Proofs and Refutations as a Model for Defining Limit

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Introduction

The development of mathematical maturity in undergraduate students includes an important transition from the introductory Calculus sequence, which typically serves as the foundation for a tertiary-level study of mathematics, to upper-division courses which require the type of advanced mathematical thinking described by Tall (1992). Students' success in higher-level courses relies less on algorithmic applications, rote memorization, and procedural skills than it does on applying abstract proof techniques, developing and utilizing logical reasoning and justification skills, and cultivating a foundational understanding of the conceptual underpinnings of mathematics. In advanced courses there is a notable cognitive shift that must be made by the student to a conceptual, and much more abstract, perspective. Indeed, the hope is that one might simultaneously develop an up-close viewpoint of the "trees" (i.e., intricacies and subtleties of foundational definitions) and a much broader viewpoint of the "forest" (i.e., how the foundational definitions support important theorems that collectively comprise the respective mathematical branches of analysis, abstract algebra, and so on). Little research has addressed which pedagogical strategies might support students in formalizing their intuitive notions of mathematics concepts, particularly in the context of advanced calculus. Bezuidenhout (2001) notes, "An important challenge to mathematics educators is to create innovative curricula and pedagogical approaches that will provide calculus students with the opportunity to…reflect on the efficacy and consistency of their mathematical thinking" (p.7). The purpose of this paper is to describe how the proofs and refutations method of mathematical discovery espoused by Lakatos (1976) can be adapted as a design heuristic that supports certain types of mathematical defining. We present two illustrative episodes of student activity from a teaching experiment conducted by the first author.

The Method of Proofs and Refutations

In his well-known book, *Proofs and Refutations*, Lakatos (1976) retrospectively analyzes the mathematical activity of famous mathematicians, distinguishing between different methods of mathematical discovery. Lakatos describes the process of mathematical discovery as being initiated by a *primitive conjecture*, in which the behavior of a particular mathematical object may be characterized. For instance, an introductory calculus student's primitive conjecture may be that if a function has a local maximum at $x=a$, then $f'(a) = 0$. The validity of the conjecture is measured via a *proof*, which Lakatos describes as a thought experiment, consisting of a sequence of lemmas. Potential *counterexamples* to the primitive conjecture often emerge, and motivate the individual to respond in various ways. Lakatos describes *monster-barring* as a response wherein the individual dismisses a counterexample as illegitimate, arguing that it does not in fact satisfy the conditions of the hypothesis. Monster barring typically involves clarifying or refining the underlying definitions. For example, if the aforementioned calculus student were asked to consider the behavior of the function $f(x) = -|x| + 2$ on the interval [-1, 1], he may dismiss the counterexample, claiming that curves that have sharp corners do not qualify as functions, and thus cannot have local maxima. *Exception-barring* refers to the process of treating counterexamples as legitimate exceptions to the conjecture or theorem, but avoiding them by reducing the scope of the conjecture or theorem. In the previous example, the exception-barring student may alter the primitive conjecture to say that if a *differentiable* function has a local maximum at x=a, then $f'(a) = 0$.

Lakatos (1976) also describes a more mathematically mature method of mathematical discovery, known as *proofs and refutations*, in which the individual responds to counterexamples by way of *proof-analysis*. In proof-analysis, the individual analyzes the proof of the primitive conjecture to identify a potentially obscured sub-conjecture for which the counterexample is problematic. Proof-analysis may result in an improved conjecture that includes a new proof-generated concept. In the case of the calculus student, the absolute-value counterexample may cause him to recognize that absolute maxima can occur when a function's derivative equals zero or fails to exist, with the new proof-generated concept being that of a critical point.

