Student Problem-Solving Behaviors: Traversing the Pirie-Kieren Model for Growth of Mathematical Understanding

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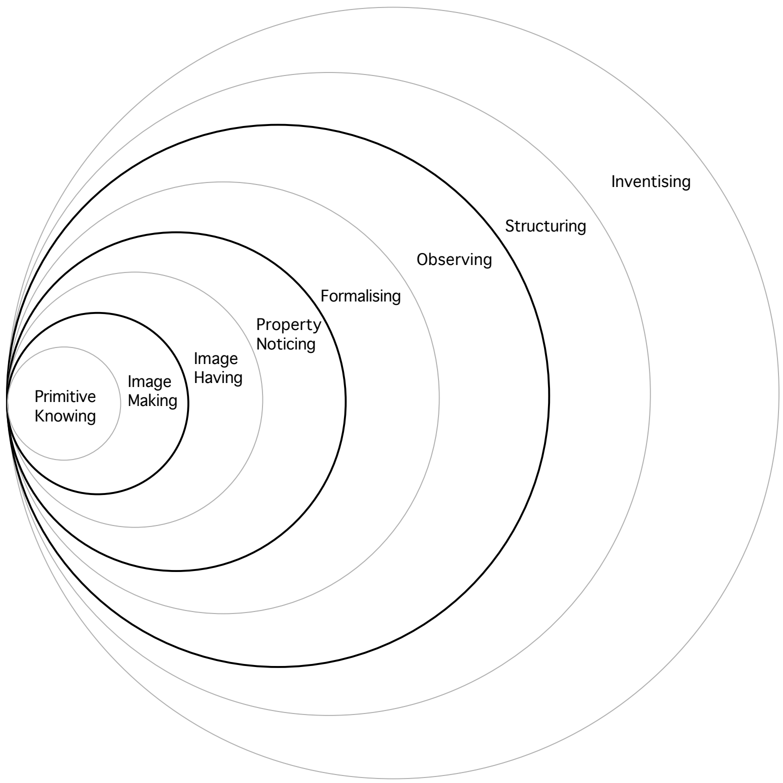
Introduction: Recent curricula emphases, which highlight the importance of thinking, communicating and instilling deep understanding in students, have encouraged the emergence of problem solving as an important mechanism for enhancing mathematics learning (Muir, Beswick & Williamson, 2008). Problem solving "is central to inquiry and application and should be interwoven throughout the mathematics curriculum" (National Council of Teachers of Mathematics, 2000, p. 256). When given opportunities to grapple with appropriately challenging tasks, students engage in problem solving to develop mathematical knowledge. The growth of understanding of elementary, secondary and university students has been well documented with regard to quadratic functions, taxicab geometry, fractions and combinatorics (Borgen & Manu, 2002; Martin, 2008; Pirie & Kieren, 1994; Warner, 2008). Several studies have focused on open-ended tasks (Empson, 2003; Speiser, Walter & Maher, 2003; Vega & Hicks, 2009).

However, there remains a need for detailed investigations into university students' growth of understanding of Taylor Series through task-based inquiries in calculus. The purpose of this qualitative study is to identify behaviors in which university calculus students engage during mathematical problem solving, and how those behaviors relate to growth of students' mathematical understanding. The Pirie-Kieren Model (1994) provides a framework for our detailed analysis of how certain problem-solving behaviors contributed to the growth of one university student's mathematical understanding of Taylor Series. Here, we focus on the mathematical problem solving activities of a student, Heber, as he worked collaboratively with his classmates on a task designed to facilitate the emergence of Taylor Series.

Theoretical Perspective: Student behaviors identified by Warner (2008) that appear to be associated with the growth of mathematical ideas and student understanding include building and explaining ideas, offering new ideas, questioning ideas, showing ideas are valid, reorganizing ideas, using multiple representations and creating new questions. Working together in groups, students have opportunities to build, offer and explain ideas to each other, determine which ideas are valid, reorganize ideas, represent understandings, provide justifications, and generate generalizations.

