Engaging in Proof Validation: Identifying Student Errors

Hortensia Soto-Johnson
Hortensia.soto@unco.edu

Kristin Noblet
Kristin.noblet@unco.edu

Sarah Rozner
Sarah.rozner@unco.edu

University of Northern Colorado

ABSTRACT

In an effort to provide multiple opportunities to validate geometry proofs, we required preservice teachers to determine the validity of alleged proofs through presentations, on homework exercises, and on exams. Using qualitative methods, we investigated preservice teachers' views of proof validation, examined their strategies and fidelity to these strategies, and compared their validation techniques to those of mathematicians. Our findings suggest our participants found proof validation challenging but welcomed the opportunity to engage in an activity that was meaningful to their future careers. Results also indicate that our participants were faithful to their claimed validation strategies and shared a number of validation strategies with mathematicians. Although faithful to their strategies, some participants were unable to respond correctly to the scenarios due to their lack of content knowledge, lack of confidence, and inability to recognize a proof different from theirs. These results may impinge on preservice teachers' pedagogical content knowledge.

Key Words: Geometry, Mathematicians, Preservice Teachers, Proof, Validation

INTRODUCTION

In the past decade research related to proof validation has become more prevalent, yet it is limited in scope. Selden and Selden (1995) defined *proof validation* as "the process an individual carries out to determine whether a proof is correct and actually proves the particular theorem it claims to prove" (p. 127). They maintained this process is complicated since it involves reading, affirming assertions, posing and answering questions of oneself, and generating subproofs. In a later study, Selden and Selden (2003) investigated undergraduate mathematics majors' ability to validate purported proofs about a number theory theorem. In their findings, they discovered the participants, mathematics majors and preservice secondary mathematics teachers, did not implement their intended proof validation strategies. Rather than integrating their claimed techniques of checking proofs step-by-step, following arguments logically, generating examples, and ensuring the proofs made sense the students relied on their feelings and surface features of the purported proof. As a result of this research, Selden and Selden (2003) hypothesized rationales for the students' infidelity to their asserted proof validation strategies, recommended instructors provide students with more opportunities to validate proofs, and suggested extending their research to include mathematicians proof validation processes.

While a few researchers have begun to investigate mathematicians' proof validation strategies (Weber, 2008; Weber & Alcock, 2005; Yang & Lin, 2008), our research centers on the implications of providing undergraduates multiple opportunities to validate proofs. The purpose of this phenomenological study is to describe how providing undergraduates with multiple opportunities to validate proofs may influence their view of proof validation and their ability to validate proofs. We also compare our participants' proof validation strategies to techniques used by mathematicians. Specifically we address the following questions:

1. What is the nature of undergraduates' perspective of validating geometry proofs?
    a. How do undergraduates define proof validation?
    b. How do undergraduates view opportunities to validate geometry proofs?
    c. What processes do undergraduates claim to incorporate for validating geometry proofs?
    d. How faithful are these undergraduates to their claimed proof validation methods?
2. How do the undergraduates' documented validation processes compare to mathematicians' validation techniques?
3. What hinders undergraduates to validate purported geometry claims?

The motivation for this study is twofold. First, researchers who have investigated proof validation have focused on number theory or analysis content (e.g., Selden & Selden, 2003; Alcock & Weber, 2005), which is typically not the type of proofs secondary inservice teachers validate. Thus, we felt a need to investigate preservice teachers proof validation skills on purported proofs related to high school geometry topics. The second reason is a culmination of Selden's and Selden's (2003) call to engage students in more proof validation activities and Weber's (2008) summary of processes used by mathematician's to validate proofs. This allowed us to explore the proof validation techniques of undergraduates who routinely validate proofs as part of the course and compare these processes to methods incorporated by mathematicians.

The National Council of Teachers of Mathematics (NCTM) *Principles and Standards for School Mathematics*, (NCTM, 2000) emphasizes the need for students to be skilled at validating proofs by the time they graduate from high school. Some may consider this a lofty goal, but it stresses the need for teachers to be proficient at both constructing and validating proofs. This in itself is challenging, given validating and constructing proofs go hand-in-hand (Selden & Selden, 2003) and students of all ages struggle constructing proofs (see Harel & Sowder, 1998; Selden & Selden, 1987; Senk, 1989; Weber, 2001). In order to better understand the role of proof validation, researchers have investigated the ability to validate proofs with various populations and various mathematical topics. Such research informed our lens for collecting, analyzing, and interpreting our data.

THEORETICAL PERSPECTIVE

Proof Validation by Undergraduates or Teachers

In one of the few quantitative studies related to proof validation, Martin and Harel (1989) investigated preservice elementary teachers' acceptance of inductive and deductive arguments based on the participants' familiarity with the situation. The arguments were about number theory statements such as "If the sum of the digits of a whole number is divisible by 3, then the number is divisible by 3." (a familiar statement) and "If *a* divides *b,* and *b* divides *c,* then *a* divides *c*" (an unfamiliar statement). In their results, Martin and Harel (1999) found the participants accepted both inductive and deductive arguments as proofs, but their acceptance did not depend on the participants' familiarity with the context.

In another study, Knuth (2002) examined inservice secondary teachers' perceptions of proof. He discovered arguments that best convinced the participants were often not proofs. The teachers tended to be convinced by concrete features such as visuals or examples, familiarity with the proof (i.e. they had seen the proof), and level of detail. Other convincing features included generality of proof, insightfulness of the proof, and use of valid methods. It is noteworthy that in the teachers' mind insightfulness was synonymous to visual representations. This may not be surprising since the context of the purported proofs was geometry, algebra, and number theory.

In both studies Martin and Harel (1989) and Knuth (2002) addressed the proof format that was acceptable and/or convincing to the participants. In a more recent study, Selden and Selden (2003) directed their research to explore students' ability to validate proofs. Specifically they asked students to determine the validity of four distinct purported proofs to the statement, "For any positive integer *n*, if n^2 is a multiple of 3, then *n* is a multiple of 3." Besides the results discussed in the introduction, Selden and Selden (2003) also found the participants were unable to determine the validity of the claimed proofs without prompting. Even after probing from the researchers, only 81% of the participants' judgments were correct.

In a similar study, Alcock & Weber (2005) explored how undergraduate mathematics majors, enrolled in an introductory real analysis course, determined the validity of a proof to the statement, "$\sqrt{n} \to \infty$ as $n \to \infty$." Unlike Selden and Selden (2003), Alcock and Weber (2005) only implemented one argument in their research. The argument is shown below.

