SUBSPACE IN LINEAR ALGEBRA: INVESTIGATING STUDENTS' CONCEPT IMAGES AND INTERACTIONS WITH THE FORMAL DEFINITION

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This paper reports on a study that we conducted for the purpose of investigating students' ways of conceptualizing key ideas in linear algebra, with the particular results presented here focusing on student interactions with the notion of subspace. In interviews conducted with eight undergraduates, we found that the students' initial descriptions of subspace often varied substantially from the language of the concept's formal definition, which is very algebraic in nature. This is consistent with literature in other mathematical content domains that indicates that a learner's primary understanding of a concept is not necessarily informed by that concept's formal definition. We utilized the analytical tools of concept image and concept definition (Tall & Vinner, 1981) in order to highlight this distinction in the student responses. Through grounded analysis, we identified recurring concept imagery that students provided for subspace, namely: geometric object, part of whole, and algebraic object. We also present results regarding the coordination between students' concept image and how they interpret the formal definition, situations in which students recognized a need for the formal definition, and qualities of subspace that students noted were consequences of the formal definition. Furthermore, we found that all students interviewed expressed the common but technically inaccurate notion that $\mathbf{R}^k$ is a subspace of $\mathbf{R}^n$ for $k < n$. We conclude by reflecting on various positive aspects of this intuition and how it can inform teaching in a productive, student-centered manner.

Purpose and Background

A first course in linear algebra is commonly seen as a pivotal yet difficult mathematics course for university students. It is one of the most useful fields of mathematical study because of its unifying power within the discipline as well as its applicability to areas outside of pure mathematics (Dorier, 1995; Strang, 1988). According to Harel (1989), linear algebra is an important subject at the college level in that it: a) can be applied to many different content areas, such as engineering and statistics, because of its power to model real situations; and b) can be studied in its own right as "a mathematical abstraction which rests upon the pivotal ideas of the postulational approach and proof" (p. 139). Thus, when considering the mathematical development of undergraduate students, a first course in linear algebra plays an important, transitional role. In many universities, a first course in linear algebra follows immediately after a calculus series and, most likely, prior to an introduction to proof course. According to Carlson (1993), the majority of students' mathematical experiences up to that point in their education have been primarily computational in nature. However, the content of linear algebra can be highly abstract and formal, in stark contrast to students' previous computationally oriented coursework. This shift in the nature of the mathematical content being taught can be rather difficult for students to handle smoothly. Carlson (1993) posited that concepts are often taught without substantial connection to students' previously learned mathematical ideas, as well as without examples or applications. Thus, students struggle to connect familiar concepts to prematurely formalized, unfamiliar ones. Dorier (1995) contended that the serious difficulties students have in learning linear algebra are a result of its abstract and formal nature. Indeed, Robert and Robinet (1989) showed that prevalent student criticisms of linear algebra relate to the course's "use of formalism, the overwhelming amount of new definitions and the lack of connection with what they already know in mathematics" (as cited in Dorier, Robert, Robinet, & Rogalski, 2000, p. 86).

As the above quote implies, the mathematical activities central to linear algebra frequently involve mathematical ideas and definitions with which students have little experience, such as subspace or linear transformation. Investigating student thinking about fundamental mathematical ideas and definitions such as these is an integral part of basic research regarding the teaching and learning of linear algebra. Much of this research has focused on student difficulties and suggestions to alleviate these difficulties (e.g., Hillel, 2000; Sierpinska, 2000; Stewart & Thomas, 2009) or on students' creative and productive ways of reasoning about key concepts of linear algebra (e.g., Possani, Trigueros, Preciado, & Lozano, in press; Wawro, 2009). A handful of these studies addressed students' experiences with subspace, a foundational idea in linear algebra. For instance, Harel and Kaput (1991) studied student responses to the following problem:

> Let V be a subspace of a vector space U, and let β be a vector in U but not in V.
> Is the set $V + \beta = \{v + \beta \mid v \text{ is a vector in } V\}$ a vector space?

Harel and Kaput found that student responses could be categorized into two main groups. Some students focused on the formal definition of vector space, checking each of the individual axioms for $V + \beta$, while others were able to view the vector space V as an entity while treating β as a shift operator that affected the existence of the zero vector. Utilizing the latter approach, the authors argued, demonstrated a high level of mathematical sophistication in that students were able to reason through properties of vector spaces rather than check them algorithmically. From the results of a large study regarding the teaching and learning of linear algebra in France, Dorier et al. (2000) found that when asked to give parametric equations representing a subspace of $\mathbf{R}^4$, students could compute a correct answer but would represent that answer with vectors in $\mathbf{R}^2$ rather than $\mathbf{R}^4$.

In a study that explicitly addressed subspace, Britton and Henderson (2009) examined students' difficulties with the notions of closure. They analyzed how students responded to the following two questions:

1. Let $V = \{t < 1,2,3 > \mid t \in \mathbf{R}\}$. Show that V is a subspace of $\mathbf{R}^3$.
2. Let X be the set of functions f: $\mathbf{R} \rightarrow \mathbf{R}$ in $\mathbf{F}$ such that the graph $y = f(x)$ passes through the point (0, 0). Show that X is a subspace of $\mathbf{F}$ [$\mathbf{F}$ is the set of all such functions, without the restriction on the graph].

The authors found that students did, as expected, evidence difficulties with showing the closure of these sets under addition and scalar multiplication. Students tried to demonstrate these properties by choosing particular rather than general vectors. Students also struggled with "the formalism required to write a convincing proof, problems with logic and set theory, [and] the difficulty of moving from the abstract mode in which the definition is phrased to the algebraic mode in which the question is framed" (p. 969). The authors concluded that, in addition to the expected difficulties with proof and logic, many of the students' difficulties centered on interpreting the meaning of the various symbols.