Pirie and Kieren (1994) developed a theoretical model for observing and describing processes through which students' build mathematical understanding. Pirie and Kieren argue that building understanding is not a linear process, but is a dynamical movement through eight embedded layers of understanding. Although the layers of understanding in the theoretical model build outward, growth in understanding occurs as students continue to work within and move back and forth through different layers of understanding (Martin, 2008). Here, we focus on four layers of understanding through which Heber moved, or traversed, that are evident in our data: *formalising*, *observing*, *structuring* and *inventising*. Students in the *formalising* layer of understanding are able to abstract methods based on previous images and develop formal mathematical ideas. In the *observing* layer, students question how formal statements about a concept are connected and look for patterns and ways to define their ideas as algorithms or theorems. *Structuring* is when students begin to think about their formal observations as theory and start to make logical arguments in the form of proof. The *inventising* layer occurs when a student has gained full understanding of a concept and can pose questions that may lead to new concepts (Pirie & Kieren, 1994, p. 171).



The Pirie-Kieren model for the growth of mathematical understanding (Pirie & Kieren, 1994, p. 167)

When students are working in an outer layer in the Pirie-Kieren model, they may be confronted with a problem that prompts them to fold back to an inner layer in order to build on existing knowledge and then move forward with their mathematical understanding (Borgen & Manu, 2002). Folding back "allows for the reconstruction and elaboration of inner level understanding to support and lead to new outer level understanding" (Pirie & Kieren, 1994, p. 172). Ideally, the result of folding back is that if a student's existing constructs were insufficient to solve a problem, then the student is able to extend incomplete understanding by developing earlier constructs which may lead to new ideas or concepts (Martin, 2008).

Another important element of the Pirie-Kieren model is the notion of "don't need" boundaries, which occur between three different layers of understanding (the darker rings in the model). Beyond these boundaries, a student may work with ideas that are not necessarily "tied to previous forms of understanding, but these previous forms are embedded in the new level of understanding and readily accessible if needed" (Pirie & Kieren, 1994, p. 173). Here, based on analysis of student behaviors, we focus on the boundary between the *observing* and *structuring* layers of understanding. When students are working in the structuring layer they think of their formal observations as theories and do not necessarily need to reference the specific images that produced those observations in the inner layers of understanding.

Generally, after crossing a "don't need" boundary with respect to a particular concept, students may continue to grow in understanding while working in the outer layer without folding back to an inner layer, but, as stated earlier, this does not imply that students cannot fold back to inner levels of understanding if needed.

Research Question: How do certain problem solving behaviors contribute to the growth of one university student's mathematical understanding of Taylor Series?

Method: We provide a qualitative analysis of Heber's problem solving behaviors with respect to relevant layers of the Pirie-Kieren model for understanding. Data were collected during the third semester of a teaching experiment in calculus at a large private university. Approximately 25 students were enrolled in the class. Students worked together in small groups on open-ended mathematics tasks. One of the tasks, The Cars Task, invited students to create a polynomial approximation of $p(t) = \sqrt{t+1}$ near $t=0$. Classes were videotaped with one camera that focused on small group problem solving activities or whole class presentations. Research team members verified verbatim transcripts of 97 minutes of one group's work on The Cars Task. Transcripts and video were linked with time codes (minutes:seconds). Transcript annotations to clarify student statements and to describe non-verbal behaviors were bracketed. Student behaviors were identified and coded (Table 1). Student behaviors were mapped onto the Pirie-Kieren model to characterize student growth of mathematical understanding.

Table 1
Codes for identified student behaviors

Building Ideas	(B)
Explaining Ideas	(E)
Offering Ideas	(O)
Questioning Ideas	(Q)
Validating Ideas	(V)
Reorganizing Ideas	(R)
Multiple Representations	(MR)
New Concepts	(NC)

Data and Analysis: Heber's group determined that they could create an accurate approximation for *p(t)* near t=0 by finding a polynomial, *g(x),* which had the same slope and concavity of *p(t)*. The group began by taking the first five derivatives of *p(t)* and evaluating those derivatives when t=0 (Fig. 1). After determining these constant values, they used anti-differentiation to produce a polynomial *g(x)* (Fig. 2) that they believed was a good approximation for *p(t).* In Episode 1, Heber explained (32:00), while talking with his classmate, Tyler, that the more derivatives they "matched up" between *p(t)* and *g(x),* the more accurately *g(x)* would approximate *p(t)*. For example, if they "matched up" 25 derivatives (meaning the first 25 derivatives of *p(t)* evaluated at zero equivalent to the first 25 derivatives of *g(x)* evaluated at zero) then *g(x)* would be more accurate.

