Investigating Student Understanding of Physics Concepts and the Underlying Calculus Concepts in Thermodynamics

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Abstract

As part of work on student understanding of concepts in advanced thermal physics, we explore student understanding of the mathematics required for productive reasoning about the physics. By analysis of student use of mathematics in responses to conceptual physics questions about thermodynamic work, as well as analogous math questions stripped of physical meaning, we find evidence that students often enter upper-level physics courses lacking the assumed prerequisite mathematics knowledge and/or the ability to apply it productively in a physics context. These results suggest advanced students are not incorporating calculus and physics into a coherent framework. We have extended our work to include assessment of mathematical concepts at the end of a multivariable calculus course. Results support the findings among physics students that some observed mathematical difficulties are not just with transfer of math knowledge to physics contexts, but seem to have origins in the understanding of the math concepts themselves.

Introduction

As researchers trained to investigate the learning and teaching of physics, the authors found themselves faced with questions about how students think about calculus concepts that were essential for a functional understanding of the content in an upper-level thermodynamics course. The answers to the specific questions we were interested in were not available in the literature of either discipline-based research field, which prompted our writing questions and has led to our participation in this conference.

The topic of thermodynamics covers a wide-ranging area that is fundamental in nature, and is utilized in many branches of physics, engineering and other natural sciences. The published research at the university level, albeit scarce, clearly indicates that students exhibit significant difficulties when learning thermal physics topics, including conceptual difficulties with heat, temperature, the Ideal Gas Law and the First Law of Thermodynamics (Rozier and Viennot, 1991; Yeo and Zadnik, 2001; Jasiem and Oberem, 2002; Loverude et al., 2002; Meltzer, 2004; Kautz et al., 2005a; Kautz et al., 2005b).

The First Law of Thermodynamics connects energy transfers—that is, energy entering or leaving the system of interest—to changes in the internal energy of that system during some thermodynamic process (e.g., the compression of a gas in a cylinder kept at fixed temperature). The internal energy of the system is a property of the equilibrium state of that system, i.e., it depends on quantities such as the temperature, pressure, and volume of the system. Internal energy, and other quantities with similar features, are known as *state functions*. The energy transfer quantities in the First Law, work (mechanically transferred energy) and heat (thermally transferred energy), on the other hand, are not state functions; they only exist when a system is undergoing some sort of process. The important point here is that changes in state functions are independent of the thermodynamic process—one only needs to know the initial and final states to determine the change in internal energy of the system, for example. In contrast, the amount of work done on (or by) the system does depend on the process. Previous results regarding student understanding of the First Law have documented several common conceptual difficulties, such as indiscriminate application of the concept of a state function to non-state function quantities such as work and heat (Loverude et al., 2002; Meltzer 2004).

For several years, members of the Physics Education Research Laboratory at the University of Maine (UMaine) have been exploring student learning in upper-level thermal physics courses, primarily taken by physics majors, extending the tools and results of research in introductory physics to the more specialized upper division. There exists a limited body of research in this area (Loverude et al., 2002; Meltzer, 2004; Meltzer, 2005b; Thompson et al., 2006; Cochran and Heron, 2006; Bucy et al., 2007; Mountcastle et al., 2007; Pollock et al., 2007; Christensen et al., 2009; Loverude 2009; Smith et al., 2009). Meltzer (2004) has suggested that particular difficulties experienced at the introductory level are also evident at the advanced undergraduate level. In thermal physics, as with most physics areas, specific mathematical concepts are required for a complete understanding and appreciation of the physics.

Our main research agenda for the learning and teaching of thermal physics deals with identifying and addressing student conceptual difficulties with the physics content. A sub-theme of this research, highlighted here, investigates the extent to which any mathematical conceptual difficulties may affect students' understanding of associated physics concepts in thermodynamics. In this area, as with many physics areas, specific mathematical concepts are required for a complete understanding and appreciation of the physics. Our research question in this context is: To what extent do students recognize, and/or understand, the relationship between the physics concepts and the underlying mathematics?