Proof. We know that $a < b \Rightarrow a^m < b^m$.
So $a < b \Rightarrow \sqrt{a} < \sqrt{b}$.
$n < n+1$ so $\sqrt{n} < \sqrt{n+1}$ for all n.
So $\left(\sqrt{n}\right) \to \infty$ as $n \to \infty$ as required. (p. 128)

While six of the 13 participants identified this as a flawed argument, only two provided legitimate mathematical reasons. Three students rejected the proof because it failed to incorporate the appropriate use of a definition. Other students focused on the truthfulness of the assertions, "rather than considering whether they were substantiated" (p. 133). Similar to the results shown by Selden and Selden (2003), 10 participants from Alcock's and Weber's (2005) study concluded the last line of the argument did not follow from the previous statements, after the researchers probed. The researchers suggested this may indicate "the ability to validate proofs may be in students' zone of proximal development and that students' abilities in this regard might improve substantially with relatively little instruction" (p. 133). Understanding how mathematicians validate proofs may facilitate such instruction. Thus, we summarize two recent studies on mathematicians' proof validation practices.

Proof Validation by Mathematicians

In his study on how mathematicians validate proofs, Weber (2008) unveiled how mathematicians use a variety of strategies to determine the validity of an argument. Along with the purported proofs to the statement used by Selden and Selden (2003), Weber posed four other statements, each with one purported proof. These statements were also related to number theory but typically found in an advanced undergraduate number theory course. Of the eight mathematicians interviewed, seven commenced by determining the proof technique used in the argument; if this method was acceptable the mathematicians proceeded to read the proof line-by-line. If the method was not acceptable, then the proof was immediately rejected. If a participant was uncertain of the sequence of claims and assertions he/she constructed a subproof or an informal justification such as using properties of the elements related to the proof. These findings coincide with the results of Weber and Alcock (2005), which also included mathematicians.

One of the most surprising results of Weber's (2008) work was that the mathematicians used validation strategies that are unacceptable in a formal proof. For example recognizing a pattern, validating a single example, or understanding why an individual example worked convinced the mathematicians of an assertion. The participants did not hesitate to use such strategies because they quickly convinced themselves by talking though the example and realizing how this generalized to other cases. The mathematicians also accepted assertions based on their inability to quickly produce a counterexample or based on the "authority of the author" (p. 452).

In another study involving mathematicians, Yang and Lin (2008) interviewed both mathematicians and middle and secondary inservice teachers about their views on reading proofs. Interview prompts relevant to our research were (1) to describe the purpose for reading proofs and (2) to indicate reading strategies implemented before, during, and after reading the proof. The mathematicians commented that they read others' proofs while forming conjectures and developing their own proofs. They also relayed that they read proofs for others in order to assist with preparing a manuscript for publication. This sometimes entailed restructuring the proof. Some mathematicians indicated they tended to prove a statement themselves, before reading a proof in a text. Others concentrated on the application of a theorem, rather than the textbook proofs. In conclusion the mathematicians expressed that the crucial components for understanding others' proofs were to distinguish between the assumptions and the conclusion and to identify a sketch of the proof. The teachers, on the other hand, claimed they read proofs to understand mathematical proofs or to recognize the proof technique such as proof by induction. A few of the teachers also indicated they attempted to prove a statement before reading it by reversing the steps from the claim and recognizing useful properties.

From this literature we borrowed the idea of only providing one purported proof per statement since this was a common technique incorporated in the literature. We also used the findings to analyze the data using analytic induction strategies (Patton, 2002). There are various strategies to implement analytic induction and we chose to use the literature-derived concepts to sensitize ourselves throughout the research while remaining open to new concepts (p. 494). We discuss this in more detail in the following two sections.

METHODS

Participants and Setting

Our participants consisted of two preservice elementary and eight preservice secondary mathematics teachers enrolled in the first semester of modern geometry. All participants successfully completed at least the first two semesters of calculus and discrete mathematics. The instructor, also the first-named author, implemented the text, *Euclidean and Non-Euclidean Geometries: Development and History* (Greenberg, 2008) for its historical perspective on topics and for its unique exercises. Along with traditional geometry exercises, the text also contains exercises that require students to rewrite an informal argument as a formal proof, to justify steps in a proof, and to indicate inappropriate steps in a purported proof.

Although these types of exercises were not in the majority, the number of exercises exceeded those found in other higher geometry texts. The instructor assigned all the exercises of this nature, even though the proof validation exercises tended to only have one error related to the inappropriate use of a premise, a definition, or a theorem. This was one of the methods used to support students' practice of proof validation. Other techniques included validating proof presentations and validating purported proofs as part of in-class exams, which we refer to as *classroom scenarios*. Each of the 20 students enrolled in the course presented distinct proofs at least twice during the semester and the remainder of the class critiqued the claimed proof as a whole. Student presentations are frequently incorporated in the mathematics classroom (e.g., Freedman, 1983; Myers, 2000; Soto-Johnson, Yestness, & Dalton, 2009), but they do not ensure all students participate in proof validation. Students may be reluctant to provide an honest critique of their peers' work or they may choose not to participate in the class discussion. It is also possible for students to base their validation on the instructor's comments, as seen in Selden and Selden (2003) and Alcock and Weber (2005). In order to guarantee all students participated in proof validation, the instructor posed classroom scenarios on each of the three in-class exams.

These scenarios differ from statements posed in the literature, because the statements are not necessarily true. They resemble scenarios from Ma's (1999) work, where a student makes a conjecture with a purported proof and it is the participants' responsibility to determine the validity of the claim and the purported proof. The scenarios related to congruent quadrilaterals, similarity, and isometries (see Appendix 1) and were connected to the exam content. The exams were approximately five weeks apart and upon returning the scored exams, the instructor asked students to share their responses to the scenarios. For every scenario, someone mentioned at least one flaw in the purported proof. The instructor did not confirm the correctness or incorrectness of the shared responses since they would be used as part of the interview process, but students remarked on the their peers' comments. For example for the first scenario someone remarked, "This is wrong because they are just looking at an example so it's not specific enough." In unison several students said, "Oh ya!"

Data

Besides the participants' work from the classroom scenarios, we also conducted video-taped, semi-structured, think-aloud, task-based interviews (Patton, 2002) that were approximately one hour in length. These interviews, which comprised of three distinct phases, served as our research data. Initially the interviewer asked the participants to define proof validation, to describe how they validated their own proofs, and to share their views of validating proofs as part of an exam. In the second phase, the interviewer presented the participants with four new classroom scenarios (see Appendix 2)[1]. Finally, the interviewer revisited the classroom scenarios from the three in-class exams. She provided the participants with their original response and asked if they wanted to modify their original response. This is similar to the protocol that Selden and Selden (2003) followed, but our process had a longer time lapse, since the interviews were conducted during the last week of the semester. From here on out we use the term participants to refer to students who partook in the interviews and we use the term student to reference a fictitious student from a scenario.