Figure 1

$$p(0)=1 \qquad p'(0)=\frac{1}{2} \qquad p''(0)=\frac{-1}{4} \qquad p'''(0)=\frac{3}{8} \qquad p^4(0)=\frac{-15}{16} \qquad p^5(0)=\frac{105}{32}$$

Figure 2

$$g(x) = \frac{105}{3840}x^5 - \frac{15}{384}x^4 + \frac{3}{48}x^3 - \frac{1}{8}x^2 + \frac{1}{2}x + 1$$

Episode 1

Time	**Speaker**	**Annotated Transcript**	**Codes**
(31:51)	Heber	If we did it [the process of finding the derivative of *p(t)*] to the sixth [sixth derivative], it [*g(x)*] would be even better. If we did it to the five hundredth, it [*g(x)*] would be almost identical probably, out to like, who knows. Forty, fifty, sixty.	(O)- Offering Ideas
(31:59)	Tyler	Thousand	
(32:00)	Heber	Yeah. Pretty far probably. And then as it, what we want to say is that as *n* approaches infinity, this function [circles his pencil around the polynomial approximation on his paper] approaches the exact value of the thing [*p(t)*].	(E)- Explaining Ideas
(32:13)	Michael	Right. As we repeat that pattern [of equivalent derivatives of *p(t)* and *g(x)*] infinitely.	
(32:18)	Tyler	We haven't proven that yet, but, that would be a lot of exhaustive proofing.	
(32:23)	Heber	Actually, we could, this is a form of proof. It's called inductive proof.	(B)- Building Ideas
(32:27)	Heber	When we show case one. We show that case one kind of approximates it. And then we show that case *n* is a better approximation. Oh no.	(E)- Explaining Ideas
(32:35)	Tyler	Than case *n* minus one.	
(32:36)	Heber	We show that *n* plus one is a better approximation than *n*. Or yeah, like you said, *n* is a better approximation than *n* minus one.	(V)- Validating Ideas
(33:06)	Tyler	How many derivatives could we take of this?	
(33:09)	Heber	An infinite number.	

Heber's behaviors of offering and explaining the pattern of matching "an infinite number" (33:09) of derivatives corresponds to the *observing* layer of understanding in the Pirie-Kieren model. Later, Heber built his idea into a theory by explaining (32:27) how to use induction to prove *g(x)* was an accurate approximation of *p(t)*. As Heber built up his observations into a theory he folded back to the *formalizing* layer of understanding to find the method of induction he needed to prove his theory was correct. By explaining and validating the method of induction, Heber moved to the *structuring* layer of understanding. By pursuing the idea of using induction to justify the method for finding a polynomial approximation that "matched" out to infinity, Heber passed through a "don't need" boundary and into the *structuring* layer of understanding. He did not need to fold back to an inner layer of understanding to find another way to develop an accurate approximation of *p(t)*.

Figure 3

$$\frac{17\cdot15\cdot13\cdot11\cdot9\cdot7\cdot5\cdot3\cdot1}{1024\cdot10!}x^{10}+\frac{15\cdot13\cdot11\cdot9\cdot7\cdot5\cdot3\cdot1}{512\cdot9!}x^{9}$$

$$-\frac{13\cdot11\cdot9\cdot7\cdot5\cdot3\cdot1}{256\cdot8!}x^{8}+\frac{11\cdot9\cdot7\cdot5\cdot3\cdot1}{128\cdot7!}x^{7}-\frac{9\cdot7\cdot5\cdot3\cdot1}{64\cdot6!}x^{6}$$

$$+\frac{7\cdot5\cdot3\cdot1}{32\cdot5!}x^{5}-\frac{5\cdot3\cdot1}{16\cdot4!}x^{4}+\frac{3\cdot1}{8\cdot3!}x^{3}-\frac{1}{4\cdot2!}x^{2}+\frac{1}{2\cdot1!}x^{1}+\frac{1}{1\cdot0!}x^{0}$$