As this research indicates, students do have difficulty with many of the mathematical ideas and activities central to the concept of subspace. In our study, however, we found that students exhibited a variety of creative and powerful ways of conceptualizing subspace and interacting with its formal definition. The research presented in this paper grows out of a study that investigated the interaction and integration of students' creative ways of thinking about key ideas in linear algebra. The broad goal of the study was two-fold: a) to investigate students' conceptualizations of key ideas in linear algebra, namely subspace, linear independence, basis, and linear transformation; and b) to investigate how students coordinate computational, geometric, and theoretical/abstract aspects of fundamental concepts in linear algebra. For us, the term *computational* means carrying out appropriate algorithms such as row reduction and understanding what the result means, *geometric* refers to reasoning with appropriate geometric structures such as planes in $\mathbf{R}^3$, and *theoretical/abstract* encompasses justifying facts and algorithms as well as being able to reason confidently in spaces of more than three dimensions. These three categories, suggested as distinct yet interrelated modes of thinking useful in linear algebra, are similar to trichotomies developed by other researchers. For instance, Hillel (2000) refers to the geometric language, the algebraic language, and the abstract language of linear algebra, and Sierpinska (2000) distinguishes between synthetic-geometric, analytic-arithmetic, and analytic-structural modes of thinking. While these categories did inform the design of our study, the results presented in this paper are not dependent upon them. Thus, a further elaboration regarding these modes of thinking is beyond the scope of this paper.

In interviews conducted with eight undergraduates, we asked students to describe how they think of subspaces of $\mathbf{R}^6$. Imagining higher-dimensional spaces can be difficult for any learner because our perceptual experience is limited to three spatial dimensions. Consider the formal definition of subspace, given below:

> A nonempty subset V of $\mathbf{R}^n$ is a subspace of $\mathbf{R}^n$ if it is closed under addition and scalar multiplication. That is, if $\mathbf{x} + \mathbf{y}$ is in V whenever $\mathbf{x}$ and $\mathbf{y}$ are in V; also if $a\mathbf{x}$ is in V whenever $\mathbf{x}$ is in V and a is a real number.

The formal definition of subspace provides the machinery to determine whether a set is a subspace of a given space and, in some cases, may allow students to construct their own examples, but it does not seem to prompt any immediate imagery. The abstraction and the lack of conventional imagery may compel students to create meaning for the term as they interact with it in different problem situations. Investigating how students deal with the issues of making sense of abstract, formal definitions was central to our study. We aim to understand how students develop imagery for concepts that may be particularly formal in nature, like subspace. In this paper, we utilize the constructs of concept image and concept definition (Tall & Vinner, 1981) to analyze these students' notion of subspace in linear algebra.

Theoretical Framework

The theoretical constructs of *concept image* and *concept definition* (Tall & Vinner, 1981) have proven to be a useful analytical tool for nearly three decades. Tall and Vinner defined *concept image* as "all the cognitive structure in the individual's mind that is associated with a given concept" (p. 151) and *concept definition* as "the form of words used to specify that concept" (p. 152). Vinner (1991) added that a person's concept image can be a visual representation, as well as a collection of impressions or experiences associated with that concept. Furthermore, any concept image may have aspects that differ from the concept's formal definition, and these distinctions may not be coherent across all problems and situations. For instance, students may think of vectors as arrows anchored at the origin, as points in a coordinate system, or as floating directed line segments. While any one of these concept images of vector may be useful in a specific situation, combining them within the same context could prove problematic for inexperienced linear algebra learners. This situation—when one aspect of a student's concept image conflicts with another aspect of that same concept image or definition—is referred to as a *potential conflict factor* (Tall & Vinner, 1981, p. 153). We return to this notion of potential conflict factors in the discussion section, in which we posit potential benefits for teaching from evoking these conflict factors within the context of coordinating concept images of subspace with that concept's formal definition.

Zandieh and Rasmussen (in press) identify two major strands of research regarding concept image and concept definition. The first focuses on cataloging learners' concept images about a particular mathematical idea (e.g., Artigue, 1992; Edwards, 1999; Zandieh, 2000). The second strand of research deals with cataloging difficulties that arise when students reason from their concept image even though they "know" the concept definition (e.g., Bingolbali & Monaghan, 2008; Edwards & Ward, 2008; Vinner & Dreyfus, 1989). For example, Vinner and Dreyfus (1989) investigated student interaction with the concept of function and noted that students did not always use the definition of function when trying to decide whether a given object was an example of a function or not. They concluded "the set of mathematical objects considered by the student to be examples of the concept is not necessarily the same as the set of mathematical objects determined by the definition" (p. 356). In a related vein, Vinner (1991) discussed three possible outcomes when a student is introduced to a new concept definition for which he already has some sort of concept image. The student could: (a) change his concept image in order to encompass the new definition, (b) keep his concept image the same and ignore or forget the new definition, or (c) use both his concept image and new definition but without integration of the two. In the case of subspace in linear algebra, our study contributes to both strands of this research. In the results section, we first present our findings regarding various student concept images for subspace. In the remainder of the section we discuss various aspects of how students interacted with the formal definition of subspace and ways in which this interaction was consistent with or in conflict with their previously described concept images of subspace.