ANALYSIS

Upon transcribing all the interviews, we analyzed the participants' responses to the classroom scenarios from the exam and the two components of the interview questions using analytic induction, open-coding, and constant-comparative methods (Patton, 2002). The analytic induction entailed searching for themes similar to those summarized in the literature. The process of open-coding consisted of creating and defining code phrases from a participant's response to each question and re-coding participant's work based on new code words. We followed through with constant-comparitive methods in order to create themes from the code phrases and synthesized the information for each task with a matrix summarizing the participant responses. In the results section we incorporated quotes to support our observations; the pseudonyms Ellie and Erika indicate preservice elementary teachers, while Sam, Sarah, Selma, Seth, Silvia, Stephanie, Steven, and Summer represent preservice secondary teachers. Two researchers coded simultaneously, which facilitated the discourse related to coding and creating new code phrases and contributed to researcher triangulation (Patton, 2002).

In the following section, we begin by summarizing the nature of our participants' perspective on validating purported proofs. A summary of the participants' responses to the new classroom scenarios follows. We culminate the results section by threading the participants' responses to the exam classroom scenarios with their interview responses. Although, the classroom scenarios require participants to explain how they would respond to the fictitious student, few participants reached this point. Thus, our analysis focused on the participants' response to the claim and the purported proof. Throughout the results we compare the participants proof strategies to those of mathematicians.

RESULTS

Participants' Perspective on Proof Validation

Overall our participants confused proof validation with the proof of a statement and found proof validation challenging yet rewarding. Eight of the participants initially confused to validate with to prove. This might be due to the fact some participants validated proofs by first attempting to prove the statements, which we discuss later. Eventually, four of these participants, such as Sarah, successfully distinguished between proving and validating a proof, with probing from the interviewer. The interviewer provided the remainder of the participants with a correct definition and allowed them to ask clarifying questions.

Sarah: Okay, I kind of look at the word validation and it's kind of going through a series of steps or a process in order to reach your conjecture. And to validate it is to use logical steps that we've already proven or that were given to us to be true in order to reach the final conclusion.

Interviewer: So, how does validation differ from proving? Are they the same in your mind?

---

[1] The angles and isometry task and the Euclidean parallel postulate task come from actual student work.

Sarah: I guess for validation, you'd, [because] to prove a proof, you'd use the steps. And then to validate it you'd have to go through and make sure that each step is logical and that you could do each step. And then make sure that it's actually correct.

For various reasons, all the participants found validating proofs as part of in-class exams challenging. Six participants indicated that identifying the student's error made the activity challenging. Four participants found the activity difficult because they claimed they would prove the statement differently. Sarah added that she attempted to prove statements prior to validating them in order to not be "swayed" by the student's reasoning. Three participants found this activity challenging because they struggled to see the "whole-picture" or to identify the missing pieces of the proof. Two other participants found the scenarios frustrating because they were unsure how to confirm a step in the student's work. Selma expressed how this activity caused her to reflect on her future as a teacher. In her remark shown below, she conveyed her concerns about her ability to validate proofs in the classroom and the pressure to be correct since she would be the expert.

Selma: I don't ever want to tell someone something wrong. Because to them, I mean, I'm always going to be right. And I don't want to stray them from the right path by informing them incorrectly because I didn't sit there and evaluate my words before I spoke.

While the participants found the validation activities challenging, they also valued the opportunity. Eight participants exclaimed the validation activities made them think through the proofs and helped to reinforce their understanding. Additionally, four participants claimed the scenarios offered a setting to respond to students and to guide students to a correct proof. Lastly, all participants remarked these activities were relevant to their future teaching since they provided some experience with proof validation.

The participants claimed to use a variety of strategies to validate others' proofs or their own. These strategies included attempting a proof prior to validating, creating visual representations of the student's purported proof, reviewing an argument step-by-step, recreating a similar argument, establishing the appropriate use of theorems, definitions, and proof techniques, and seeking assistance from others. After constructing their own proof, the participants claimed to compare the student's alleged proof to their own. In validating their own proofs, some participants asked peers for feedback. All of these strategies, except for creating a visual representation are documented, which is not surprising since students rely heavily on diagrams in geometry.

From our data (discussed below) it appeared our participants were faithful to their claimed strategies. At some point during the interviews, each of the participants validated proofs using visual representations, reviewing the arguments step-by-step, and establishing the appropriate use of theorems, definitions, and proof techniques. While Sarah, Selma, and Silvia all claimed to attempt a proof prior to validating, Sarah was the only one who used this technique during the interview. None of the participants recreated a similar argument to check the logic for a second time and due to the constraints of the interview no one sought assistance from others. We now synthesize the results to each of the new classroom scenarios.

Validating New Classroom Scenarios Tasks

SSSS Task. When asked to validate the claim that SSSS congruence proves that two quadrilaterals are congruent, nine of the participants believed the claim to be invalid. One participant, Selma, thought the claim was invalid and attempted to create a counterexample but was unable to do. Thus, she resigned herself to believe the claim was valid. Although the mathematicians in Weber's (2008) work used the same tactic, they were able to construct counterexamples. Of the participants who answered the scenario correctly, one provided a counterexample involving Saccheri quadrilaterals while two participants gave a rectangle-and-parallelogram argument. Most of the participants (6) illustrated a counterexample using a square and rhombus. These participants claimed that although the square and rhombus could have all four corresponding sides congruent, their corresponding angles do not need to be congruent. Seth's use of definitions (Alcock & Weber, 2006) helped him to produce the square-and-rhombus counterexample:

Seth: Okay, so if all four sides were equal, then by definition it would be a rhombus. And, oh. Well, no, then they wouldn't be congruent cause you could also have a square. So, then they wouldn't be congruent at all.

Part-Whole Task: All participants stated the student's claim that triangles ABD and ABC were congruent was invalid, but they presented a variety of reasons for their beliefs. Three participants recognized that triangle ABD was part of triangle ABC, which made it smaller, since the part is always smaller than the whole. Four participants claimed that the inclusive angles weren't equal, or that angle ABD was less than angle ABC. And two participants claimed that segment AD was part of, and thus less than, segment AC. All of these explanations make use of deductive reasoning, based on the common notion that the part is smaller than the whole. Silvia was unable to justify why the two triangles were not congruent, and seemed to rely on a feeling (Selden & Selden, 2003). When asked what misconceptions the student might have that influenced the claim, half of the participants insightfully identified that both triangles "share" angle A. Thus, both triangles have two corresponding sides and one congruent corresponding angle. With disregard for order, this could constitute as SAS congruence. Below, Summer explains why one might incorrectly invoke the SAS theorem.

Summer: Well they do share this angle A and they do [share] this side $\overline{AB}$, but they don't share this side $\overline{AC}$. Well, they share the side $\overline{AC}$, but it's cut in two. He can't prove it by SAS.

Interviewer: So why do you think he might think it's SAS?