By the next time the class met, Heber had expanded *g(x)* to the tenth degree (Fig. 3). Heber's group decided not to pursue the development of an inductive proof, but instead returned to the *observing* layer of understanding and focused on Heber's notations and representations to find a general expression that would determine the coefficients of *g(x)* to approximate *p(t).* At first, the students in Heber's group decided to use a question mark as a placeholder for the numerator in the general form for the coefficients of *g(x)*, while they determined a notation for the pattern in the denominator, $2^n n!$ (Fig. 4), of the general expression. After a few minutes of problem solving, they developed the notation for the question mark they had been using as a placeholder (Fig. 5). They generalized the expression in Figure 5 so it would be accurate for all n, except when n was zero, and then substituted the expression for the question mark into the numerator (Fig.6). Finally, they simplified Figure 6 to produce their final formula (Fig. 7) to describe any coefficient of the nth degree polynomial approximation, *g(x)*. The mapping of Heber's growth of understanding in Episode 1 is provided below.

Figure 4

$$\frac{?}{2^n n!}$$

Figure 5

$$? = \frac{(2n)!}{2^n n!}$$

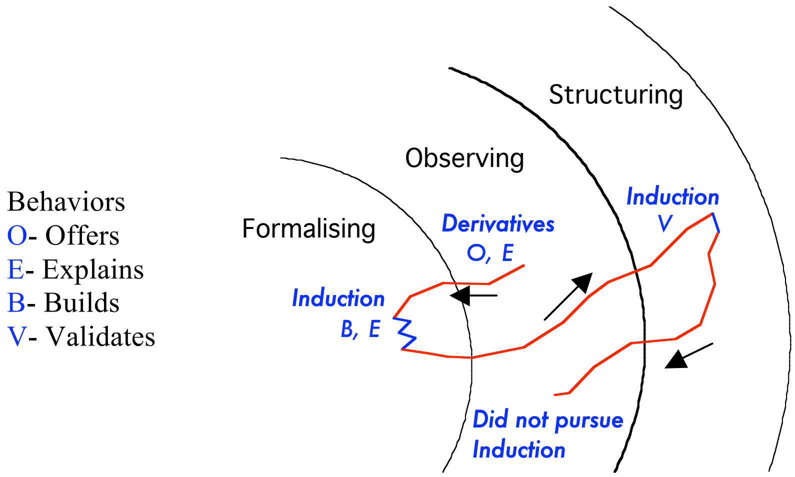
Figure 6

$$\frac{\dfrac{(2(n-1))!}{2^{(n-1)}(n-1)!}}{2^n n!}$$

Figure 7

$$(-1)^{(n-1)}\left(\frac{(2(n-1))!}{2^{(2n-1)}((n-1)!)^2 n}\right)$$

Mapping of Heber's growth of understanding in Episode 1



Later, in Episode 2, while Heber and his group were presenting to the entire class the development of their general form for the coefficients (Fig. 7), a comment (39:41) made by the professor prompted Heber to justify (40:14) that their notation accurately reflected the processes of finding the *nth* derivative evaluated at zero and then dividing by *n!*. Heber's justification corresponds to the *structuring* layer of understanding. Heber's professor then asked him (40:25) to come up with "short cut" notation that would be equivalent to the formula in Figure 6.