There also exists a fair amount of research regarding how one could use the concept image/concept definition framework as a tool to influence pedagogy at the undergraduate level. Tall and Vinner (1981) argued that, in the area of real analysis, weak understandings of a formal definition, coupled with a strong concept image, can make the process of formal proof difficult for students. Vinner (1991) made similar claims and offered as a possible solution training students to make explicit use of a definition, stating that this could only be achieved if the students are given tasks that cannot be solved without using a formal definition. "As long as referring to the concept image will result in a correct solution," Vinner states, "the student will keep referring to the concept image since this strategy is simple and natural" (p. 80). Bingolbali and Monaghan (2008) made explicit the assumption that students bring a variety of experiences and ideas to their work on mathematical tasks. The authors studied student concept images of the derivative, arguing that departmental affiliation (mechanical engineering or mathematics) affected the development of concept images and should not be ignored. Finally, Zandieh and Rasmussen (in press) make use of the concept image/concept definition framing to extend the notion of defining as a mathematical activity. The authors examine how students actually use and create concept images and concept definitions in an undergraduate non-Euclidean geometry course in order to offer a structured way to aid practitioners in planning for and analyzing the ways in which students define as they transition from intuitive ways of reasoning to more conventional, formal ways of reasoning in mathematics. In the discussion section, we offer implications for teaching that involve productively making use of students' intuitive (but technically incorrect) notion that $\mathbf{R}^k$ is a subspace of $\mathbf{R}^n$ for $k < n$. We posit this as an example of a pedagogical opportunity to foster students' creation of a concept image of isomorphism that may be further refined to the formal definition of isomorphism.

Methods

The subjects for this study were first-year university students enrolled in an honors calculus course at a large public university in the southwestern United States. All students had completed AP Calculus in high school and had earned a 5 on the AP (BC) exam. Introductory linear algebra was the content covered in the first of the university course's three quarters, and the course emphasized geometry in $\mathbf{R}^n$ as a foundation for multivariable calculus, rather than systems of equations. Approximately three weeks after the conclusion of the linear algebra quarter, eight students participated in hour-long, individual, semi-structured interviews (Bernard, 1988). Of the 24 that volunteered (total enrollment was 33) to be interviewed, the course professor personally chose these eight in order to represent the class's diversity appropriately in the categories of course grade, gender, and ethnicity. Data sources consisted of video recordings of the interviews, students' written work, and researcher field notes. All of the interviews were transcribed completely.

Recall that the broad goal of the interview was to investigate students' conceptualizations of foundational ideas in linear algebra (subspace, linear independence, basis, and linear transformation), as well as to investigate how students coordinate computational, geometric, and theoretical/abstract aspects of their reasoning about these fundamental concepts. In line with these goals, the interviews we conducted used the following five questions as a broad frame for inquiring into student thinking:

1. How do you think about what a subspace of $\mathbf{R}^6$ is?
2. How do you think about what it means for a set of six vectors to be linearly independent?
3. Michelle would like to create a basis of $\mathbf{R}^4$. She has already listed two vectors she wants to include in her basis, and would like to add more to her list until she obtains a basis. What instructions would you give her on how to accomplish this?
4. A student used row operations to reduce the augmented matrix of a system of linear equations to the following form $\left[\begin{array}{cccc|c} 0 & 1 & 0 & -2 & 5 \\ 0 & 0 & 1 & 4 & 6 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right]$. What can you tell about the original system from this?
5. How do you think about the meaning of the equation $A\mathbf{x} = 2\mathbf{x}$, where $\mathbf{x}$ is an n x 1 vector and A is an n x n matrix?

These questions served as a frame for the interview in that each question also had a set of prepared follow-up questions that further investigated student thinking regarding the main concept of the question. The main follow-ups for question one above, which deals with subspace, were as follows:

a. Consider the vectors $\begin{bmatrix} 1 \\ 2 \\ 3 \\ 5 \end{bmatrix}, \begin{bmatrix} 2 \\ 5 \\ -3 \\ 1 \end{bmatrix}$ and $\begin{bmatrix} 2 \\ 4 \\ 5 \\ -7 \end{bmatrix}$. Do these vectors form a subspace?

b. On this sheet of paper is your textbook's definition for subspace. Read it aloud and explain how you think about it.
c. How does the textbook definition relate to your previous description of how you think of subspace?

In follow-up (a) above, one may notice that to say three vectors "form" a subspace is mathematically imprecise. It would be a more proper question to ask whether these vectors form a basis for a subspace, or to ask for an alternative description of the subspace these vectors span. This question was intentionally vague in order to investigate how students thought about how subspaces are "built." This vagueness also allowed us to see how students interacted with the vectors as elements of a vector space: for instance, deciding whether these vectors are in $\mathbf{R}^3$ (because there are three vectors) or $\mathbf{R}^4$ (because each vector has four components) was not trivial for some.

In further reflection on our interview design, we created prompts that we felt asked students about their thinking rather than giving them problems to solve. By doing so, students were given the opportunity to describe their visualization of ideas to us and reflect on their own knowledge and thought processes rather than get "bogged down" in long calculations or numerical errors. For example, one student, Shelley, shared how she personally sees the utility of both her concept image and the formal definition of subspace: "If I had to prove a subspace I would use this [the definition], but when I just think about it, I just think about it as a space." Furthermore, we were able to retrospectively examine the data to see what questions prompted students to use a formal definition as opposed to only a concept image, as well as to see whether students spontaneously referred to definitions and for what purposes they did so. Thus, the design of our interviews allowed us to not only examine students' concept images but also how students connected these images to and found use for the formal definition of subspace.