Mathematics is a vital part of solving many physics problems, often used to condense a complicated conceptual problem into a relatively simple relationship between variables. Many convenient representational tools exist in physics (e.g., equations, graphs and diagrams) that simplify analysis of a complex problem. Appropriate interpretation of these representations requires recognition of the connections between the physics and the mathematics built into the representation and subsequent application of the related mathematical concepts (Redish, 2005).

Meltzer (2002) has shown a link between mathematical acumen and success in an algebra-based physics course. Only a handful of studies in PER have attempted to investigate mathematical difficulties with calculus concepts among physics students (Thompson et al., 2006; Black and Wittmann, 2007; Bucy et al., 2007; Pollock et al., 2007; Rebello et al., 2007.)

In thermodynamics, two- or three-dimensional graphical representations of physical processes are especially useful in helping to understand these processes. These diagrams can contain information about the thermodynamic “path” followed in a process, regions of different phase, and critical behavior. Graphs of pressure *versus* volume, known as *P-V* diagrams, are used extensively as representations of physical processes as well as of the corresponding mathematical models. Information can quickly and easily be obtained from these representations. For example, the work done on a system undergoing a thermodynamic process is defined as the integral of pressure with respect to the volume:

$$\mathrm{W} \equiv \int P\, dV$$

Thus physicists can quickly determine the work done by a calculation, if the function $P(V)$ is known for a given process. Qualitative determinations can be made by evaluating the area under the curve for a particular process.

Due to the physics content areas that we investigate, we have focused on student understanding of pressure-volume (*P-V*) diagrams in the context of determining quantities related to the First Law of Thermodynamics (Pollock et al., 2007), and on student understanding of relationships between material properties using mixed second-order partial derivatives (Bucy et al., 2007). We sought to isolate mathematical difficulties that may underlie observed physics difficulties by asking physics questions that are completely stripped of their content—we call these “physicsless physics questions”—and focus on the calculus concepts under investigation (i.e., integration and partial differentiation). We administered physics questions as well as the analogous math questions to the students in our thermodynamics course. A comparison of student responses and reasoning on these paired questions allows us to pinpoint the domain(s) in which confusion may lie.

Brown et al. (1989) provide a cognitive model in which students frame activities based on the environment in which they are located. Tuminaro and Redish (2007) have coined the term *epistemic framing* to describe students’ expectations in different pedagogical situations. These expectations are cued by the setting, explicitly and implicitly. Epistemic framing answers the question “What are we going to do here?” or “What do I need to be successful here?” It may be that students exhibiting mathematical difficulties were not able to access their ideas about the mathematical concepts appropriately in a physics classroom. With the long-term goal of improving student learning through effective curriculum implementation, it is imperative that we ascertain whether these students possess the requisite calculus knowledge or if they are simply unable to access and use that information effectively in a physics classroom. To explore this question of transfer, we developed and administered a series of questions as a brief survey in a multivariable calculus course (the final prerequisite math course for thermal physics) after all relevant instruction.

The survey we administered consisted of six primary questions, some with subquestions. The first three questions were math analogs – the physicsless physics question companions – to physics questions asked in thermodynamics. The last three covered the more basic concepts of derivative, slope, and the product and chain rules of differentiation. In this paper we discuss the results from the first integral question and those on slope and derivative.

Context for research

The data are collected from two primary sources. The results on the physics questions come from students in UMaine's upper-level *Physical Thermodynamics* course, taught in the fall semester. The course meets for three 50-minute sessions each week, for 14 full weeks. This course has been taught by the same instructor for more than ten years, using the same textbook (Carter, 2001) for more than five years. Students turn in homework approximately once every two weeks and are given two cumulative exams and a comprehensive final exam during the semester. The class size is small, usually between 8 and 15 students. We have administered our questions in 5 semesters of this course, yielding paired data from a total of 35 independent students.

The majority of the students in thermodynamics are either physics or engineering physics majors, with an occasional math or earth sciences major and an occasional physics graduate student. For many students in the class, this is their first exposure to thermal physics in college, since UMaine's introductory calculus-based physics sequence does not cover thermal physics. Introductory chemistry, which typically covers the First Law and related topics, is not mandatory for some of the students in the course. However, all students in the data set had completed three semesters of calculus, including multivariable calculus, as well as a course in ordinary differential equations; some students had taken additional mathematics courses.

