus/diagonals or Leibniz/cofactor formula to calculate determinant of *A* is zero
    2. *Geometric – 2-d parallelogram with no/zero volume*
    3. *Geometric – 3-d parallelpiped with no/zero volume*

G. Column vectors of A span R^3
    1. *3 x 3 sized matrix spans R^3* – Size of matrix dictates whether matrix spans; n vectors for n dimensions
    2. *Procedural – RREF of matrix is identity matrix*
    3. *Geometric– can get to every point in the dimension*
    4. *Algebraic – Linear combination for all points in R^3*

H. Column vectors of A do NOT span R^3
    1. *Non 3 x 3 sized matrix cannot span R^3* – Size of matrix dictates it cannot span; do not have n vectors for m dimensions
    2. *Procedural – RREF of matrix is not the identity matrix*
    3. *Geometric – Cannot get to every point in the dimension*
    4. *Algebraic – No linear combinations for some points in R^3*

I. Other

Figure 4. List of Concepts and Understandings Invoked by Students

In the course of addressing each part of the interview question, the students expressed their personal understandings of the concepts in question and used these understandings to connect the concepts and directly relate one concept to another. For example, a student might explain they see the determinant of a 3 x 3 matrix as the volume of the parallelpiped formed in space or linear independence as three vectors that are neither collinear nor coplanar. The student might relate linear independence to a set of three vectors not being collinear or coplanar. So the vectors would form a parallelpiped with nonzero volume and the determinant with these three column vectors would be nonzero. Thus, these understandings became the means by which students connected the concepts and justified their true/false answer to the claims in the prompts.

Josiah

We begin with an excerpt from one student, Josiah, regarding the invertibility of the 3 x 3 matrix *A* and the spans of the column vectors of *A*. As subsequently explained in more detail, each connection between concepts is coded according to the lettering and numbering of concepts in Figure 4.

> If it's [the matrix *A* is] invertible, those 3 vectors [the column vectors of *A*] should be able to reach any point. AG3 Because you should be able to reach that point with a combination of these vectors and scalars. G3G4 Which is exactly what you're doing with column space. G4G And then those combinations should be able to create a 3-D object. G4E3 Which is what you're talking about with the determinant, E3E it's, in my mind at least, 3 dimensions. And the column vectors being linearly independent is a prerequisite in order to be able to create 3-D objects. CE3 So in a way, their linear independence, it's the same thing as they're spanning R^3. CG And the fact that they span R^3 is what allows the determinant to be a nonzero number. GE

In the previous quotation, Josiah relates various understandings of invertibility, linear independence, determinant and span, often using personal meanings of each to connect these main concepts. The adjacency matrix *M*, listing just those concepts addressed by Josiah in this excerpt, is shown in Figure 5. In order to make the matrix more easily read, all 0 entries in the matrix (or lack of a connection) are omitted, instead denoted by a blank entry.

	A	C	E	E3	G	G3	G4
A						1	
C				1	1		
E							
E3			1				
G			1				
G3							1
G4				1	1		

A: Matrix *A* is Invertible
C: Column vectors of *A* are linearly independent
E: Det(A)$\neq$0
E3: Vectors form 3-D parallelpiped with nonzero volume
G: Column vectors of *A* span R^3
G3: Can get to any point in R^3
G4: Can make a linear combination for any point in R^3

Figure 5. Adjacency Matrix *M* for Josiah's Excerpt

For each coded connection in the excerpt, there is a number filled into the row corresponding to the first concept and in the column corresponding to the second concept within the connection. Though the students frequently did not explicitly state their understandings as implications, most responses are generalized as either an implication (as in Josiah's connection CE3) or as one main concept and an understanding of that concept (G4G). In either instance, an entry in the adjacency matrix is intended to chart Josiah's connection from one concept or understanding to another, not necessarily record explicitly stated logical implications in his interview. Within Josiah's matrix we see examples of meanings and images given to objects, denoted by entries within the block diagonal submatrices, and many understandings that cross between main concepts, denoted by entries not within the block diagonal submatrices.