For the purposes of analysis, we used grounded theory (Strauss & Corbin, 1990). During our first round of analysis, we independently created open codes to categorize the kinds of language that students used when discussing the various concepts in the interview. From these codes, we decided to investigate how students reason about the concept of subspace and interact with the definition. We then engaged in axial coding, where we independently created and fleshed out the concept image categories for subspace that we saw emerging from the data. Following each round of analysis, we met together to discuss our notes, summaries, and codes. Transcripts were re-read, video data re-examined, and codes and inferences were discussed to ensure investigator triangulation (Denzin, 1978). Once we agreed on the existence of three main categories that captured the various ways in which students described subspace, we began the process of selective coding (Strauss & Corbin, 1990), during which we focused our further investigation on three specific topics: (a) coordination between students' language to describe subspace and to interpret the formal definition, (b) instances of students spontaneously invoking a use for the formal definition of subspace, and (c) situations in which students generated consequences of the definition. Finally, we returned to the data in order to investigate instances in which students explicitly referred to $\mathbf{R}^k$ as a subspace of $\mathbf{R}^n$. Each of these investigations is discussed in the results section, with the latter investigation serving as the foundation for a discussion regarding positive implications for teaching.

Results

From our analysis, we found that the eight students interviewed used a variety of language to describe their personal notions of subspace. From this variety, we identified three main categories for students' concept images of subspace. We also present results regarding the coordination between students' concept image and how they interpret the formal definition, situations in which students perceived a need for the formal definition, and qualities of subspace that students noted were consequences of the formal definition.

Categorization of Students' Concept Images of Subspace
We now describe the three categories for concept image that emerged from our data. These three categories are: Geometric Object (e.g., subspace is a *plane* in a space), Part of a Whole (e.g., subspace is *contained within* a space), and Algebraic Object (e.g., subspace is a *collection of vectors*). In this section we define and elaborate on each of these categories and provide examples of each.

Subspace as Geometric Object. Seven of the eight students referred to subspace as some familiar object in geometry, such as a plane, line, cube, or sphere. Many of the students used multiple familiar geometric terms in order to describe how they think of subspaces. For example:

Interviewer: How do you think about what a subspace of $\mathbf{R}^6$ is?

Henry: I typically just think about a line, a plane, and the 3-D space. Because obviously you can't imagine $\mathbf{R}^4$, $\mathbf{R}^5$, $\mathbf{R}^6$. So I just kind of imagine those spaces.

Throughout the interview, Henry repeatedly mentioned that it is not possible to imagine anything of a higher dimension than $\mathbf{R}^3$. Indeed, Henry noted that he visualizes $\mathbf{R}^6$ as having six orthogonal coordinate axes, later saying, "If you ask me something in ... more than $\mathbf{R}^3$, I'll have to think about it in $\mathbf{R}^2$ and $\mathbf{R}^3$ and sort of expand it, ... and then hopefully it'll be the same in $\mathbf{R}^6$." At one such instance early in the interview, the interviewer prompted Henry about this:

Interviewer: What about if I asked about subspaces of $\mathbf{R}^3$, can you talk about that?

Henry: I just think about planes and lines.

This geometric imagery remained constant throughout Henry's interview. The main interview prompt for subspace (asking how students think about a subspace of $\mathbf{R}^6$) and its follow-ups were directly focused on eliciting student imagery. Henry's responses to other questions in the protocol—questions that did not explicitly ask about his imagery—were consistent with this geometric language, as we illustrate in subsequent sections.

Let us consider another student whose responses exemplified the Geometric Object view of subspace. When asked how she thought about subspaces of $\mathbf{R}^6$, Nancy's first response was, "I guess sort of any plane that lies in, somewhere in $\mathbf{R}^6$, which you can't really visualize ... But I guess a plane or some other multi-dimensional space or object that would fit into whatever the $\mathbf{R}^6$ would look like." Her first description relied on a geometric conceptualization of space, even though "you can't really visualize" $\mathbf{R}^6$. Within the first five minutes of the interview, Nancy refers to subspaces as planes six more times. One of these occurred in the following example, where the interviewer asked Nancy to switch from thinking about subspaces of $\mathbf{R}^6$ to those of $\mathbf{R}^3$.

Interviewer: ...How about $\mathbf{R}^3$?

Nancy: I don't know, I sort of visualize it as almost a sphere, but with sort of fuzzy edges, I guess.

Interviewer: Can you tell me what a subspace of $\mathbf{R}^3$ would look like?

Nancy: I mean $\mathbf{R}^2$ would also be, so a plane through $\mathbf{R}^3$.

Interviewer: So $\mathbf{R}^2$ and a plane through $\mathbf{R}^3$, those things are the same thing?

Nancy: Yeah.

In this example, Nancy was explicit in her thinking that $\mathbf{R}^2$ is a subspace of $\mathbf{R}^{3,}$ rather than the mathematically correct understanding that two-dimensional subspaces of $\mathbf{R}^3$ are isomorphic to

$\mathbf{R}^2$. We return to this issue in the discussion section.

Because of the frequency with which Nancy referred to a subspace as a plane, we claim that her main descriptor for the notion of subspace was Subspace as Geometric Object. In summary, student utterances that fit into this category include those that featured familiar geometric objects such as lines, planes, spheres, and cubes. Students' imagery focused primarily on reasoning in $\mathbf{R}^2$ and $\mathbf{R}^3$, and students frequently discussed difficulties in imagining $\mathbf{R}^6$—as evidenced by Henry's earlier utterances. Finally, it is noteworthy that Nancy's description also utilized Part of a Whole imagery (the subspace would *fit into* $\mathbf{R}^6$). We describe this second category now.

Subspace as Part of a Whole. Six of the eight students described subspace in light of a relationship to some larger, encompassing "parent space" (our term). The terms that were used to reflect this relationship were *contained within, part of, smaller than,* and *inside of.* What separates elements of this category from those of the previous one is the lack of explicit reference to a particular geometric shape. For instance, when Shelley was asked to describe how she thought about what a subspace of $\mathbf{R}^6$ is, she replied:

Shelley: A subspace. I just think of it as a part of being in $\mathbf{R}^6$. You have different dimensions in that, but it can be all of $\mathbf{R}^6$ as well. I don't think of it as a space, exactly, I visualize 3-D but not exactly like, yeah.