We also have data obtained in the last week of instruction in *Calculus III,* which includes multivariable differential and integral calculus. Total populations of the course are on the order of 100-150 students, typically taught in lecture sections of 35-50 students. The course meets in four 50-minute lectures each week, and is taught in both the Fall and Spring semesters. Students in this course are typically majoring in a physical science, engineering, or mathematics. The responses from calculus surveys are not matched across all questions.

Research perspective

We are primarily interested in analyzing student understanding of the physics content. We view our research data using the idea of *specific difficulties* (Heron, 2003) with the material to describe student reasoning. These difficulties can manifest themselves as incorrect or inappropriate ideas expressed by students, or flawed patterns of reasoning to specific questions. We start with targeted, context-dependent results and then generalize across contexts and seek larger patterns of student responses in our data. While we are familiar with more cognitive models from both physics (e.g., Hammer and Elby 2003) and mathematics (e.g., Schonfeld 1992) education research traditions, we are not currently considering cognition in our analysis. Fundamentally, our emphasis is on gathering and interpreting empirical data that can act as a foundation for future studies on reasoning in advanced physics and curriculum development to address student specific difficulties.

Student Understanding of the Physics and Mathematics of Thermodynamic Work as Represented on *P-V* Diagrams

Background

Meltzer (2004) developed a set of questions probing student understanding of the First Law and related quantities based on processes shown on a *P-V* diagram, relating to findings by Loverude et al. (2002). In answering these questions students are expected to recall that work is the integral of *PdV*, and thus the area under the curve is associated with the work done by the gas. Both Meltzer and Loverude et al. reported on student conceptual difficulties with the physics concepts, but we wondered if some of the conceptual difficulties might originate in the confusion over the mathematics at play. In addition to replicating Meltzer's experiment in our upper-level thermodynamics course, we designed qualitative questions regarding comparisons and determinations of the magnitudes and signs of integrals without any physics context that are analogous to Meltzer’s questions.

We are aware of a limited body of work pertaining to the topic of student understanding of integral calculus concepts in mathematics education research. Previous findings of which we are aware indicate that students do not possess the necessary concept image of an integral, definite or indefinite, to allow them to successfully complete problems involving concepts of integration, in particular an understanding of the relationship between a definite integral and the area under the curve, the ability to find areas under the curve when the curve crosses an axis, and the general relationship between a definite integral and the area under the curve (Orton, 1983; Vinner, 1989; Thompson, 1994; Bezuidenhout and Olivier, 2000; Rasslan and Tall, 2002; Grundmeier et al., 2006; Sealey 2006; Thompson and Silverman, 2007). We should point out that while the term *concept image* is not used in the physics education research community, we do similarly describe student ideas about a particular concept.

Comparison of Work and of Integral Magnitudes by Physics Students

We have previously reported on comparisons of student responses and reasoning for one pair of questions dealing with the physics and mathematics of work in thermodynamic processes (Pollock et al., 2007). Based on results from Loverude et al. (2002), Meltzer (2004) asked students to compare energy transfers and changes (work, heat transfer, internal energy changes) for two ideal gas processes (#1 and #2), shown on a *P-V* diagram, that have the same beginning states and the same ending states (A and B, respectively; Fig. 1(a)). For the work question students are asked to compare the works done by the gas in the two processes. The correct response can be obtained by recognizing that the work done by the gas is defined by the integral of the pressure with respect to the volume, $\int PdV$, and that this integral is represented by the area under the curve for that process.

We paired Meltzer's work question with an analogous mathematics question (Fig. 1(b)), designed to elicit any underlying mathematical difficulties in this context. In this "integral question," we asked students about the magnitudes of integrals of two functions that have identical beginning and end points. The two thermodynamic processes are replaced with two different functions. We explicitly chose to label the starting and end points as "*a*" and "*b*" just as was done for our states on the *P-V* diagram, even though this is not a typical mathematics convention. We asked the math question usually in the first week of class, and the physics question in the second or third week. We also interviewed students to get more information on our written results.

