Effective Folding Back via Student Research of the History of Mathematics

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Abstract: The purpose of this qualitative study is to explore the growth of students' understanding when they conduct research of mathematical concepts through the historical development of those concepts. Research of and expository writing about a particular concept affords students the opportunity to reexamine and deepen their current levels of understanding about a mathematical concept. Students' written reflections provide insight into their self-awareness of what they learn from their research and writing efforts and whether the activity resulted in growth of mathematical understanding.

Introduction

Traditionally, many undergraduate mathematics programs include a variety of courses which present mathematics as a fixed and formalized body of knowledge. Instructional practices in these programs emphasize the teaching and learning of facts, algorithms, and theorems. The teaching of mathematical concepts in a formalized manner can result in disjoint understanding among students (Katz, 2007; Martin, 2000; Pirie & Kieren, 1994a; Sfard, 1991). In particular, teaching mathematics as a static body of knowledge based on procedures ignores the complexity of learning to communicate mathematically. Pedagogical approaches that provide students opportunities to deepen their understanding of mathematical concepts are needed if students are to gain a more complete knowledge and appreciation of mathematics. Since learning occurs when students connect new information to what they already know, providing students diverse opportunities to communicate about mathematics creates a potential tool for students to make such connections. Furthermore, these connections are made through self-awareness of their own learning processes while learning to communicate. The overarching goal of this study is to evaluate the process of writing a research paper as a mathematical task which contributes to students' growth of mathematical understanding. Specifically, this preliminary study seeks to explore and characterize the ways student research of the historical development of a particular mathematical concept contributes to deepening a student's level of understanding concerning that concept.

Background

Writing in Mathematics

Students are increasingly required to write as part of their mathematics education experiences, typically with the goal of helping students learn to *communicate* about mathematics and not just *do* mathematic. When students write to communicate mathematically, they must think about mathematics in a new way. This requires a level of self-awareness on the part of the students who must now think about *how* to transform their mathematical knowledge into written communication. The process of writing about mathematics encourages students to organize the mathematics in their own terms, thereby supporting conceptual learning and retention (Keith, 1990). Learning and writing are both considered activities which actively involve the student in making connections between what is being learned and what is already known (Mayher, Lester, & Prandl, 1983).

History of Mathematics

The role of using the history of mathematics in the teaching and learning of mathematics is a major area of research which has made important contributions to the field of mathematics education (Fauvel & Maanen, 2000). For example, the use of the history of mathematics in instruction provides opportunities for students to deepen their mathematical understanding (Bartolini Bussi & Sierpinksa, 2000) and can help students gain a better understanding of the meaning of mathematical concepts, methods, theories, and proofs (Tzanakis & Arcavi, 2000). In most institutions of higher education, mathematics is taught as an abstract science without much attention given to teaching what mathematics is about and how mathematical ideas developed. In order to bridge this gap, the history of mathematics "can serve as a major pedagogical tool for the teaching of higher mathematics" (Swetz, 1995, p. 103). Concerning the use of the history of mathematics as a pedagogical tool, there exists a plethora of classroom resources for teachers (i.e., MAA, 2004; NCTM, 1989; Swetz, Fauvel, Bekken, Johansson, & Katz, 1995). However, there is currently a limited amount of research focused on *how* the historical development of mathematics contributes to students' growth of mathematical understanding.

Theoretical Framework

The Pirie-Kieren Theory for the Dynamical Growth of Mathematical Understanding (Pirie & Kieren, 1994b) describes eight embedded levels of understanding based on the cognitive changes that occur during the processes of learning mathematics. These processes are dynamic, recursive and non-linear. The Pirie-Kieren theory is first and foremost a theory about how a specific student's mathematical understanding grows and develops concerning a particular mathematical concept or topic. *Folding back* is a key feature of the theory and an activity vital to growth in understanding (Martin & Pirie, 1998; Martin, 2000; Pirie & Kieren, 1994b). When a student folds back to an inner, localized level of understanding, not only is the current, more formalized level of understanding extended, but the inner level is also changed because it has been informed by that outer level. This process "thickens" the inner level of understanding and thus, a student gains a broader and deeper understanding of a concept. Furthermore, the process of folding back can allow for the reconstruction of inner levels of understanding (Pirie & Kieren, 1994b).

Folding back is considered effective if it results in growth of understanding, but this requires a level of self-awareness of the learner's own understanding. According to Martin

(2000), a learner needs to be “self-aware of the nature of his or her existing understandings if the folding back is to be effective” (p. 145). Effective folding back is more likely when a student is given the opportunity and time to work with their inner levels of understanding in order to enrich it, usually resulting from open-ended teacher interventions or prompts (Martin, 2008; Martin, 2000; Martin and Pirie, 1998). Through the study of the historical development of mathematics, students are provided opportunities to fold back and thicken their inner levels of understanding of topics they have been exposed to during their previous mathematical education experiences. This process of folding back allows the inner levels of understanding to be informed by the outer levels of understanding resulting from more formalized experiences with the mathematics. In the context of this study, researching the historical development of a particular concept affords students the opportunity to reconnect, reexamine and enrich what they already understand and know about that mathematical concept.