Summer: Because he thinks these two sides $\overline{BC}$ and $\overline{BD}$ are congruent. So, he's saying that since they are congruent, he's saying that they must be equal because then they share two sides and an angle, but it would be SSA, and because of this length $\overline{AC}$, he can't use that as the reasoning, because those lengths are different.

Two participants also reasoned that the student may have been thinking that the triangles were similar, rather than congruent. Thus, the participants made conjectures (Yang & Ling, 2008), however, neither of the two participants proved or disproved whether the triangles could be similar, which in fact is not possible.

Altitude, Median & Angle Bisector Task: Nine participants realized the student was correct in her idea, but drew the altitude, median, and angle bisector from the wrong vertex. Stephanie, however, insisted that the altitude, median, and angle bisector are only the same if the triangle is equilateral.

Stephanie: Oh I think she just has it mixed up with an equilateral triangle. Because I think those are properties of equilateral triangles.

Interviewer: Are you saying, that this is not a property of an isosceles triangle?

Stephanie: Correct.

Although Steven eventually answered the scenario correctly, he initially struggled with it and in fact articulated several incorrect statements. Steven believed the altitude was the longest segment and this influenced his response, as shown below. Figure 1 illustrates his work. Both Stephanie and Steven made mistakes due to incorrect use of terms (Alcock &Weber, 2005), which Steven eventually recognized.

Steven: Well, for one, the way she's going from this point (O) to this line ($\overline{IS}$), is going to change because you would think that if it was an equilateral triangle, it could be, because then, everything would be equal in the triangle. Since she is looking from O to $\overline{IS}$, um $\overline{IS}$ is not equal to $\overline{OS}$, so this has its own line out here, so that's going to change the distance from O to $\overline{IS}$. So the altitude is going to be the highest point between O and $\overline{IS}$.

Interviewer: And what do you mean by the highest point?

Steven: The largest distance from O to $\overline{IS}$.

Interviewer: So you're saying that the altitude is the largest distance from that vertex to the line. So is $\overline{OA}$ longer than $\overline{OM}$?

Steven: Yeah because if you were to extend this line here $\overline{OS}$, A would travel this way (towards I). No, I'm wrong. It would travel this way (towards S). If you were to extend this line $\overline{OS}$ down to here it would continue to travel down to here (towards $\overline{OS}$), while M would stay in the middle of the line $\overline{IS}$, since it's the midpoint. It would adjust to whatever the line is. So, it would kind of curve like this. Well, they wouldn't be the same because the maximum distance is going to change because of how long this side $\overline{OS}$ is. And, that midpoint isn't always going to be the highest distance.

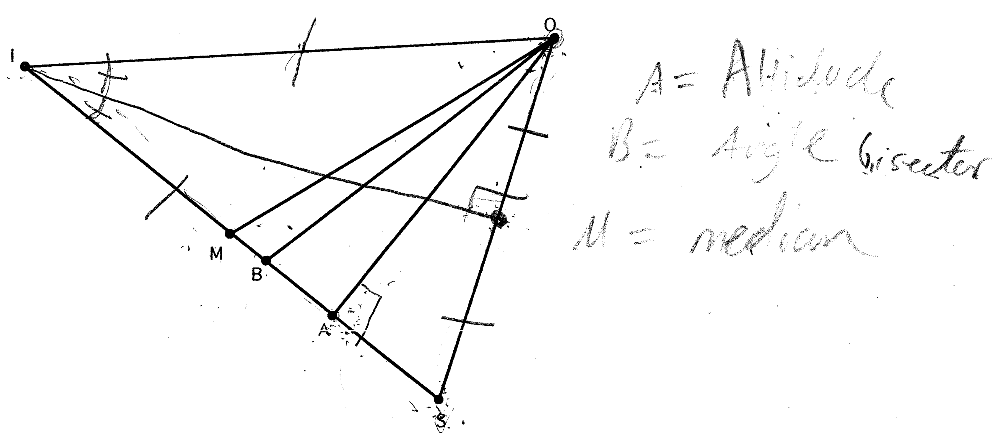


Figure 1. *Steven's work on the Altitude, Median, & Angle Bisector Task.*

Angles and Isometry Task: Six participants thought the purported proof was too specific and lost generality (Knuth, 2002) by only considering translations. Many of these students, such as Silvia, suggested the student prove the statement for each transformation, which may indicate an incomplete understanding of the term isometry. Silvia and another participant also appeared to interpret "translations" as "transformations."

Silvia: Alright, well he didn't say which translation he would do, so I would say his proof is not finished. He needs more information, as in if he puts this under a reflection, it's gonna be, I guess like this (reflects ABC to get $C'B'A'$). $C'B'A'$ and he also has to prove that under a reflection that the distance between AB and $A'B'$ is the same, BC and $B'C'$ and then AC and $A'C'$ are the same. And I would say that you would have to do that under slide as well (draws another triangle).

With probing, three participants claimed the student could construct a general proof for any isometry rather than for each transformation. Ellie's comment supports our observation.

Ellie: I think he's being too specific. He's just proving that this is true under a translation. … I wasn't sure if you needed to go through and prove it for all four of them…you just need to do it in general

While there were many areas in which the student needed to be clearer, only Sam mentioned all of them. He recognized the loss of generality and suggested using the definition of isometry. Additionally, he suggested providing reasons for the congruency of angles and acknowledged the student failed to use the intended proof by contradiction strategy. Eight participants made sense of the student's proof using the definition of isometry, but only two explicitly suggested that he include it in his proof. Three participants also proposed the student provide reasoning for why angles ABC and $A'B'C'$ are congruent. Four participants realized the student did not make use of the "proof by contradiction" statement, although they agreed it was a good strategy. These suggestions indicate how the participants considered the correct use of definitions, constructed a subproof, and regarded the appropriateness of the proof strategy (Knuth, 2002; Alcock & Weber, 2005; Weber, 2008; Yang & Lin, 2008).

Euclidean Parallel Postulate Task: Given that students struggle proving bi-conditional statements (Selden & Selden, 1995), it was not surprising our participants found this task challenging. Adding to this challenge was the fact that the proof was correct except for in the first part of the proof, $\overline{PF}$ should be $\overline{PD}$. Six participants identified this error, but two of these six recognized it only after suggestive probing from the interviewer. Although two other participants were unsure of the first part of the proof, they did not identify the mistake. Stephanie offered a unique response to the errors present in this scenario. In response to how she should address a typographical error, Stephanie admitted that is would be difficult to determine whether the student miswrote or misunderstood. She went on to say that she would make a decision based on the student's prior performance in class or she would give the student the benefit of the doubt. In other words, she would make a decision based on the authority of the author (Weber, 2008).