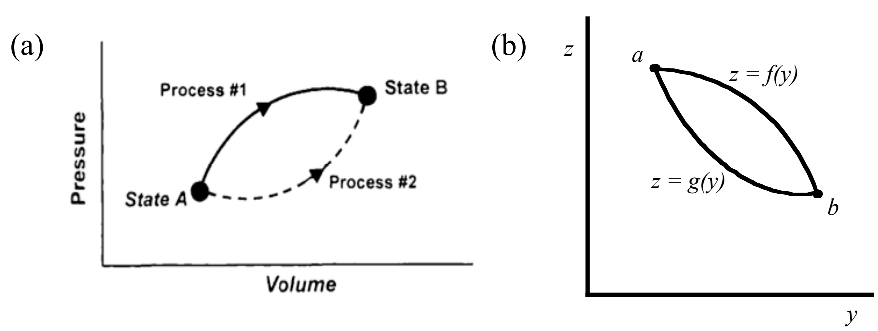

Figure 1. Diagrams used on questions about (a) comparing work done by identical systems over two different thermodynamic processes, from Meltzer (2004); (b) comparing the values of the integral of the two functions shown over the same interval.

The paired results are shown in Table 1. About half of the students gave correct responses to both questions. The next most common response pair was that the integrals were different (correct) but that the works were equal, given by one fifth of all students. All 7 students used an *area under the curve* argument for the integral comparison. The reasoning on the work comparison question, however, dealt explicitly with physics concepts. Some stated explicitly (incorrectly) that work is a state function. Some wrote that work was path independent—a reasonable idea in the context of conservative forces as typically taught in introductory mechanics; in fact, one student explicitly wrote “assuming conservative force.” Another student assumed “zero dissipative processes,” which is another reasonable-but-incorrect application of ideas learned at the introductory level.

Student Response Pairs	Physical Thermodynamics N=35 (N_{class} = 5)
$I_1 > I_2$ and $W_1 > W_2$	18 (51%)
$I_1 > I_2$ and $W_1 = W_2$	7 (20%)
$I_1 = I_2$ and $W_1 = W_2$	5 (14%)
other	5 (14%)

Table 1. Results from paired responses to integral comparison question and work comparison questions.

Another group of students equated the integrals on the math question and equated the works, indicating conceptual difficulties with the math and possibly, but not necessarily, the physics.

We have found that students were less likely to provide detailed explanations for their responses at the upper level than at the lower level. So in addition to the written data, we conducted individual student interviews to get more information on reasoning used to answer these questions. Eight students were interviewed on the integral comparison question, five before any instruction in thermodynamics (e.g., from the sophomore class of physics majors) and three after all instruction in thermodynamics. The five “pre-thermodynamics” students were also asked the work comparison question. In general, the interview results supported the written data: similar explanations were given with little additional insight into the reasoning, especially with regard to *state function* and *path independence* reasoning for the physics question. However, one additional line of reasoning came to light in the interviews in both contexts. Two student statements are given below:

> *"I guess that the top is the same. It's the same curve in reverse. So I am going to go with equal. ... symmetry across the line through a-b."*

> *"they are in fact equal to each other... And while the processes may be different, uhh, they both reflect about the straight-line going between point a and point b and therefore they must be symmetric."*

These students relied on the symmetry of the graphs to decide that the integrals/works were equal. These arguments suggest that students are relying on the length of the path and not the area under the curve in these cases to determine the integral value. We saw no evidence of this reasoning on the written responses in physics; however, it may be that this reasoning was tacit in the cases where no reasoning was given, and may be more prevalent than we thought.

Overall, the results from these questions indicate that some of the difficulties that arise among physics students when comparing thermodynamic work based on a *P-V* diagram—e.g., treating work as path independent—may be attributed to difficulties with the mathematical aspect of the diagram, in particular with the correct application of an understanding of integrals or a misinterpretation of the graphical representation, rather than simply physics conceptual difficulties (e.g., treating work as a state function).