Episode 2

Time	**Speaker**	**Annotated Transcript**	**Codes**
(39:41)	Prof.	So you're telling me that I could find the coefficients of each of the terms in my polynomial approximation to the [*p(t)*] function if I simply found the derivative and then divided by *n* factorial?	
(40:10)	Heber	Yes.	
(40:11)	Prof.	Wait, could we write that out somehow?	
(40:14)	Heber	Um, okay. It's kind of written out right here [Points to Fig. 7], this is our derivative, this is our value evaluated at zero.	(B)- Building Ideas (R)-Reorganizing Ideas
(40:25)	Prof.	But I'm thinking short cut notation for me, if I wanted to go back to my original function, I could just find the derivative, the *n*th derivative?	
(40:40)	Heber	Yeah okay, so we could do it [writes the	(MR)- Multiple

		expression in Fig. 8] f of n [the nth derivative] evaluated at zero, divided by n factorial.	Representations (NC)- New Concepts
(40:56)	Prof.	And that's equivalent, but those two expressions [Fig. 7 & Fig. 8] are equivalent for this function?	
(41:04)	Heber	Yeah, this and this [Points to Fig. 7 & Fig. 8] are equivalent, I just multiplied this one out [Fig. 7].	(E)- Explains Ideas
(41:23)	Michael	So I guess that [Fig. 8] could be useful, because that's well and it's more general.	
(41:28)	Heber	Yeah that's definitely much more general.	

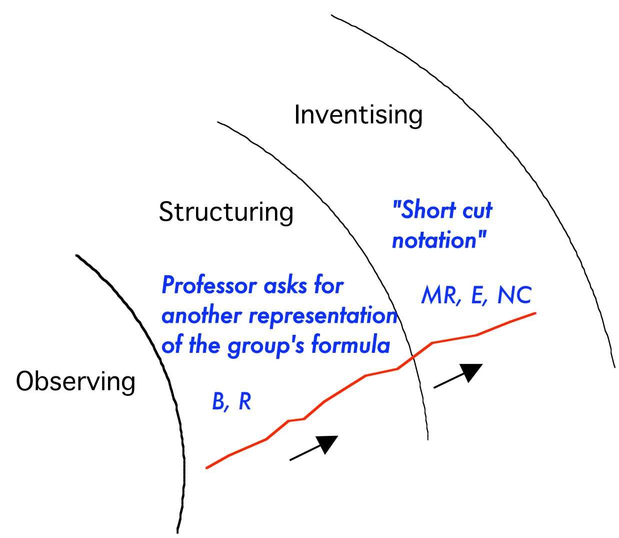
The sequence of interventions by Heber's professor helped Heber to produce a new expression (Fig. 8) for the generalized formula they had created. During this episode (40:40), Heber responded to the instructor's intervention and developed the notation for a new expression equivalent to the general expression they had developed previously (Fig. 7). In developing the new expression, Heber reorganized his ideas to generate a new representation of the generalized formula that resulted in the discovery of a new concept, the general formula for finding coefficients for Taylor Polynomials. By explaining an idea, reorganizing his ideas, and creating a new expression, Heber moved from the *structuring* layer into the *inventising* layer of understanding. By developing the notation for their formula in Figure 7 first, Heber was able to explain the new concept and attribute meaning to the new symbolism in Figure 8. The mapping of Heber's growth of understanding in Episode 2 is provided below.

Figure 8

$$\frac{f^{n}(0)}{n!}$$

Mapping of Heber's growth of understanding in Episode 2

Behaviors
B- Builds
R- Reorganizes
MR- Multiple Representations
E- Explains
NC- New Concept



Findings: The problem-solving behaviors Heber demonstrated in each layer of the Pirie-Kieren model for the growth of mathematical understanding throughout the Cars Task and the frequency of those behaviors are displayed in Table 2. The layers of understanding in which Heber demonstrated the largest variety of behaviors were *image having*, *formalizing*, *structuring* and *inventising*. After Heber passed through a "don't need" boundary (represented by the shaded rows in Table 2) he demonstrated the behaviors of multiple representations, questioning ideas, reorganizing ideas, and validating ideas. These behaviors do not occur in any other layers except for *inventising*, which is the layer in which students culminate their understanding. The most frequent behaviors demonstrated by Heber throughout the task are offering, building and explaining ideas.