In their 2007 paper, Larsen and Zandieh noted that in an environment "in which students see themselves as responsible for the development of the mathematical ideas, [the students'] mathematical activity may be strikingly similar to that of creative mathematicians" (p. 208). Larsen and Zandieh analyzed students' reinvention efforts in an undergraduate abstract algebra course, recasting Lakatos' descriptions of mathematical discovery as a useful framework for making sense of students' mathematical activity. In particular, Larsen and Zandieh analyzed the focus of their students' attention in response to counterexamples, as well as the outcome of the students' activity. Important distinctions arose. For instance, faced with counterexamples, monster-barring students made modifications to the underlying *definitions* of their conjectures, whereas exception-barring students acknowledged the counterexamples as legitimate, and subsequently modified their *conjectures* rather than the underlying definitions. Table 1 summarizes Larsen and Zandieh's findings.

Type of Activity	Focus of Activity	Outcome of Activity
Monster-Barring	Counterexample and underlying definitions	Modification or clarification of an underlying definition
Exception-Barring	Counterexample and conjecture	Modification of the conjecture
Proof-Analysis	Proof, counterexample, and conjecture	Modification of the conjecture and sometimes a new definition for a new proof-generated concept

Table 1 (Larsen & Zandieh, 2007)

The purpose of our work is to build upon the contributions of Larsen and Zandieh (2007), and to account for students' activity when the definition under consideration is in the foreground and rather than underlying some conjecture. In other words, our intent is to consider the case where the conjecture involved is a conjectured *definition*. As a result, we will be exploring in detail the monster-barring process and unearthing additional complexity in what is at first glance a fairly simple mathematical activity.

Reinventing Limit

Relatively little is known about how students come to reason coherently about the formal definition of limit. While some (e.g., Cottrill et al. (1996)) have provided conjectured models of student thinking about limits, there is a dearth of empirically-based research that traces student thinking to the point of coherent reasoning about the conventional ε-δ definition. In an effort to address this gap in the literature, the first author conducted a task-based Informal Limit Reasoning Survey with twelve undergraduate students from a large, urban university in the Pacific Northwest region of the United States. Each of the survey participants were students in two or more of the courses forming a three-quarter introductory Calculus sequence taught by the first author during the 2006-2007 academic year. Four of the twelve students were selected for two teaching experiments. Both teaching experiments consisted of ten, 60-100 minute paired sessions, and one 30-60 minute individual exit interview. The paired sessions were conducted in a classroom, with the pairs of students responding to instructional tasks on the blackboard in the front of the room. Only the participating students, researcher, and a research assistant were present for each session. Each session was generally separated by a span of 6-10 days, allowing time for ongoing analysis between sessions and subsequent construction of appropriate instructional activities based on the ongoing analysis. All sessions, including the individual exit interviews, were videotaped by a research assistant. These twenty-two videotaped sessions were the primary source of data for the study.

The four students selected for the two teaching experiments were chosen on the basis of possessing robust informal understanding of limit, as well as our estimation of their ability to work effectively in tandem to reinvent the definition of limit. Evidence of these criteria existed in the students' responses to the task-based survey, as well as in their written work throughout the three-term introductory Calculus sequence. In addition, students selected to participate in this study had demonstrated a greater effort and desire, relative to other students, to consistently make sense of their experiential world as it relates to complex mathematical ideas. Working with such students made it possible to trace student thinking to the point of coherent reasoning about the conventional ε-δ definition of limit.

Demographic background for the teaching experiment participants is as follows: one female and three males, with an age range from 19 to 28 years of age. None of the students had been exposed to the formal definition of limit prior to participating in the study. Additional background information is provided in Table 2.[1]

Name	Academic Major	Calc 1 Grade	Calc 2 Grade	Calc 3 Grade
Amy	Linguistics	A	A	A
Mike	Mathematics	A	A	A
Chris	Computer Science	A	A-	B+
Jason	Philosophy	A	A-	P

Table 2 – Background Information of Teaching Experiment Participants

In both teaching experiments, the central task was for the students to generate a precise definition of limit that captured the intended meaning of the conventional ε-δ definition. Instructional activities were primarily focused on discussing limits in a graphical setting, in hopes that the absence of analytic expressions might support the enrichment of the visual aspects of the students' respective concept-images. Tasks included the students generating prototypical examples of limit, which subsequently served as sources of motivation for the students as they attempted to precisely characterize what it means for a function to have a limit. The majority of each teaching experiment, then, comprised a period of iterative refinement for the students; as they attempted to characterize limit precisely, the examples and non-examples of limit that they encountered created cognitive conflict for them, which they sought to resolve by refining their characterization.