Although the second part of the proof was valid, many students struggled with it. Two confused the statement of the Euclidean Parallel Postulate, and four were unsure whether the students could make use of the fact that perpendiculars are unique from a given point. Sam suggested the student consider the sum of the interior angles, and that it may be less than 180 degrees. While this path is another valid way to construct the proof, the participant neglected to recognize the validity of the proof that was given to her. Although many of the participants were either unsure or skeptical of the proof's validity, none of the participants completely constructed an alternative proof. One such alternative proof might have employed proof by contradiction to demonstrate that any line through P, distinct from line m, would intersect line l, a concept that many participants used on the *Parallelism and Isometry Task*.

Summary of Validations from Exams

As described earlier, each in-class exam contained a classroom scenario where the participants determined whether the students' comments were correct, explained how they would correct misconceptions, and provided an explanation as to how they might guide the students in the correct direction. Unfortunately during the in-class exams, the participants did not provide rich responses to the fictitious student: common responses tended to be "good try." Each of these scenarios were revisited during the interview.

Congruency of Quadrilaterals Task: During the in-class exam, nine of the ten participants commented that dissecting the quadrilaterals into triangles was an appropriate proof strategy. Unfortunately, only two participants criticized the student's purported proof because it failed to generalize to all quadrilaterals (Knuth, 2002). During the interview the same seven participants who identified the *Angles and Isometry Task* as losing generality also identified this task as being too specific. However, four of those participants identified the loss of generality, only after suggestive probing from the interviewer. Three participants never indicated the loss of generality resulted in an incorrect proof. Conversely, eight participants recognized the incorrect assumption that the diagonal of a quadrilateral is also the angle bisector, and thus claimed the argument invalid. At the time of the interview nine participants recognized this flawed reasoning. Two of these participants provided proofs if one assumed that the diagonal of a quadrilateral was also the angle bisector of a quadrilateral. Seth was the only participant who stopped reading the argument as soon as he found a mistake, similar to the process of a mathematician (Weber, 2008).

At the time of the interview all the participants claimed SASA is not a valid method for proving two quadrilaterals congruent and attempted to provide a conjecture and proof through a variety of ways. Summer, one of the three participants who did not identify the loss of generality as an error, used the rectangle to prove that SASA was enough to determine the congruencey of quadrilaterals. She outlined a correct method, along with accurate reasoning, for proving two parallelograms congruent using side-angle-side and properties of a parallelogram. Although her strategy was correct, it did not generalize to all quadrilaterals.

Both Silvia and Steven initially claimed two quadrilaterals were congruent if all the corresponding sides are congruent. Steven independently realized his mistake by recalling his square and rhombus counterexample to the SSSS task. Unlike Steven, Silvia required reminding about the SSSS task. During the interview it became clear Silvia had a misunderstanding about convexity; she believed convex polygons only have angles of 90 degrees or less. Thus familiarity (Knuth, 2002) with a similar task influenced both of these participants' decision, but neither was able to determine conditions for which two convex quadrilaterals are congruent.

Seth and Stephanie both conjectured interesting, yet incorrect, methods for proving two convex quadrilaterals congruent. Seth started with the concept of similar quadrilaterals, reasoning that if he would prove two quadrilaterals were similar, then he would only need to show that one pair of corresponding sides were congruent in order to establish a one-to-one ratio. While this was a viable idea, Seth incorrectly claimed that the angle-angle-angle theorem was sufficient for establishing similarity for quadrilaterals. Although he did not investigate the idea further, the counterexample involving a square and rectangle may have encouraged him to refine his thinking. Stephanie considered other aspects of quadrilaterals, such as their diagonals. She reasoned that having corresponding diagonals congruent is sufficient to prove two quadrilaterals congruent, because "a parallelogram… would not have the same diagonal lengths as a rectangle." Like Seth, Stephanie did not devote time to finding a counterexample to her conjecture.

Curiously, the five participants who identified a correct method for proving two convex quadrilaterals congruent all identified the side-angle-side-angle-side. (Another equally valid method is angle-side-angle-side-angle). In the following Sam describes his reasoning, which was similar to the other correct responses.

Sam: Side-angle-side-angle-side, … because the only thing that's not making these quadrilaterals congruent is that, … So this [vertex], we'll say C, is always going to be equal distance from D. So you know this vertex is going to stay in the same spot, so this leg connecting the length of it (BC) is gonna be determined by where this vertex is. And then, so this vertex could be in infinite spots along this ray AB and A'B' which would make the quadrilaterals different. (See Figure 2.)

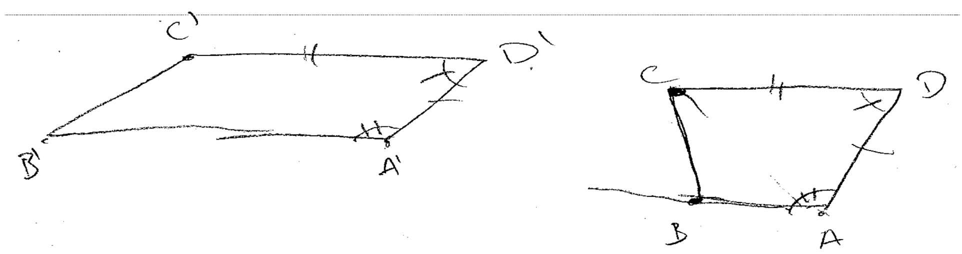


Figure 2. *Sam's diagram for proving Side, Angle, Side, Angle, Side for quadrilaterals*

Similarity Task: During the in-class exam, seven participants explicitly stated the student was incorrect due to her erroneous belief that the angle bisector was the perpendicular bisector of the opposite side. One participant illustrated this fallacy with right and obtuse triangles. Another student verified how this was only true for isosceles triangles. During the interview, nine of the participants addressed this mistake. In order to determine whether the student's hint was helpful, four participants attempted the proof on their own (Alcock & Weber, 2005; Knuth, 2002; Weber, 2008). Although they all made progress, no-one completed the proof and no-one expanded on the appropriateness of the student's proof strategy.

With probing by the interviewer, nine participants expressed belief that the provided strategies were viable. While not all were successful, each participant attempted to expand on the students' method. Some participants proved the existence of an isosceles triangle in the diagram, and others were able to verify that triangles ACD and AEB are similar. Of our participants, two almost completed the proof but only Ellie and Seth provided a complete and correct proof.

Parallelism and Isometry Task: The in-class results for this scenario mimicked the results of the congruency of quadrilaterals scenario. Half the participants felt the purported proof was partially correct and one participant claimed it was a correct proof. Five participants commented that proof by contradiction was an appropriate proof technique for this statement, but only two recognized the lack of generality in the proof. Seth was the only participant who recognized the lack of generality in both the first and third classroom scenarios. Other participants sought to determine the flaws in the argument by attempting to prove it themselves, but in most cases they simply re-iterated the student's argument and explained the definitions of a line of reflection and isometry. Since this task was quite similar to the *Angles and Isometry Task*, the interview responses to the task were relatively brief, and most of the participants were successful in ways similar to the earlier task. However, more participants (nine, as opposed to six) identified the loss of generality by only considering reflection, and less participants (seven, as opposed to eight) stated or suggested using the definition of isometry. Seven participants suggested the student focus on the distance between the parallel lines versus the distance between their images.