Interviewer: You don't visualize it, you don't visualize $\mathbf{R}^6$, is that what you're saying?

Shelley: Yeah, I can, but I just, the vision I get is $\mathbf{R}^3$ I guess, but I just think of a smaller space inside that.

Note that Shelley was explicit in describing how she does not necessarily see a specific space, but rather that she thinks of a subspace of $\mathbf{R}^6$ as a *part of being in* and a *space inside* $\mathbf{R}^6$. Because of the frequency with which Shelley referred to a subspace in relation to some other parent space but without a strong object-like description, we claim that her main descriptor for the notion of subspace was Subspace as Part of a Whole. It is noteworthy that Shelley, throughout the entire interview, never described subspace as a geometric object. This lends reliability to the identification of the Subspace as Part of a Whole image as her primary one. Consider the following excerpt as a final note to demonstrate Shelley's consistency. Later in the interview, when asked how she would explain subspace to her roommate, her answer was similar.

Shelley: Um, well, I guess that's the only way I can explain it, because I just think of it as a space that's smaller than or all of the given $\mathbf{R}^n$.

Subspace as Algebraic Object. Four of the eight students referred to subspace in a way that we categorized as Subspace as Algebraic Object. In broad terms, descriptions that fit into this category are those that have a strong focus on mathematical or formal objectification. Students used terminology such as *element of* rather than *part of*, language more consistent with an abstract, set theoretic view of subspace. Students may have described subspace as a set of vectors or a matrix satisfying some condition, by a list of components, or by a system of equations it satisfies. For instance, when asked how he thinks of a subspace of $\mathbf{R}^6$, Mark responded by first describing how he imagines $\mathbf{R}^6$ and then how he envisions subspaces of $\mathbf{R}^6$.

Mark: I think of, I mean originally when I entered the class, I tried to think of everything in terms of 3-D, and then of course when we got to $\mathbf{R}^6$, yeah, I started re-imagining it. I don't know, I still like to think of it spatially as just, you see some of those complex boxes that mathematicians draw every now and then to try to illustrate. So I tried to think of it like that. But after linear algebra, it's really just a matrix with 6 entries for me, I really couldn't work with anything else. So I really, when I see $\mathbf{R}^6$, it kind of now has become just an algebraic term for me, it's a subspace in this case would be any dimensional or geometric thing that could fit into that matrix with 5... With 5 entries, 5, if it's $\mathbf{R}^6$, it fits anything with **R** being less than 6, it'll fit any object or any sequence with values that are, and then of course the largest subspace will be $\mathbf{R}^6$ itself.

Note that, when asked how he thinks of subspaces in $\mathbf{R}^6$, Mark felt it pertinent to first describe how he used to find it helpful to imagine subspaces spatially in terms of three dimensions, but that he no longer does so. He is then very pointed and clear in stating that he now sees subspaces as algebraic objects. He also seems to say that the pertinent qualities of subspace are, for him, based on characteristics of vectors and matrices. He mentions needing to know how many components each have, etc. This is in contrast to the two previous categories that were based on imagery that could be used independently of the content domain of linear algebra if one so chose. The imagery in the first two categories was heavily visual, whereas here—although we do see Mark mention visual imagery to a small extent—it is mainly based on mathematical objects such as vector and matrix.

We find it interesting that only four of the eight students used algebraic terminology to describe how they thought about subspaces, given that the formal definition of subspace is very algebraic and set-theoretic in nature. We wondered, then, if students hold their concept image and concept definition in isolation from each other. Do students use consistent language when describing how they think of subspace and when interpreting the formal definition? Because of the definition's strong set-theoretic wording, did more students utilize algebraic language to interpret the formal definition? These questions are the focus of the following sections.

Coordination between Concept Image and Formal Definition

During the interviews, students were first asked to describe how they thought about subspaces of $\mathbf{R}^6$. Later, we presented them with their textbook's definition of subspace and asked them to interpret it. Invariably, students interpreted the concept definition in ways that were consistent with their earlier image of subspace. That is, in each interview, the language students used to interpret the formal definition was consistent with that previously used to describe how they thought about subspace. Regarding the first and third concept image categories—Geometric Object and Algebraic Object—students used language consistent with these categories to interpret the definition only if their prior description of subspace had used the same language. We found this somewhat surprising given the inherent algebraic formulation of the formal definition itself, which is restated below:

A nonempty subset V of $\mathbf{R}^n$ is a subspace of $\mathbf{R}^n$ if it is closed under addition and scalar multiplication. That is, if $\mathbf{x} + \mathbf{y}$ is in V whenever $\mathbf{x}$ and $\mathbf{y}$ are in V; also if $a\mathbf{x}$ is in V whenever $\mathbf{x}$ is in V and a is a real number.

A second result concerns the remaining category—Subspace as Part of a Whole. We found that every student utilized language consistent with this concept image category when interpreting the formal definition. It must be noted, however, that the formal definition itself utilizes language consistent with Part of a Whole imagery. The definition includes closure under addition and scalar multiplication, and the word "in" is prominently used throughout. This may account for the fact that every student used part-whole imagery in his or her interpretation of the concept definition.

As stated above, we found it surprising that more students did not utilize Algebraic Object imagery when interpreting the formal definition. Most students consistently connected their geometric imagery to their interpretation of the definition. As an illustrative case, consider Henry's explanation of how he interprets closure under addition and scalar multiplication geometrically. Italics were added to highlight Henry's repeated geometric language.