Proofs and Refutations as a Model for Defining

Our retrospective analysis of the data from these teaching experiments reveals that the students' attempts to mathematize their informal understandings of limit followed closely the four stages described by Lakatos (1976), and adapted by Larsen and Zandieh (2007). The

[1] Jason chose to take the third term of Calculus under the pass/no pass grade option because Calculus III was not a requirement for his major. His level of demonstrated understanding of the material in the third course was similar to that of the first two Calculus courses.

students' *primitive conjecture* in this case was that a particular number *L* is the limit of a function if and only if it satisfies their constructed definition. The students' *proofs* were sometimes implicit thought experiments, wherein they considered the extent to which their evolving definition of limit captured the mathematical behavior found in their self-generated prototypical examples of limit. These thought experiments led to the emergence of *counterexamples*, which played a significant role in the evolution of the students' definition.

Lakatos' (1976) description of *proof-analysis* suggests that it is a more sophisticated method of mathematical discovery, distinct from that of *monster-barring*. Larsen and Zandieh (2007) suggest, however, that, monster-barring may be fundamentally important in the context of defining.

> The method of monster-barring by itself could…be used to support students' defining activity. Here the students' attention would be focused on the counterexamples and the definition. For example, in their description of students' defining of triangle, Zandieh and Rasmussen (2007) identify the important role of non-examples of triangles during the negotiation of the definition; the students revised their definitions to bar these monsters (p. 215).

Zaslavsky and Shir (2005) echo these sentiments, reporting that students viewed the classification of examples and non-examples of a concept as one of the central purposes of mathematical definitions, commenting that "the students pointed to its power in 'refuting functions'" (p. 334). In this sense, the students in the study by Zaslavsky and Shir developed definitions of analytic concepts via a process of monster-barring.

Our analysis of the data in the two teaching experiments suggests that monster-barring played a significant role in the students' reinvention of a definition of limit capturing the intended meaning of the conventional ε-δ definition. An important distinction is worth making, however. While Larsen and Zandieh describe monster-barring as an activity that is focused on the underlying definition and the counterexample, we found that in our situation, the students' activity involved something akin to the method of proof-analysis. In the case of limit, the definition provides a recipe for proving that *L* is or is not the limit of a function *f*. We noticed that as part of their monster-barring activity, the students appeared to imagine the proving process suggested by their definitions and to analyze this process in order to refine these definitions. Thus, monster-barring was not a response to counterexamples mutually exclusive from that of proof-analysis. On the contrary, the students' monster-barring activity appeared to involve the method of proof-analysis. Specifically, in an effort to bar problematic counterexamples, the students engaged in proof-analysis, which spurred refinements to their definition. Their refined definitions were then tested against the prototypical examples of limit they had previously generated. The act of testing their refined definitions led to the emergence of new counterexamples, which inspired more monster-barring, and yet more proof-analysis. This iterative process ultimately supported the students in their reinvention efforts. We share with the reader illustrative examples from the first teaching experiment.

Example 1: A Dampened Sine Wave

In the process of reinventing the definition of *limit at a point* (i.e., what it means for a function f to have a limit L at x=a), the students (Amy and Mike) reached a point of diminishing returns due to two central cognitive challenges described in Swinyard and Larsen (submitted). In hopes of resolving these challenges in a less cognitively complex context, the first author shifted the students' focus to defining *limit at infinity* (i.e., what it means for a function f to have a limit L as $x \to \infty$). At the outset of the seventh session of the teaching experiment, Amy and Mike generated prototypical examples of limit at infinity in response to the following prompt.

> **Prompt:** Please generate (draw) as many distinct examples of how a function f could have a limit of 4 as $x \to \infty$. In other words, what are the different scenarios in which a function could have a limit of 4 as $x \to \infty$?

In response to the task, Amy and Mike drew the examples shown in Figure 1.