Results from Multivariable Calculus

The integral comparison question was asked on the survey in the multivariable calculus course described above. Just over 60% of students (22/35) in the thermodynamics course provided correct responses, and just over half (98/183) of the calculus students did so. In both data sets, around one quarter of students said that the integrals were equal. Endpoint-based reasoning was similar in both courses. However, in the calculus course, some students gave written responses that either explicitly or implicitly used symmetry to explain why the integrals were equal, implying that path length was important, similar to the reasoning from the physics student interviews described above.

In fact, when presenting the integral comparison question to researchers in undergraduate mathematics education (at this meeting), some indicated that their initial reaction was to consider line integrals, since the figure labels the points on the graph (points *a* and *b*) rather than on the axis (y_a and y_b or $y=a$ and $y=b$ as the endpoints of the integral). While we used this hybrid representation deliberately—distinguishing this as a physicsless physics question rather than strictly a mathematics question—the point is well taken. To follow up on this comment, we intend to vary features of the graph and investigate any differences in responses. We have already made the graphs asymmetric in order to eliminate the symmetry argument altogether: in both cases the lower curve is now longer than the upper curve, so that *path length* reasoning should yield a different response (i.e., $I_1 < I_2$) than *area under the curve* reasoning. We have not received sufficient data to make any claims about these modifications as of yet.

As mentioned earlier, the RUME literature contains evidence of student difficulties with integration, both definite and indefinite. Our questions removed the opportunity for students to calculate an integral explicitly, which makes comparisons to the RUME results difficult. Our sense of the literature is that the concept image of an integral as area under the curve is problematic in several situations in mathematics; Thompson and Silverman (2007) suggest that that image may limit the applicability of the conception of integrals in different contexts. Thompson and Silverman advocate for the idea of the *accumulation function* to connect integration (and the Fundamental Theorem of Calculus) to rates of change (i.e., the derivative), and to redirect student thinking back to fundamental ideas of *function*. In our opinion, the predominant concept image of integrals in physics is as the area under a curve—even a negative area. To our knowledge there is no evidence in the PER literature of students failing to recognize negative area in a physical context. One possibility is that researchers did not know to look for it, so have not seen it. It may also be that the ubiquity of graphical representations in physics, and the frequent analysis of integrals of curves with different units and different physics contexts, somehow affects students to recognize areas “under” functions below the axis as negative quantities (e.g., moving backwards leads to less displacement, negative acceleration leads to a less positive velocity, etc.). Note that in thermodynamics, pressure and volume are always positive (until one deals with cavitation or other advanced fluid dynamics where “negative pressures” are discussed), so we are not likely to see evidence of *area*-based difficulties of that nature in our students. While the *area* model is predominant, it may be that physicists also use *accumulation* in analysis involving integrals, since integrals of the functional form of the accumulation function described by Thompson and Silverman are prevalent in physics. It is worthy of an interdisciplinary investigation to explore the concept images of integrals in physics in general, across physics contexts.

We have a number of additional tasks planned for future study in this area. We aim to probe student thinking in a similar context of physicsless physics questions about integration that do not rely on the pictorial representation of the *P-V* diagram. Multiple variations (like those highlighted above) on both our math and physics questions will be explored as well, in some cases making the questions more like the types of math questions students are used to seeing, and in some cases more like the physics questions. Furthermore, we have probed student thinking in a similar thermodynamics context by asking about the change in internal energy, which, unlike work, does *not* depend on the process that the goes undergoes, as well as an analogous mathematics question. We intend to report on those results in a future publication.

Intermediate mathematics students' responses to questions about slope and derivative

Among the earliest findings in the physics education research literature are those difficulties reported by Trowbridge and McDermott concerning student understanding of kinematics (Trowbridge and McDermott, 1980; Trowbridge and McDermott, 1981). A significant portion of this work was done through the analysis of student ideas about graphical representations of various kinematic processes and was followed a few years later by the work of McDermott et al. (1984). Beichner developed a multiple-choice survey to assess student knowledge of graphical representations in the context of kinematics called the Test of Understanding Graphs – Kinematics (TUG-K) (1994). Beichner confirmed a number of items that had been previously reported by McDermott et al., including "slope/height/area confusion" in the context of kinematics among students in the introductory sequences.