Henry: Basically I think of a *plane* again, it's the *x-y* plane. And then I think just pick two vectors, whatever vectors, don't have to go orthogonal, whatever length. And then if you add them. And then I'll *imagine the parallelogram rule*, you add them, and then if it's *in the plane*, and then it's a subspace. Then it's closed under addition. Then I'll think about multiplication. I'll just pick a vector and then I'll *go forever in one direction* [extends an arm out and points towards the distance], you multiply by a whole bunch of numbers in *one direction*. And then multiply by some negative number, and you'd go *another direction*. And that should be in the subset.

In this explanation, Henry used a plane as a generic example of subspace. He described both vector operations, and the meaning of closure, in geometric terms. First, Henry discussed imagining closure under addition as creating a parallelogram on a plane. Henry then illustrated how scalar multiplication can be thought of going forever in both the positive and negative directions. This description of how the definition can be imagined geometrically is consistent with the concept image of subspace that Henry exhibited before he was given the definition.

Some students did use algebraic language in interpreting the definition, in a variety of ways. We illustrate this variation with two examples. In the first, Amy tried to use the definition to determine whether a set of three given vectors in $\mathbf{R}^4$ (see follow-up question (a) under Methods) form a subspace:

Amy: I'm just trying to put the vectors, name each of the vectors, so it's **x** and **y**. And I was thinking of scalars, and then I was thinking of adding them, and if they would still be in the subspace? Then do I need to, is there, do I need to say it's a subspace of $\mathbf{R}^n$, is this what it's asking, is this a subspace of $\mathbf{R}^n$ or is this a subspace of anything, or?

When given the definition, Amy literally tried to match up the three given vectors with the two vectors appearing in the definition. The mismatch caused her to struggle in interpreting the definition. Amy's literal interpretation of the definition algebraically was consistent with her concept image, as she frequently discussed subspace in terms of dimensions and adding additional basis vectors. When dealing with the definition, Amy saw the **x** and **y** not as arbitrary vectors within the space, but as specific vectors that needed to be tested in a procedural way.

Mark's interpretation of the definition also demonstrated his algebraic concept image, but in a different manner. Instead of focusing on the presence of two vectors in the concept definition, he interpreted closure under addition and scalar multiplication as meaning all linear combinations of the vectors. The following excerpt is again from follow-up question (a).

Interviewer: How would you describe that subspace then?

Mark: I feel it's just points, actually, it's just the point (1,2,3,5). Under this, it's not all of what, it's not all possible additions or multiplications, or sums and products of these matrices, it's just the points, the three points. Unless we define the subset as all linear combinations of the two. So it's just three points in the $\mathbf{R}^4$, just existing in space around, in $\mathbf{R}^6$.

Mark noticed that the set that he was given was not all linear combinations of the vectors, but only three points of $\mathbf{R}^4$. The interviewer suggested that the question could be changed to say that the given vectors formed a basis for a subspace, and Mark's response combined algebraic and geometric imagery:

Mark: Yes, that would change it, because then it would be all combinations of those, and all multiples of those vectors [points to the set of three vectors]. Then it would be in fact a, it would be like a geometric figure kind of. I don't really know how I, I can't really visualize it in $\mathbf{R}^4$, but it would be kind of, I think the best metaphor I can think of would be a small bowl inside a cube, kind of like that. It would be a geometric shape, or like a plane through 3-dimensional space sort of deal. Because then this [points to the set of three vectors again] would be, this would go, would be a form in $\mathbf{R}^4$ that actually, I guess this would span if it's linearly independent of one another, it would span $\mathbf{R}^4$. So you'd have like an $\mathbf{R}^4$ plane going through $\mathbf{R}^6$ subspace.

A basis implied for Mark that the set includes all linear combinations of the vectors, allowing for a different shape, specifically "a small bowl inside of a cube." The algebraic imagery of all of the linear combinations of the given vectors lead him to draw the conclusion that the set is no longer just three points, but instead is a geometric figure, which he characterized as "a geometric shape…a plane through three-dimensional space sort of deal." Mark interpreted the definition using the algebraic imagery of linear combinations, which he connected to his geometric imagery.

Mark's responses illustrate the powerful role that definitions can play in solidifying and developing a student's concept image. Other students also combined their understanding of the definition with their concept image. In Amy's case, her literal interpretation of the definition displayed her reliance upon algebraic imagery and computation to make sense of subspace. Henry showed how he created a geometric understanding of the definition by relating it to his earlier images, whereas Mark showed how the concept definition can bring together disparate aspects of the concept image into a more cohesive whole. Mark's understanding of the definition combines his algebraic and geometric reasoning through the mechanism of closure under addition and scalar multiplication.

Necessity for and Consequences of the Formal Definition

Knowing when a definition would be a valuable resource is important for learning mathematics. We analyzed when the students first requested, mentioned, or produced the definition of subspace, as an indicator of the purposes definitions can serve for them. Tall and Vinner (1981) noted that students relied upon their concept images rather than the concept definition to solve problems and make sense of mathematics. They stated that this reliance could be a hindrance to the students. We have found in our analysis that our students frequently saw the use and value of the definition and knew when they would use it in order to make sense of the mathematics.

Seven of the eight students invoked the definition prior to being given the textbook definition, some doing so multiple times. Four of the seven mentioned or gave the definition during the task of determining whether the given set of three vectors in $\mathbf{R}^4$ form a subspace. Amy went so far as to refuse to answer the question until the definition was provided, stating, "Would you tell me the definition of a subspace? Because that would help!" This was consistent with

Amy's overall responses to many of the interview questions; she relied upon literal readings of the definitions and computational methods. We contend that her procedural understanding made the logical structure of the definition hard to grasp. The other three students were not as rigid, but they also requested the definition in order to solidify their responses. Another student, Shelley, gave the definition to support her claim that a certain set of vectors was *not* a subspace. This response, along with the others previously mentioned, indicate that students see the need for the definition to affirm, mathematically, that an object or set of objects is in fact a subspace.