DISCUSSION

Son and Sinclair (2010) claim, "… preservice teachers should have an opportunity in teacher education programs to examine student errors, to discuss possible strategies they could use, and to modify their strategies based on the discussion to respond to student errors appropriately" (p. 41). Proof validation is a venue for such opportunities, but several obstacles prevent preservice teachers from correctly validating purported proofs. From our research we found preservice teachers' content knowledge, confidence level, and own proofs interfered with their ability to validate geometry proofs.

In order to determine the validity of the fictitious students' statements and their alleged proofs, participants needed knowledge about concepts related to triangle congruence theorems, similarity theorems, Euclid's parallel postulate, and isometries. It also entailed recognizing true statements can be proved using various approaches. The lack of this knowledge manifested when participants indicated they did not feel confident validating purported proofs when a student "proved" the statement differently. This indicates the inability to enact flexible thinking while validating students' work. This was apparent in Sam's inability to focus on the purported proof at hand for the Euclidean parallel postulate task. It also seemed this lack of content knowledge could result in convincing some participants that illogical claims were true. Both of these apprehensions may affect preservice teachers' *pedagogical content knowledge* (Ball, Thames & Phelps, 2008) and result in providing incorrect feedback to students. This is especially prevalent when preservice teachers determine the validity of a statement based on the authority of the author, which Stephanie confessed she would do for the Eulcidean parallel postulate task.

Participants' lack of content knowledge also emerged in their responses to the classroom scenarios. In the part-whole task we heard students claim the triangles might be similar, when in fact this is not possible. In the altitude, median, and angle bisector task, Stephanie believed these parts of the triangle could only be congruent in an equilateral triangle, while Steven claimed the altitude of the triangle was longer than the sides of a triangle. With probing, Steven realized the altitude forms a right triangle and was then able to see the "whole picture." Some participants remarked that the inability to see the whole picture made validating the claims more difficult. Our participants also struggled with the use of definitions, which materialized in responses to the angles and isometry task, Silvia's misunderstanding of convex, and our participants' interchange of validating and proving. Other mathematical errors were the incorrect belief that angle-angle-angle is sufficient to prove two quadrilaterals are congruent and the belief that the diagonal of a rectangle is the angle bisector. Both of these errors occurred in the congruence of quadrilaterals task.

We recognize that focusing on the lack of content knowledge may be due to the slight use of a deficit model, but we also acknowledge our participants appropriately incorporated several proof validation techniques cited in the literature. Our participants appeared to rely on their feelings when they couldn't provide a mathematical explanation. They considered the generality of the purported proof, the use of definitions, as well as the use of appropriate proof strategies. A few participants generated a proof for the intended claim, but this sometimes hindered their ability to validate the student's proof. We also witnessed one participant who was unable to provide a counterexample in validating a claim and another who claimed she would account for the authority of the author. Our participants were also influenced by the familiarity of tasks. For example the SSSS task helped the participants determine the validity of the SASA task. Similarly, participants relied on knowledge about the angles and isometry task to respond to the parallelism and isometry task. This may indicate that providing students with opportunities to validate proofs allows them to become more proficient – at least if the flaws are similar. Finally, with probing, the participants formed conjectures in response to congruency of quadrilaterals task.

Our participants' techniques for responding to the classroom scenarios were similar to several of the proof validation techniques documented in the literature, but there were a few processes that were not incorporated. For example, we did not witness our participants consider the insightfulness of a proof, focus on surface features of the purported proof, recognize patterns to justify claims, or validate a single example in order to convince themselves of a student claim. This may be a result of the fact that these techniques were not modeled in the classroom. On the other hand the instructor routinely informed the participants that considering a single example was not a proof.

Although our participants sometimes struggled with validating the alleged proofs, there were benefits to these activities. The classroom scenarios provided another venue for the instructor to assess her students' errors and allowed the preservice teachers to recognize their own content deficiencies. They may also contribute to the preservice teachers' pedagogical content knowledge and provide a forum to strengthen preservice teachers' proof validation abilities.

LIMITATIONS AND FURTHER RESEARCH

Initially we set out to investigate how our participants would respond to the classroom scenarios and to the students, but few if any participants shared how they would react to the student's comments. Thus, it was difficult to identify what the participants felt was necessary for students to include in their proofs or arguments. While participants attempted to make sense of student proofs, they often provided the reasoning behind students' statements. In most cases, however, the participants did not explicitly state whether the reasoning was a necessary component of the proof. For example the definition of isometry is a necessary component of the proof that angles are invariant under any isometry, but the student did not state or explicitly use this definition in the sample proof. While making sense of the sample proof, most participants described isometry and how it supported the student's argument, but few specifically recommended the student use it in the proof. It is unclear from the data whether participants who did not make this suggestion would have, had they been responding to an actual student. For future studies, it may be helpful to ask participants to be complete in their suggestions to students, in order to determine what information and reasoning each participant deems necessary to include. This would allow valuable experiences for preservice teachers to combine their content knowledge and pedagogical content knowledge. Further research may also investigate how we can assess the relationship between content knowledge and pedagogical content knowledge in the area of creating and validating proofs.

References

Alcock, L., & Weber, K. (2005). Proof validation in real analysis: Inferring and checking warrants. *Journal of Mathematical Behavior, 24*, 125-134.

Ball, D., Thames, M. H., & Phelps, G. (2008). Content knowledge for Teaching: What makes it special? *Journal for Teacher Education, 59*, 389-407.

Greenberg, M. J. (2008). *Euclidean and non-Euclidean geometries: Development and history* (4th ed.). NY, NY: W. H. Freeman and Co.

Harel, G., & Sowder, L. (1998). Students' proof schemes: Results from exploratory studies. In A. Schoenfeld, J. Kaput, & E. Dubinsky (Eds.), *Research in collegiate mathematics education III* (pp. 234-283). Providence, RI: American Mathematical Society.

Knuth, E. J. (2002). Secondary school mathematics teachers' conceptions of proof. *Journal for Research in Mathematics Education, 33(5)*, 379-405.

Ma. L. (1999). *Knowing and teaching elementary mathematics*. Mahwah, NJ: Lawrence Erlbaum Associates.

Martin, W. G. & Harel, G. (1989). Proof frames of preservice elementary teachers. *Journal for Research in Mathematics Education*, 20(1), 41-51.

National Council of Teachers of Mathematics. (2000). *Principles and standards for school mathematics*. Reston, VA: National Council of Teachers of Mathematics.