Furthermore, the latter connections rely mostly on Josiah's understandings of each main concept, for example, determinants as being the volume of the parallelpiped formed by the column vectors of *A*. These are the entries not in the upper-left corner of the block submatrices. These patterns indicate Josiah has not only a firm understanding of how the main concepts relate or personal understandings of main concepts like determinants and span, but also can use these personal understandings to relate the concepts and provide informal justifications to his claims.

The sophistication of Josiah's connections is further understood by looking at the powers of the adjacency matrix *M*. Figure 6 depicts M^3, which can be read to determine the number of walks of length 3 between the listed concepts.

	A	C	E	E3	G	G3	G4
A				1	1		
C							
E							
E3							
G							
G3			1				
G4							

A: Matrix *A* is Invertible
C: Column vectors of *A* are linearly independent
E: Det(*A*)≠0
E3: Vectors form 3-D parallelpiped with nonzero volume
G: Column vectors of *A* span R^3
G3: Can get to any point in R^3
G4: Can make a linear combination for any point in R^3

Figure 6. Matrix M^3 for Josiah's Excerpt

Among the 3 walks of length 3 indicated by M^3 is the connection between the invertibility of matrix *A* and the column vectors of *A* spanning all of R^3. This connection is clearly seen in the argument given by Josiah in the first half of the excerpt, but was previously lost in adjacency matrix *M*. By taking natural number powers of *M* we can see this more sophisticated arguments that connect concepts and concept understandings, as well as predict potential connections students could make. The matrix M^3 suggests Josiah can connect concept A, the invertibility of matrix *A*, with understanding E3, that there is a parallelpiped of nonzero volume formed by the column vectors of A. This suggests Josiah can articulate such a connection in three steps with the understandings he has already stated, even if he has not explicitly done so.

Similar patterns indicating a sophisticated understanding of concepts and connections between these understandings are seen in the adjacency matrix J in Figure 7, which denotes all connections made by Josiah in Question 1 claim a) through c) from the interview problem.

	A	A1	A2	A3	A4	A5	B	B1	B2	B3	B4	B5	C	C1	C2	C3	C4	C5	C6	D	D1	D2	D3	D4	D5	D6	E	E1	E2	E3	F	F1	F2	F3	G	G1	G2	G3	G4	H	H1	H2	H3	H4	I
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Figure 7. Adjacency Matrix J for Josiah's Interview

There are several aspects of Josiah's adjacency matrix *J* that indicate a sophisticated understanding of concepts and a flexible use of understandings to build connections. For example, the majority of the entries lie off of the block diagonal. These entries indicate a large portion of Josiah's explanations connected understandings of different concepts, such as a geometric view of linear independence (C4) with a geometric interpretation of determinants (E3). Main concepts are frequently related through these personal understandings, as seen in the infrequency of entries in the upper –left corner of each block submatrix, entries which often indicate either summaries of claims or justification by external authority. Instead, Josiah only made such connections, such at the three in row C and column G, when he had already justified such a connection through other understandings (see the previous excerpt). Lastly, note that all the connections made by Josiah lie in the unshaded blocks of the matrix. Entries within the white blocks indicate correct connections or implications within Josiah's reasoning. Thus, through *J* we can quickly summarize Josiah's understanding as accurate, full of sophisticated personal understandings of concepts that can be readily used to justifying connections.

Bethany

For contrast, we will now look at Figure 8 for the adjacency matrix *B* resulting from the interview with another student, Bethany.

	A	A1	A2	A3	A4	A5	B	B1	B2	B3	B4	B5	C	C1	C2	C3	C4	C5	C6	D	D1	D2	D3	D4	D5	D6	E	E1	E2	E3	F	F1	F2	F3	G	G1	G2	G3	G4	H	H1	H2	H3	H4	I
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Figure 8. Adjacency Matrix *B* for Bethany's Interview

Bethany's adjacency matrix suggests a more superficial understanding of the concepts and connections made to concepts in the Invertible Matrix Theorem, in comparison to Josiah. All but two of Bethany's entries lie within the block diagonal. These entries suggest that the majority of Bethany's understandings are used only to understand the main concepts of invertibility, linear independence, determinant and span. She does not use these understandings to build connections between these concepts, as would be indicated in any entries off the block diagonal.