In other examples of seeing the need for a formal definition, students referred to it in order to elaborate or explain features of their concept image. Two consequences of the concept definition that were cited by students were that a subspace must contain the zero vector and must be "flat." Both had been emphasized in the class. After Nancy provided her initial thoughts about subspaces of $\mathbf{R}^6$, the following exchange occurred:

Interviewer: What is necessary for something to be a subspace?

Nancy: I don't remember the definition of a subspace off the top of my head...Usually I think of it as like some sort of flat thing that, I remember it has to go through the origin. But usually a flat, either a plane or a line or something that's 2-dimensional.

After being provided with the definition and asked how it related to her previously given image of subspace, she made the point more explicitly:

Nancy: It's sort of related to the picture I had in my head about some sort of flat object, or a plane, or a line or whatever. Because a subspace has to be flat or else it won't be closed under addition.

For Nancy, the intuitive concept of flatness is captured by the definition. Furthermore, Nancy's statement evidences that the definition and the image are in fact connected in meaningful ways. The intuition is that any two linearly independent vectors in a subspace determine a plane containing them, their sum, and indeed all their linear combinations. Closure means that the subspace must contain this plane, rather than bending away from it as a curved surface would, so nonzero curvature is inconsistent with the definition.

Flatness was not the only consequence of the definition that was mentioned by the students before being given the definition. Many students mentioned inclusion of the zero vector as part of their concept image before being given the definition. Two of them specifically cited this as a consequence of the definition.

Interviewer: Could any plane be a subspace of $\mathbf{R}^3$?

Henry: It has to pass through the origin.

Interviewer: Why so?

Henry: Because that's the definition of a subspace...It's not a definition, but it's a little thing that comes along with the definition. It has to be closed under vector addition, and closed under scalar multiplication, and if you're both of those, then you pass through origin.

Henry first gave the definition after being asked whether any plane could be a subspace. This implies that Henry saw the definition as being important for specifying what is necessary for a plane or other object to be a subspace. More specifically, he invoked the definition to explain why the zero vector must be part of the subspace. He first suggested that this *is* the definition, but then clarified that it is a *consequence* of the definition. Henry parsed for himself an aspect of the relationship between his concept image and the concept definition by explicating the relationship between a subspace passing through the origin and being closed under scalar multiplication and addition. Note that Henry said a subspace must “pass through the origin,” as opposed to “must contain the zero vector.” This distinction is additional evidence for his geometric object, rather than algebraic object, image of subspace. Two other students, Shelley and Nancy, also pointed out that any subspace must contain the origin. In addition, they both gave a proof as to why this had to be the case by illustrating that the subspace would not be closed under scalar multiplication (where the scalar is zero) if that were not true.

Discussion

Tall and Vinner (1981) pointed out that a person’s concept image may contain elements that conflict with the formal definition. These *potential conflict factors* may obstruct students’ learning of the formal theory built around the concept in question. We describe such an example from our data and suggest that this particular conflict also affords useful pedagogical opportunities.

Our example of such a conflict between concept image and concept definition is found in a certain “misconception” exhibited by all the students interviewed. This is the belief that the spaces $\mathbf{R}^n$ are nested, in the sense that $\mathbf{R}^2$ is a subspace of $\mathbf{R}^3$, which is a subspace of $\mathbf{R}^4$, and so forth. It often includes the view that *any* 2-dimensional subspace of $\mathbf{R}^n$ “is” $\mathbf{R}^2$. Adhering strictly to the definitions of these vector spaces, this view is false. $\mathbf{R}^2$, for example, is the set of ordered pairs of real numbers (x, y), while $\mathbf{R}^3$ is the set of ordered triples (x, y, z), so the former is not even a subset of the latter. However, based on most students’ (and mathematicians’) image of $\mathbf{R}^2$, the *x-y* plane is certainly a subset and a subspace of three-dimensional space equipped with a Cartesian coordinate system. All planar subspaces of $\mathbf{R}^3$ can be obtained by rotating the *x-y* plane, and one often views rotations as not changing the identity of a geometric object. Thus, which planar subspace of $\mathbf{R}^3$ “is” $\mathbf{R}^2$?. Furthermore, vectors, as used in physics, represented by arrows in space, do not change their character as three-dimensional vectors just because they happen to lie in a horizontal plane. More generally, a vector having $k < n$ components can be “padded” with final zeros so as to become an element of $\mathbf{R}^n$. Observation in linear algebra classes has demonstrated, for instance, that students often conceive of a vector in $\mathbf{R}^3$ with a zero in the z component as being part of $\mathbf{R}^2$. The idea that padding with zeros does not change a vector may be related to early conceptions of zero as “nothing.” The dimension of a subspace is its most important characteristic and gives the number of parameters needed to determine a vector in that subspace. Because of this, we *want* students’ concept images to include the idea that subspaces of the same dimension are essentially interchangeable.