We decided to include physicsless physics questions about concepts of slope and derivative on our math survey in order to shed additional light on the student thinking regarding these concepts without the burden of the physics context. As described above, students in thermal physics were struggling with fundamental ideas of integration; we also have evidence of student difficulties with partial derivatives in other thermodynamics contexts (Thompson et al.

2006; Bucy et al. 2007). Given these results, we thought it best not to make any assumptions about the understanding of our students on slope and (single-variable) derivative without assessing them. The result was two questions that are very similar to questions from the PER literature, but stripped of any physics content: the Slope Ranking Task and the Derivative Sign and Ranking Task.

The Slope Ranking Task (Fig. 2(a)) requires students to order the slope of the drawn function at four different values of *x*. The question attempts to dissuade ranking absolute values and contains a great deal of language explaining what the form of the desired response is. A correct response on the Slope Ranking Task requires students to associate slope with the steepness of the curve at the four given points.

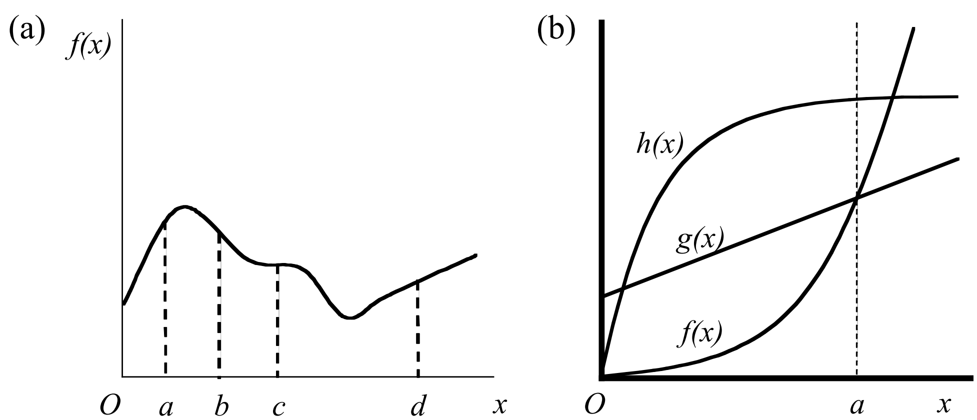

Figure 2. (a) Slope Ranking Task; (b) Derivative Sign and Ranking Task.

Over three semesters, roughly 15% of students were unable to complete this task successfully, less than half provided any reasoning for their correct rankings (see Table 2). The most common incorrect response (accounting for one fifth of incorrect responses) was a ranking that is consistent with the average slope between points rather than the instantaneous slope at a point. This type of confusion has been documented in the PER literature in kinematics (Beichner, 1994; Shaffer and McDermott, 2005): students interchanging average and instantaneous velocity for objects that are not experiencing constant acceleration.

Correct Responses	Fall 2007 (*N* =70)	Spring 2008 (*N* = 31)	Fall 2008 (*N* = 37)
Slope of *f(x)* @ (*a*>*d*>*c*>*b*)	79%	90%	89%

Table 2. Correct responses on the Slope Ranking Task for students after all instruction in a Calculus III course.

A few students (<5%) gave a ranking consistent with the value of the function at each point rather than the slope at each point. This confusion has been previously reported in the PER literature by Beichner in the context of graphical interpretation of kinematics (1994). It is unclear to what extent the errors reported by Beichner may occur due to student confusion with mathematical understanding rather than the physical context. Additionally, PER has many examples of students confusing a quantity and its rate of change, whether in a graphical context or in the context of a physical demonstration, most notably confusion between position and velocity (Trowbridge and McDermott, 1980) and between velocity and acceleration (Trowbridge and McDermott, 1981).