There are four instances in which Bethany connects concepts, the first two of which are seen in row A and column C, as Bethany relates invertibility of A to the column vectors of A being linearly independent. In justifying the first of these claims, Bethany states, "I know there is the Invertible Matrix Theorem, and I just can't remember whether it talked about independence or dependence; but I'm pretty sure it was independent." When asked if she can explain this connection beyond referencing the Invertible Matrix Theorem, Bethany said no "because I don't understand the definitions of linearly dependent and independent enough to really know." Thus, Bethany's connections depend not on her personal understandings but on external authority. Frequently such justifications were indicated by entries in the upper-let corner of a block submatrix without other entries in the block or corresponding row and column. It is also noteworthy that Bethany has two entries in the shaded regions, indicating incorrect connections and further evidence that Bethany's connections are more superficial than those justified through understandings of the main concepts.

Conclusions

A major contribution of this work centers on the innovative use of adjacency matrices as a systematic methodological tool for detailing students' understandings and connections. The cases of Josiah and Bethany provide concrete examples for the usefulness of this approach to compare strengths and weaknesses of the connections between ideas that students do and do not make. In comparison to concept maps, the methodological approach that we developed allows for a more systematic and nuanced analysis into student understanding.

As evidenced in the data we presented, students develop a large number of personal understandings of concepts central to a first course in linear algebra. In the case of Josiah, the methodology of adjacency matrices highlighted the multitudes of understandings he exhibited for each concept, and these personal understandings enabled him to create multiple connections between concepts and accurately respond to the claims and justify his reasoning. In the case of Bethany, adjacency matrices highlighted the less sophisticated understandings she had for each concept. Furthermore, the lack of density of her matrix and the limited number of entries off of the block diagonal indicated the minimal number of connections made between concepts.

An issue for further consideration is the extent to which taking higher powers of the larger matrices would indicate connections each student has made or could make, using their personal understandings of each concept. In the case of using an excerpt from Josiah's interview, higher powers of the matrices indicated connections between concepts that might otherwise be missed. However, in the case of a larger matrix, in is difficult to know which powers should be taken of the adjacency matrix to see this connection and which of these entries of the higher powered adjacency matrices indicate connections the student explicitly made versus connections the student potential could make.

References

Bernard, R. H. (1988). *Research methods in cultural anthropology*. London: Sage.

Chartrantm G., & Lesniak, L. (2005). *Graph & digraphs* (4th ed.). Boca Raton, FL: Chapman & Hall/CRC.

Dorier, J. L. (2000). *On the teaching of linear algebra*. Dordrecht: Kluwer.

Harary, F., Norman, R. Z., & Cartwright, D. (1965) *Structural Models: An introduction to the theory of directed graphs*. New York, NY: Wiley.

Lay, D. C. (2003). *Linear algebra and its applications* (3rd ed.). Reading, MA: Addison-Wesley.

Ore, O. (1990). *Graphs and their uses*. Washington, DC: The Mathematical Association of America.

Schvaneveldt, R. W., Durso, F. T., & Dearholt, D. W. (1989). Network structures in proximity data. In G. Bower (Ed.), *The psychology of learning and motivation: Advances in research and theory*, *24*, pp. 249-284.

Skemp, R. (1987). *The psychology of learning mathematics.* Hillsdale, NJ: Lawrence Erlbaum Associates.

Strom, D., Kemeny, V., Lehrer, R., & Forman, E. (2001). Visualizing the emergent structure of children's mathematical argument. *Cognitive Science, 25*, pp. 733-773.

Williams, C. G. (1998). Using concept maps to access conceptual knowledge of function. *Journal for Research in Mathematics Education, 29* (4), pp. 414-421.