Students expressed these beliefs about “nested subspaces” in our interviews in several ways. They often volunteered $\mathbf{R}^1$ or $\mathbf{R}^2$ as examples of subspaces of $\mathbf{R}^3$. Some seemed to use $\mathbf{R}^k$ as a way of referring to any k-dimensional subspace of $\mathbf{R}^n$, for example stating that three linearly independent vectors in $\mathbf{R}^6$ would span a subspace which would be $\mathbf{R}^3$. For many students, the nested subspace idea played a useful explanatory role. Asked why two vectors could not span $\mathbf{R}^3$, Nancy said “they can only describe points with two variables,” or that the z component of every vector in their span would necessarily vanish. This is indeed a helpful image, but not consistent with the definitions. Kim, when pressed, realized that two vectors could span a plane inclined relative to the coordinate axes, on which x, y, and z could all be nonzero. Mark (see below) realized that three vectors in $\mathbf{R}^4$ cannot literally span $\mathbf{R}^3$ because the vectors have four components, but he still asserted that $\mathbf{R}^4$ is nested inside $\mathbf{R}^5$ and so forth. Ted, when asked why a basis for $\mathbf{R}^n$ must contain n vectors, replied that if there were only n-1, they would lie in $\mathbf{R}^{n-1}$ instead.

Mathematicians are familiar with the subtleties of uncritically regarding two mathematical objects as the same and deal with this conflict between image and definition by means of the concept of *isomorphism*. An isomorphism of vector spaces is an invertible linear mapping between them. Such a mapping between $\mathbf{R}^2$ and the *x-y* plane of $\mathbf{R}^3$, for instance, could be given by sending (x, y) to $(x, y, 0)$. Of course, there is no reason the extra zero must appear in the third position—the *y-z* or *x-z* plane would serve as well. Any plane subspace of any $\mathbf{R}^n$ is isomorphic to $\mathbf{R}^2$, but there is no standard isomorphism between them: a choice of basis $\{\mathbf{v},\mathbf{w}\}$ for the subspace must be made in order to identify (x, y) with $x\mathbf{v} + y\mathbf{w}$. An isomorphism allows two subspaces to be regarded as the same by providing the extra information of how their points are to be matched with each other. This essentially requires the specification of a coordinate system in each subspace; points having the same coordinates are then matched. There is no "canonical" or preferred way of doing this, but infinitely many possible choices.

The notion of isomorphism was not covered in the course taken by these students. Thus, rather than diagnosing them as having the "misconception" that $\mathbf{R}^k$ is a nested subspace of $\mathbf{R}^n$, we view them as utilizing imagery important for constructing an early concept image of isomorphism. In the development of their images of subspace, we have seen evidence that students want to regard some subspaces as "the same." Instructors should be aware of the opportunities to see the usefulness of this, as well as the confusion it can potentially cause for students. The problem is to retain the usefulness of the notion of two spaces being "the same" while avoiding the confusion it generates. Harel's Necessity Principle (Harel, 2000) states that in order for students to learn, they must see the intellectual necessity for the concepts and procedures being taught. Hence, identification and elaboration of this particular problem can help teachers to necessitate the concept of isomorphism. Teachers who are aware of this developing issue can pose questions that focus on it and help students to analyze and refine the notion of "the same" toward that of "isomorphic." In this way the conflict between image and definition becomes a perturbation that affords pedagogical opportunities, allowing teachers to build from students' informal concept images of isomorphism in order to develop a concept definition of isomorphism.

Conclusion

The goal of this study was to investigate the ways that students conceptualize fundamental ideas in linear algebra and how those conceptualizations are connected to the ideas' formal definitions. Subspace was an appropriate concept to explore because of the contrast between the abstract, algebraic language used in the formal definition and the concrete geometric image (at least in low dimensions) provided in students' explanations. Our analysis showed that, despite this, students were able to interpret the formal definition in terms of their rich concept images. They could connect specific aspects of their images, such as flatness and inclusion of the zero vector, to the algebraic closure specified by the definition. They realized the usefulness of the definition for identifying examples and nonexamples and for justification of properties of subspaces. Thus, this study demonstrates how, in linear algebra, definitions can be an important tool in developing overall intuitions about mathematical concepts.

The students also demonstrated how the concept definition could become a catalyst for tying together seemingly disparate and potentially conflicting aspects of the concept definition, like their own geometric and algebraic intuitions. Our results suggest that emphasizing where definitions can be used to explicitly address this is a productive practice for teachers who want students to better understand the definitions for themselves.

We also saw problems that students have in understanding the ways that parent spaces and subspaces are and are not connected. The students demonstrated how the geometric images used to describe spaces such as $\mathbf{R}^2$ and $\mathbf{R}^3$ can lead to the conclusion that they are nested, a potential conflict factor in the terminology of Tall and Vinner. This can necessitate the idea of isomorphism. But our data has also shown how the definition could possibly be used to deal with this confusion. Consider Mark's response below as an example:

Mark: I just realized that they have, I think I made the foolish assumption because there were three vectors that they were in $\mathbf{R}^3$...There's actually four coordinates in each vector, which means they're contained in $\mathbf{R}^4$...They actually can't even be contained in $\mathbf{R}^3$. And so, any addition of these two can't be in $\mathbf{R}^3$, because they're going to have four coordinates.

By reflecting on the definition, Mark has been able to pinpoint where his image misled him. His understanding of the definition allowed him to deal with the subspace "misconception" in a meaningful and appropriate way.

Zandieh & Rasmussen (in press) discuss the dual role that definitions can play in students' development of mathematical knowledge. Definitions can not only play the role of describing certain mathematical objects, but can also play the role of aiding in the development of further intuitions about those same mathematical objects and the systems that they are a part of. In the case of subspace, the definition can act as a unifying factor between concept images and other concepts and problems related to subspace, such as isomorphism. Being aware of students' common concept images and interpretations of associated definitions has positive pedagogical implications. Not only are teachers made aware of how students think about fundamental ideas in linear algebra, but they can also use this information as the foundation for a meaningful, student-centered development of subsequent mathematical ideas.

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