Patton, M. Q. (2002). *Qualitative research and evaluation methods* (3rd ed.). Thousand Oaks, CA: SAGE.

Selden, A., & Selden, J. (1987). Errors and misconceptions in college level theorem proving. In J.D. Novak (Ed.), *Proceedings of the Second International Seminar on Misconceptions and Educational Strategies in Science and Mathematics* (pp. 457-470). Ithaca, NY: Cornell University.

Selden, A., & Selden, J. (2003). Validations of proofs considered as texts: Can undergraduates tell whether an argument proves a theorem. *Journal for Research in Mathematics Education, 34(1),* 4-36.

Selden, J., & Selden, A. (1995). Unpacking the logic of mathematical statements. *Educational Studies in Mathematics, 29*, 123-151.

Senk, S. L. (1989). Van Hiele levels and achievement in writing geometry proofs. *Journal for Research in Mathematics Education, 20(3)*, 309-321.

Son, J. & Sinclair, N. (2010). How preservice teachers interpret and respond to student geometric errors. *School Science and Mathematics, 110(1),* 31-46.

Soto-Johnson, H., Yestness, N., & Dalton, C. (Spring 2009). Assessing multiple abstract algebra assessments. *Investigations in Mathematics Learning, V. 1(3),* 1-26.

Weber, K. (2001). Student difficulty in constructing proofs: The need for strategic knowledge. *Educational Studies in Mathematics, 48(1),* 101-119.

Weber, K. (2008). How mathematicians determine if an argument is a valid proof. *Journal for Research in Mathematics Education, 39(4)*, 431-459.

Weber, K., & Alcock, L. (2005). Using warranted implications to understand and validate proofs. *For the Learning of Mathematics, 25(1),* 24-51.

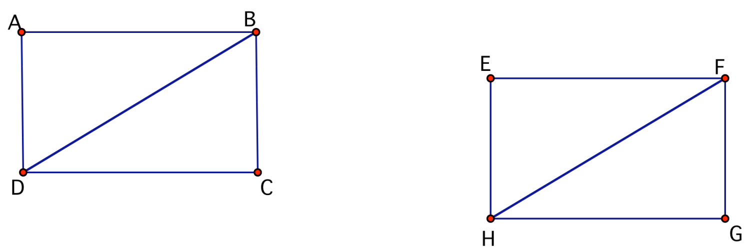
Yang, K., & Lin, F. (2008). A model of reading comprehension of geometry proof. *Educational Studies in Mathematics*, *67*, 59-76.

APPENDIX 1: EXAM CLASSROOM SCNENARIOS

<u>Congruency of Quadrilaterals Task</u>*:* You are a high school geometry teacher who just finished discussing the different propositions to prove that two triangles are congruent. In an effort to extend students' knowledge to other shapes, you pose the following question. *Under what conditions can we verify that two convex quadrilaterals (quadrilaterals whose angle measure is not obtuse) are congruent?*

Your student Miguel makes the following claim and provides the following proof to his conjecture. Miguel says, "*Two convex quadrilaterals are congruent if they satisfy the SASA proposition i.e. two convex quadrilaterals ABCD ad EFGH are congruent if* $AB \cong EF$*,* $BC \cong FG, \langle B \cong \langle F, \& \langle C \cong \langle G$*. Here is my proof.*"

> Pf: *Let quadrilaterals ABCD and EFGH be rectangles, which are obviously convex since none of the angles are greater than 90 degrees and such that* $AB \cong EF$*,* $BC \cong FG, \langle B \cong \langle F, \& \langle C \cong \langle G$*. Construct segment BD, which bisects angle B and D* → $\langle ABD \cong \langle CBD \ \& \ \langle BDC \cong \langle BDA$*. Similarly, if we construct segment FH, it will bisect angles F and H* → $\langle EFH \cong \langle GFH \ \& \ \langle GHF \cong \langle EHF$*.*
> *Since angles B and F are right angles and we bisected them we have the following equality:* $\langle DBC \cong \langle HFG$*. Thus by ASA we have that triangle BCD is congruent to triangle FGH.*



> *Also by SSS we have that triangle ABD is congruent to triangle CDB and triangle EFH is congruent to triangle GHF. Thus, by the transitive property we have that triangle ABD is congruent to triangle EFH.*
>
> *Since all the corresponding sides and angles of the two quadrilaterals are congruent then the two quadrilaterals are congruent. Thus, SASA holds for convex quadrilaterals.*

Is Miguel's proof correct, partially correct, or incorrect?

a. If correct, explain how it could be improved (if at all) and how you might relay this to Miguel.
b. If it is partially correct, explain aspects that are correct and incorrect and how you might relay this to Miguel.
c. If incorrect, explain what misconception(s) Miguel is portraying in his proof and provide a correct proof for Miguel.

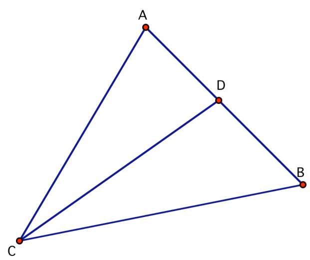
<u>Similarity Task</u>: In you high school geometry class you just finished defining similarity and providing the proofs for the AAA, SSS, and SAS similarity theorems. In groups, your students work on the proof for the following statement.

"The bisector of an angle of a triangle divides the opposite side into segments proportional to their adjacent sides."

As you go by Miguel's group you hear the following conversation.

Logan: Well, I am not really sure what to do, this doesn't make sense. What does adjacent mean?

Olivia: Remember, it means next to, so let's draw a picture. We have triangle ABC and $\overline{CD}$ bisects angle C. So, we want to show $\frac{\overline{AD}}{\overline{BD}} = \frac{\overline{AC}}{\overline{BC}}$.

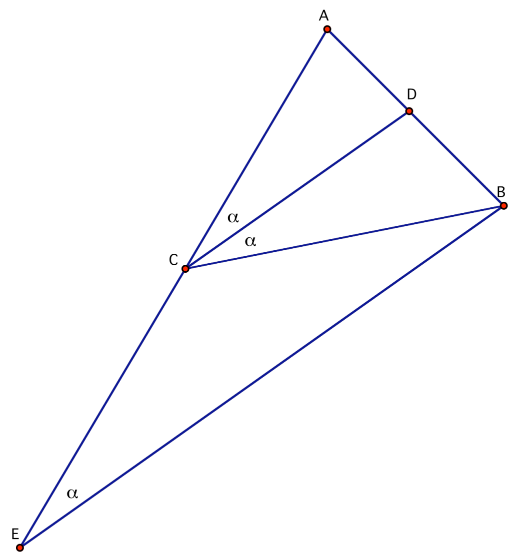


Megan: Oh this will be easy. We will use the AAA for similarity because CD bisects angle C and we know the angle bisector is perpendicular to the opposite side. Thus, we have $\Delta ADC \sim \Delta BDC$. So we know that $\frac{\overline{AD}}{\overline{AC}} = \frac{\overline{BD}}{\overline{BC}}$.