The Derivative Sign and Ranking Task (Fig. 2(b)) asks students to determine the signs of and to compare the magnitudes of the derivatives of three different functions of the independent variable x at the same value of x, based on a set of graphs of the three functions. This question requires students to make a connection with a derivative and either the slope or the change in the function, which must then be read off the graph. This task was specifically written to overlay the assessed concept in the previous question, but also to differentiate between students who could rank the slopes of a line but might not be able to connect “derivative” with the slope of the line.

The curves of the three functions were drawn to allow common incorrect reasoning to be more clearly determinable.We found that more than half of the students were able to state that the derivatives for all three functions were positive (see Table 3). Due to the potentially misleading flatness of curve *h(x)* some students stated the derivative at $x = a$ was zero or not determinable. A more prominent curve was added in place of *h(x)* after the first administration which seems to significantly eliminate this "alternative" correct explanation.

	Fall 2007 (N =82)	Spring 2008 (N = 34)	Fall 2009 (N = 39)
	Original Version	Modified Version	
All Correct Responses	71%	62%	77%
Preferred Correct Response *df/dx, dg/dx, dh/dx* are postive	51%	62%	69%
Alternative Correct Response *df/dx, dg/dx* positive*; dh/dz* zero or nei	20%	0%	8%
***2nd derivative* responses**	7%	18%	13%

Table 3. Response rates for Derivative Sign and Ranking Task. Percentages in the All Correct Responses row are the sum of the Preferred and Alternative Correct Responses; *2nd derivative* responses are those students whose sign choice is consistent with the 2nd derivative, but not necessarily identified by the student as a *2nd derivative* answer.

The most common incorrect response on the original version (7%) was a ranking consistent with that for the values of the 2nd derivative of these curves. We were concerned that the question wording may have been unclear. In response, we altered the language to more clearly indicate we sought the derivative of the functions and gave the expressions for the derivatives in the response area (e.g., $\left.\frac{df}{dx}\right|_{x=a}$) for emphasis. However, the Spring 2008 students gave *2nd derivative* responses at a much higher rate (18%) with the modified version, implying that the question wording and presentation were not the issue, and that this is still a significant difficulty for students in Calculus III.

We can cast additional light on the thinking of those students that gave responses consistent with *2nd derivative* reasoning by examining their responses to the Slope Ranking Task. Nearly all (95%) of these students gave a correct ranking on the slope question. Therefore we can hypothesize that these students are able to make sense of the slope of a curved surface, but do not match the idea of derivative with slope of the curve. One possible explanation (among many) would be that a student might carry two notions of derivative: that of the change of the function and that of the slope of the function. Students may use one notion or the other as the student sees fit in a given context, but may, at times, use them simultaneously. Thus, a question about derivative might cause them to think of the "change in the slope" and give a *2nd derivative* response.

Most students who gave a correct sign for the derivatives in part a) correctly ranked the values of the derivatives in part b). There were some instances (<5%) of student rankings that were consistent with a ranking of the areas under the curves of the functions, with at least two instances of students explicitly justifying their answers as "areas under the curve." If this question were given in a kinematics context, researchers would likely identify student confusion with kinematics; instead, this physicsless physics question points to at least a few students who use notions about area to answer questions about derivatives.

Conclusions

By analysis of student use of mathematics in solutions to conceptual physics questions involving integration as well as analogous math questions stripped of physical meaning, we find evidence that students often enter upper-level physics courses lacking the necessary (and assumed) prerequisite mathematics knowledge and/or the ability to apply it productively in a physical context. Taken as a whole, these results point to difficulties among advanced students in incorporating calculus and physics into a coherent framework.

Preliminary results from questions about integrals and about slopes and derivatives administered in a multivariable calculus course suggest that the observed mathematical difficulties are not just with transfer of math knowledge to physics contexts. Some of these difficulties seem to have origins in the understanding of the math concepts themselves.

Recent discussions suggest that it may be possible that this confusion may be due in part to the differences in the canonical representations used in physics and in mathematics. This aspect of our results will be explored further in future research.

We are continuing to collect and analyze data from written questions and interviews, and to expand the scope of our investigation to additional populations at the introductory level, both in physics and in mathematics.

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