In an effort to address a lack of significant data examples which demonstrate students’ actions of folding back, Martin (2008) developed a theoretical framework for describing these actions. The framework allows an observer to trace the actions of folding back from an initial prompt, through the forms of actions students use at an inner layer of understanding, and whether the outcomes of these actions result in effective or ineffective folding back. In consideration that this is a preliminary study, it is beyond the scope of this study to describe this framework in detail. Readers interested in a detailed description of the framework are referred to Martin (2008). Since the overarching goal of this study is to ascertain the contribution of students’ research and writing in terms of their growth of mathematical understanding, analysis is focused on the outcome of the actions of students’ folding back as reported in students’ written reflections. In order for folding back to be effective, a student must be able to use their newly extended understanding to overcome the original obstacle that induced the folding back. Ineffective folding back is the result of a student being unable to use their extended understanding with the original obstacle. Ineffective folding back does not imply folding back did not occur. The student may have modified their understanding in some way, but is unable to apply it to their outer layer of understanding of a concept. Using Martin's (2008) framework for folding back as a lens for examining students' self-awareness of learning, the following research question guides this study: Does the process of researching the historical development of mathematical concepts contribute to undergraduate students' growth in mathematical understanding through effective folding back?

Method of Inquiry

In an upper level history of mathematics course designed for mathematics and secondary mathematics education majors, various mathematical concepts are explored within the context of their historical development. Learning about the historical development of these concepts can contribute to the growth of students' mathematical understanding. Specifically, conducting research concerning the historical development of a particular concept and learning to communicate that information in written form provides students with opportunities to acknowledge and become more self-aware of their current levels of understanding and to possibly extend that understanding though the action of folding back.

The participants of this study are 14 junior and senior level undergraduate students enrolled in a semester-long course, The Historical Development of Mathematics. The course is a requirement for secondary mathematics education majors, but is an elective course for mathematics majors. Nine of the participants were secondary mathematics education majors and five were mathematics majors. Primary data were taken from students' written reflections collected at the end of the semester as part of the research paper assignment. Students' written reflections allow them to explicitly state how they were able to connect the concept to what they already know. For the purposes of this study, the reflection assignment was analyzed for emergent themes concerning students' thinking about and awareness of the concepts they learned during their research and writing experiences. The use of reflection supports the idea that the processes of reorganizing one's own understanding during the action of folding back require a degree of self-awareness (Martin, 2008). The emergent patterns of folding back are framed through students' awareness of their own growth in mathematical understanding. These written reflections were completed at the end of the course and were based on the following prompt:

> How has the process of writing a research paper on the historical development of a topic contributed to your knowledge of that topic?

Results and Discussion

Initial analysis reveals that the students who completed the reflection assignment were able to deepen and extend their mathematical understanding of the concept that was their topic of inquiry. Several students stated they were explicitly aware of the limitations of their own understandings in terms of the mathematics, particularly when beginning the research assignment. A few students did not explicitly state their learning in terms of the mathematics, but did address the nature of what they did learn from the experience. The results illustrate the level of self-awareness these particular students held concerning what they learned about their research topic and also reveal where gaps may exist in their mathematical understanding.

Effective Folding Back

Key components of effective folding back include students' awareness of the limitations of their own mathematical understanding and the need to engage in relevant mathematical activity (Martin, 2008). These components guide the analysis of students' written reflections in order to find evidence of effective folding back. Of the students who explicitly addressed their self-awareness of their current levels of understanding of their research topic, several of these students also included comments about the mathematics they learned and how their understanding is now extended or newly generated. Examples from students' written reflections follow in order to illustrate the connections to the components of effective folding back (all names are pseudonyms):

> "Going into my research, what I remembered and understood was very muddy…the proofs were a little over my head, but the applications were overwhelmingly interesting…the little bit I remembered about the conics was the algebraic side, but it was interesting to see its development through geometry…I enjoyed see all the applications of conic sections more than anything else."
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> -Betty, Conic Sections

> "When I was first assigned the topic, I knew very little about it and its effects on the field of mathematics. Now I fully understand that this theorem gives birth to the idea of prime numbers…after understanding this important theorem, many concepts from my Modern Algebra course became clearer and seemed to make sense. Discussing concepts as the Euclidean Algorithm and of numbers being relatively prime both enhanced my mathematical knowledge thanks to this paper topic."
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> -Brad, Fundamental Theorem of Arithmetic

Ineffective Folding Back

Even if folding back occurs, it may be ineffective. A student may extend their mathematical understanding of a concept, but is unaware of his/her own understandings or cannot use the extended understanding in the current mathematical activity (Martin, 2008). Several students' written reflections indicate the action of folding back was not effective for these students. These reflections are considered evidence of ineffective folding back if a student revealed a lack of self-awareness of learning mathematics from the activity. For example:

> "Through this paper, I learned that mathematics can be a very confusing and mind blowing subject…What I learned through this experience is that one must be very careful with the ways that you define a new topic…In the end, I learned that we all need to be careful of our phrasing of our thoughts, or the things we do mathematically."
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> -Staci, Mathematical Paradoxes

> "Everything covered in my paper and presentation was new to me…because I do not recall having ever heard of them. One of the reasons I chose them for my topic was because of their beauty…an interesting mathematical revelation to me was the way that Euler's formula may be used in correlation to the number of faces, vertices, and edges…besides the mathematical part, the beauty of the Platonic Solids is what captured my attention."
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> -Barbara, Platonic Solids

The information obtained from this preliminary analysis is particularly useful and informative for both novice and experienced instructors as it addresses a potential pedagogical practice that encourages students' effective folding back via self-awareness of and reflection on their individual mathematical understanding and knowledge. Furthermore, the use of writing is increasingly used as a way to help students learn to communicate mathematically. The teaching and research communities are now faced with the challenge of improving the incorporation of writing in mathematics if it is to be used as a tool for learning mathematics and increasing students' growth of understanding.

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