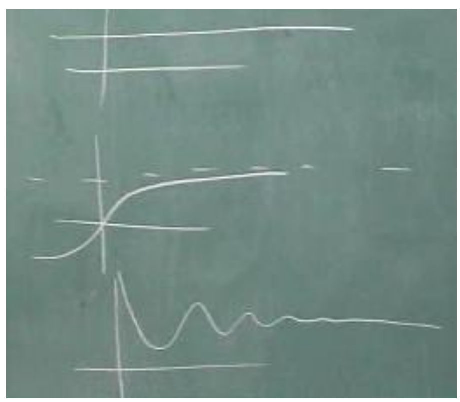

Figure 1 – Examples of Limit at Infinity

The students were then asked to construct a precise description/definition of what it means for a function f to have a limit of 4 as $x \to \infty$. In considering the examples drawn in Figure 1, Amy and Mike provided the following primitive conjectured definition: "On the interval (b, ∞) the function needs to approach some finite value L." Similar to the students in Zaslavsky and Shir's (2005) study, Amy and Mike subsequently "modified their definition to better reflect the concept image they held" (p.328), paying particular attention to describing what it means for a function to approach some finite value L. The presence of the prototypical examples on the board supported Amy and Mike in utilizing a method resembling proofs and refutations – after each modification to their definition they appeared to imagine the proving process suggested by their definition in light of the examples on the board. These *proofs* led to the emergence of counterexamples. For instance, after a modification to their primitive conjecture resulted in the refined definition "As *x* gets larger, the distance $|L\text{-}y|$ between *L* and your corresponding *y*-values continues to decrease," Amy recognized that the dampened sine wave shown in Figure 1 was problematic.

Craig: And you're saying that…as *x* gets larger, they continue to decrease, so the distance from your *y*-values to *L* gets smaller and smaller?

Amy: I don't know about that…Because I mean, obviously if it's going to equal – in this situation *y* is going to equal *L* quite frequently…when it crosses it, so – so that's no good…I mean the oscillations are tightening, but that's not a good way to describe it.

It is worth noting that in response to the dampened sine wave, Amy appeared to conduct something analogous to *proof-analysis*, in which she "intended to make the proof work rather than simply exclude the counterexample from the domain of the conjecture" (Larsen & Zandieh, 2007, p.208). Specifically, instead of reducing the domain of validity for her definition, Amy identified where her process for validating a limit candidate was breaking down so as to articulate why the counterexample was problematic. In this sense, Amy's attempt to bar the dampened sine wave as a counterexample (i.e., make it so that the dampened sine wave was ***not*** a counterexample) prompted proof-analysis. Amy's analysis led to further refinement of their definition, which, in turn, led to a different kind of counterexample.

Example 2: Eliminating Extraneous Limit Candidates

In response to the dampened sine wave function, Amy and Mike again refined their definition of limit at infinity in an effort to more accurately describe what it means for a function *f* to approach a finite value *L*. Amy suggested that "there needs to be some interval from *a* to ∞ where the function is continuous and…where the maximum distance between the *y*-values and *L* show a pattern of decreasing as *x* increases." After a moment's reflection, during which she appeared to check her suggested definition against the dampened sine wave, Amy recognized that extraneous *y*-values close to *L* would serve as counterexamples to such a definition.

Amy: Is this going to be enough? I mean, because what if, if we leave it at that, then isn't that true for like, for *L*…like bigger than or equal to 4? You know? Like doesn't that make it true for like, every single,

Mike: Hmm? Say it again.

Amy: So, okay, we need some interval from *a* to ∞ on which *f* is continuous and the maximum distances between *y*-values and some finite number *L* show a pattern of decreasing as *x* increases….[W]hat I'm having trouble with is just, is this [definition] specific enough to, like, to *L* being 4? I mean, isn't this thing that we just said also true for 5 and 6 and 9.2, because…on that interval, *f* is continuous, and the maximum distances between *y*-values…and 10 are also decreasing…So, I think maybe that's not quite specific enough...It's too broad.