Table 2

Observed problem-solving behaviors throughout 97 focus minutes

Layer	(Behavior)Frequency							
Image Making	(B)3	(E)3		(O)2		(Q)1		
Image Having	(B)1	(E)7	(MR)1	(O)3		(Q)4	(R)1	(V)5
Property Noticing	(B)2	(E)8		(O)2				
Formalizing	(B)2	(E)6		(O)5		(Q)1	(R)2	(V)3
Observing		(E)2		(O)2				
Structuring	(B)2	(E)1	(MR)1				(R)4	(V)2
Inventising	(B)3	(E)3		(O)2	(NC)2	(Q)1	(R)2	(V)2

The problem solving behaviors demonstrated by Heber were central to his growth of mathematical understanding and provided evidence of his progression through the levels of the Pirie-Kieren Model and the development of his understanding of the Taylor Series. The intervention of the professor was enacted in response to and in concert with students' behaviors and was pivotal in helping Heber produce multiple representations of the same expression and to transition from the *structuring* to the *inventising* layer in the development of an abbreviated notation for finding coefficients in Taylor Series polynomial functions.

Discussion and Implications: Student problem-solving behaviors, in these episodes, were shown to be significant in facilitating the growth of mathematical understanding. When given challenging and engaging mathematical tasks, students' growth of mathematical understanding can be traced through analysis of problem-solving behaviors and by mapping those evolving behaviors to different layers of understanding as a dynamical process. During the growth of mathematical understanding, students build their own notations to attribute meaning to the symbolisms that emerge.

In order for mathematics educators to ascertain how problem-solving behaviors facilitate growth of students' mathematical understanding, students need opportunities to collaboratively problem solve and participate in effective mathematical discourse with their peers and teachers. The Pirie-Kieren model provides a powerful structure for determining layers in which students work and across which they traverse in the dynamical growth of mathematical understanding. When the Pirie-Kieren model is partnered with an analysis of student problem-solving behaviors, a useful lens emerges through which teachers might view students' growth of understanding. By analyzing problem-solving behaviors of students' and how those behaviors correspond with different layers of understanding, teachers may more effectively evaluate and intervene in students' mathematical discourse. Finally, it is vital that teacher interventions are deeply grounded in students' mathematical problem-solving behaviors to support and facilitate student learning.

Future research could include investigations into how other populations, including high-school students, build mathematical understanding by mapping learners' problem-solving behaviors to the Pirie-Kieren model.

References

Empson, S. B. (2003). Low-performing students and teaching fractions for understanding: An interactional analysis. *Journal for Research in Mathematics Education, 34*(4), 305-343.

Borgen, K. L., & Manu, S. S. (2002). What do students really understand? *Journal of Mathematical Behavior, 21*(2), 151-165.

Martin, L. (2008). Folding back and the dynamical growth of mathematical understanding: Elaborating the Pirie-Kieren theory. *Journal of Mathematical Behavior, 27*(1), 64-85.

Muir, T., Beswick, K., & Williamson, J. (2008). "I'm not very good at solving problems": An exploration of students' problem solving behaviours. *Journal of Mathematical Behavior, 27*(3), 228-241.

National Council of Teachers of Mathematics. (2000). *Principles and standards for school mathematics.* Reston, VA: National Council of Teachers of mathematics.

Pirie, S., & Kieren, T. (1994). Growth in mathematical understanding: How can we characterise it and how can we represent it? *Educational Studies in Mathematics, 26*(2), 165-190.

Spando, J. (1996). Developing problem solving behaviors in secondary mathematics education through homework. *New England Educational Research Organization,* 1-30.

Speiser, B., Walter, C., & Maher, C. (2003). Representing motion: An experiment in learning. *Journal of Mathematical Behavior, 22*(1), 1-35.

Vega, E., & Hicks, S. (2009). The catwalk task: Reflections and synthesis: Part 2. *Journal of Mathematical Behavior, 27*(4), 255-263.

Warner, L. B. (2008). How do students' behaviors relate to the growth of their mathematical ideas? *Journal of Mathematical Behavior, 27*(3), 206-227.