Olivia: But, that's not what I have written down for what we wanted to show.

Miguel: But, we can just cross multiply and divide accordingly and we will have $\frac{\overline{AD}}{\overline{BD}} = \frac{\overline{AC}}{\overline{BC}}$. Wow, that was easier than I thought it would be.

Olivia: Huh, something doesn't seem right. These problems aren't always that easy. What if we construct a parallel to $\overline{CD}$ at point B and extend side $\overline{AC}$. So, we have something like this.



Logan: Maybe, but how does that help? I think that Megan has the correct proof – remember sometimes it really is that easy.

The students notice you listening to their conversation and Miguel asks you if Megan's proof is correct and if not how does Olivia's diagram help? How do you respond?

<u>Angles and Isometry Task</u>: You just finished discussing transformations, isometries, and the idea of invariant in your high school geometry class. On a quiz you have the following item:

Prove that angles are invariant under any isometry.

Miguel provides the following proof.

Pf by contradiction: Suppose that angles are not invariant under a given isometry. Let A, B and C be three non-collinear points that form angle ABC. Now the three points form a triangle ABC and if we translate this triangle we get triangle $A'B'C'$ *which is congruent to triangle ABC by SSS. Thus,* $\langle ABC \cong \langle A'B'C'$ → *angles are invariant under an isometry.*

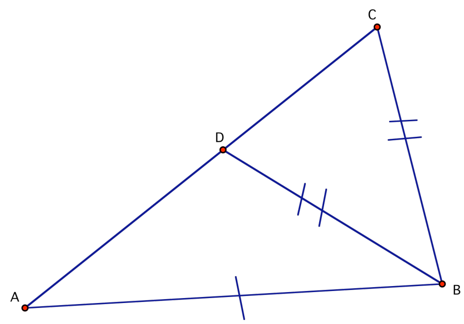
Is Miguel's proof correct, partially correct, or incorrect?

d. If correct, explain how it could be improved (if at all) and how you might relay this to Miguel.
e. If it is partially correct, explain aspects that are correct and incorrect and how you might relay this to Miguel.
f. If incorrect, explain what misconception(s) Miguel is portraying in his proof and provide a correct proof for Miguel.

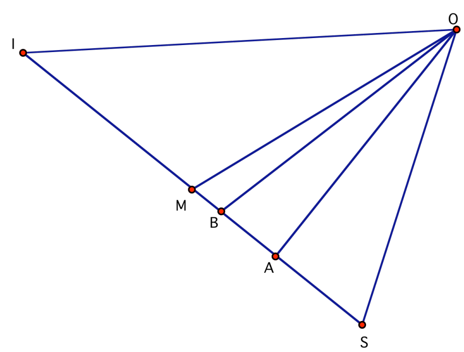
APPENDIX 2: NEW CLASSROOM SCENARIOS

<u>SSSS Task:</u> The SSS congruence property is sufficient to show that two triangles are congruent. Is it true that SSSS is sufficient to show that two quadrilaterals are congruent? Explain your answer.

<u>Part-Whole Task:</u> Paul looks at the figure below and claims that triangle ABD and triangle ABC are congruent by SAS. Is he correct? Why or why not? (Musser, Burger & Peterson, 2000, p. 687.) What misconceptions may Paul be bringing to this question?



<u>Altitude, Median & Angle Bisector Task:</u> Gwenette is looking at an isosceles triangle ($\overline{IS} \cong \overline{IO}$). She says, "I thought in an isosceles triangle, the altitude (A), the median (M), and the angle bisector (B) were all supposed to be the same line segment. Mine are all different." How would you respond? (Musser, Burger, & Peterson, 2000, p. 711)



<u>Angles and Isometry Task:</u> You just finished discussing transformations, isometries, and the idea of invariant in your high school geometry class. On a quiz you have the following item:

Prove that angles are invariant under any isometry.

Miguel provides the following proof.

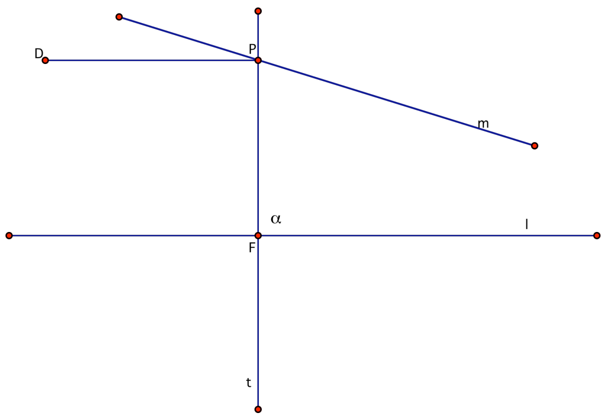
Pf by contradiction: Suppose that angles are not invariant under a given isometry. Let A, B and C be three non-collinear points that form angle ABC. Now the three points form a triangle ABC and if we translate this triangle we get triangle $A'B'C'$ which is congruent to triangle ABC by SSS. Thus, $\langle ABC \cong \langle A'B'C'$ → angles are invariant under an isometry.

Euclidean Parallel Postulate Task: You just finished discussing the Euclidean Parallel Postulate in class and required your students to prove statements are equivalent to the Euclidean Parallel Postulate. One of the tasks that you assigned was:

Prove the Euclidean Parallel Postulate is true iff if t is a transversal to l and m, $l \parallel m$, *and* $t \perp l$, *then* $t \perp m$.

Talisa provided the following proof.

Assume the Euclidean Parallel Postulate. WTS $t \perp m$, *when* $t \perp l$, *and* $l \parallel m$. *Suppose we have a line l and a line || to l called m. We also have a transversal called t ⊥ to l. Let's assume t is not ⊥ to m → we can create a point D such that* $\langle FPD \cong \langle \alpha$. *Since* $\alpha = 90°, \langle FPD = 90°$. *This implies* $\overleftrightarrow{PF}$ *is ⊥ to t. However,* $\overleftrightarrow{PD}$ *is also || to l by alternate interior angle theorem. This contradicts the Euclidean Parallel Postulate because only one line can pass through P and be || to l and m already does → that* $\overleftrightarrow{PD}$ *is = line m, and thus line m ⊥t.*



Assume $l \parallel m$, *and* $t \perp l$, *and* $t \perp m$. *WTS the Euclidean Parallel Postulate. Let's assume we can create another line n through P || to l.*
Since $t \perp l$, *this line must be ⊥ to t to be || to l by corollary 1 of the alternate interior angle theorem. However, m is already ⊥ to t and passes through P and perpendiculars are unique. Thus m must = n and there can be at most one line || to l through P.*