The emergence of extraneous *y*-values as counterexamples was a watershed moment during the teaching experiment, for it ultimately led to Amy and Mike employing proof-analysis in a manner that resulted in them articulately defining limit at infinity. In response to Amy's identification of the problematic component of their definition (i.e., it was allowing for extraneous *y*-values within an ε-neighborhood of *L* to serve as limits), the first author prompted Amy and Mike to consider what it would mean for the function *f* to get *close* to *L*, with *close* being defined as within a single unit of *L*. Amy suggested bounding the function, which in turn supported her and Mike in addressing the extraneous *y*-values.

Amy: Well, I feel like it would be useful to talk about it being, being bounded….[W]hat if we were to say that there is some *y*-value that this function will never exceed, and there's some *y*-value that it will never get…smaller than, you know?

Craig: So if we wanted to…show that this function is within 1 of 4, what would your bounds be, then, Amy?

Amy: Uh, within 1, well, I guess that then it would be between 3 and 5.

…

Craig: Okay, if I wanted to be within ½ of 4, then what would my bounds be? Can you draw them?

Mike: Yup. So you'd have 4 ½ and 3 ½. [drawing the bounds shown in Figure 2]…[W]e know the limit isn't 5 anymore, because it's bounded by 4 ½ and 3 ½.

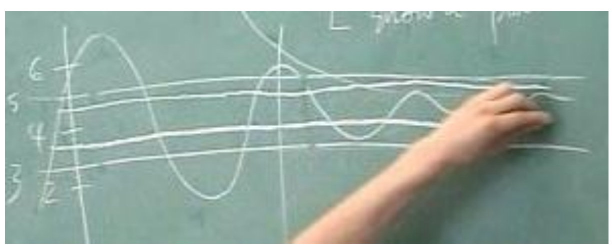

Figure 2 – Dampened Sinusoidal Function with *Close* = ½

As the conversation continued, Mike appeared to extrapolate the process of choosing iteratively restrictive definitions of *close*, and recognized that he and Amy had developed a new proof-generated process for eliminating extraneous limit candidates.

Mike: And we can keep doing that.

Craig: What do you mean we can keep doing that?

Mike: ...We can keep making our bounds closer and closer to 4, and the function will keep lying within those new bounds that we make….And so, with these new bounds around the limit now, we eliminated all these *y*-values as being a limit.

Craig: And you kept which one as your limit, then?

Mike: We still have 4.

Craig: But everything else won't work anymore?

Mike: Yeah. So we eliminated limits.

Mike's observation was followed by Amy subsequently recognizing that using the notion of *arbitrary closeness* allowed for the encapsulation of the infinite process of excluding extraneous *y*-values infinitely close to, but not equal to, the limit L. The notion of *arbitrary closeness*, then, served as a proof-generated concept for Amy and Mike, and resulted in the following definition: "It is possible to make bounds arbitrarily close to 4 and by taking large enough *x*-values we will find an interval (a, ∞) on which $f(x)$ is within those bounds."

Our analysis suggests, then, that Amy and Mike's reinvention efforts benefitted from iterative monster-barring and proof-analysis activity. Specifically, in an effort to bar problematic counterexamples, Amy and Mike engaged in proof-analysis, which spurred refinements to their definition. Their refined definitions were then tested against the prototypical examples of limit on the board. The act of testing their refined definitions led to the emergence of new counterexamples, which inspired more monster-barring, and yet more proof-analysis. This cyclic process ultimately resulted in a definition of limit at infinity capturing the intended meaning of the conventional ε-N definition.

Final Articulation of Limit at Infinity: $\lim_{x \to \infty} f(x) = L$ provided for any arbitrarily small positive number λ, by taking sufficiently large values of x, we can find an interval (a, ∞) such that for all x in (a, ∞), $|L - f(x)| \leq \lambda$.

A Heuristic for Supporting Defining

The nature of the discussions in which the students in the first author's study (Swinyard, 2008) engaged is reminiscent of the mathematical-social interactions characterized initially by Lakatos (1976), and more recently illustrated by Larsen and Zandieh (2007) within the context of abstract algebra. Although the instructional trajectory in first author's study was both heavily guided and strongly scaffolded, the students nevertheless were provided a context in which they could be mathematically creative. In a manner consistent with the dynamic set forth by Lakatos, at various times all four students took on the roles of conjecturer and refuter, seeking to build upon their informal understandings of limit by iteratively refining a self-constructed definition of the concept. Interestingly, upon reflection, the students acknowledged the uniqueness of this type of active role in mathematical learning. One particularly illustrative excerpt from the second teaching experiment highlights the potential benefits of engaging students in a learning environment that aims to stimulate authentic mathematizing, and provide them with an experience reminiscent of that of mathematicians such as Cauchy and Weierstrass in the historical development of analysis.

Craig: What did you guys accomplish in these past ten weeks?

Jason: …[W]hat did we accomplish? A level of mental gymnastics not encountered in any of my studies ever….[C]lass is usually taught in the, "I'm [the] lecturer, I'm going to tell you, now you learn this. And then recite it back to me later when I ask." And that's not what we were doing in these interviews….So I know that the limit is this value that the function intends to reach…Now deconstruct that into some fundamentals, some elements, and try to describe those elements, and try to put them together in a cohesive picture. So take what you know, and then figure out why you know that….Yeah, so, first and foremost [it] was getting the opportunity to do those mental gymnastics. Start at the real general level – height function intends to reach – getting down to the nitty-gritty specifics. And then, okay, the limit is this. And then having Craig go, well what if I just do this, and erase one dot that makes a critical change to what we now know? Well, now…we've got to get back into the nitty-gritty and get even more nitty and grittier. Restate, come back, and look for counterexamples….I've never had the experience of feeling like my study was more fruitful with others around….I can always get through the material more sensibly and more efficiently if it's just me dealing with the material. But that's probably because up until this point I've only ever been trying to understand the broad general level of any given subject.

Jason's response underscores the incongruence between his experience as a participant in the study and his previous experience with mathematics. All four students commented during the teaching experiments that the type of learning with which they were engaged was different than anything they had previously experienced in an academic setting. For instance, on more than one occasion, Jason commented that he and Chris were acting like "real mathematicians." We believe that a similar approach may pay dividends in developing an advanced calculus curriculum that could have a transformative effect on undergraduate mathematics education. We agree wholeheartedly with the stance taken by Tall (1992) –"[T]rue progress in making the transition to more advanced mathematical thinking can be achieved by helping students reflect on their own thinking processes and confront the conflicts that arise in moving to a richer context where old implicit beliefs no longer hold" (p.508).

We suggest that the careful construction of definitions that reflect robust concept images may be the type of mathematical activity conducive to students making the transition to more advanced mathematical thinking. The purpose of this paper is to illustrate how monster-barring (Lakatos, 1976) can be employed as a sophisticated design heuristic that supports certain types of mathematical defining. Based on our analysis of the data from the first author's research

(Swinyard, 2008), we offer the following suggestions to undergraduate mathematics educators interested in supporting students' defining efforts. First, prior to engaging students in mathematical defining, we suggest establishing that the students possess a strong concept image of the mathematical object of interest – indeed, without robust concept images, we doubt that students' construction of rigorous definitions at the advanced calculus level is even possible. Second, once a robust concept image has been established, we suggest having the students generate as many qualitatively distinct examples and non-examples of the mathematical object as possible. Doing so provides the students sources of motivation for iterative refinement of their conjectured definitions. Third, as conjectured definitions emerge, engage students in "running" the verification process described by their definitions for the previously generated examples and non-examples. By analyzing these thought experiments, the students stand to gain insight into how to refine their definitions to bar the offending monsters. Thus, monster-barring and the powerful method of proof-analysis can unfold in tandem – as students aim to bar problematic monsters, their proof-analyses can motivate refinements to their conjectured definitions, which they can check against their prototypical examples and non-examples of the object. These checks can lead to the emergence of more monsters, which the students can address via more proof analysis.

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