The Role of Quantitative and Covariational Reasoning in Students' Orientation to Novel Precalculus Problems

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Introduction

In this paper, we present findings and results from an investigation into the development of students' mathematical problem solving abilities as they participated in a semester long precalculus course. Our investigation built on results from previous studies into how students solve novel mathematics problems, and was grounded in learning theory on function from a quantitative and covariational reasoning perspective. We provide an overview of problem-solving literature related to how students solve novel word problems. Then, we describe how our present research question contributes to the broader knowledge base for student problem-solving. We then present data supporting major conclusions from the investigation. Finally, we attempt to describe implications our results have for practicing teachers, researchers, and students.

Background

Problem-Solving

In Polya's (1957) study of students' problem solving abilities, he proposed that the problem solving process involved four stages: understanding the problem, devising a solution method, carrying out the solution method, and looking back. In Polya's view, understanding the problem involved first... His description of problem solving is powerful for documenting what students are *doing*, but does not address how students are thinking so what they are doing makes sense. For example… These stages also suggest Polya believed making sense of the problem and building a solution method was important. Our work is about quantitative reasoning, and what does it take to conceptualize these knowns and unknowns.

Schoenfeld (1992), described four categories of knowledge that are needed to be a successful problem solver in mathematics that emerged from his protocol analysis of students as they solved mathematics problems. These four aspects were: resources, including proposition and procedural knowledge, heuristics, which are strategies and techniques for problem solving, control, and beliefs. Schoenfeld argued that successful solution of mathematics problems depended on a combination of resources, heuristics, control, and beliefs. He believed that these problem-solving processes were to be learned and taught. For example, Schoenfeld believed problem solving was something that could be transmitted from teacher to student. His framing of problem solving, as with Polya, did not take into account the mental actions of individual students. As with Polya, he highlighted making sense of the problem and constructing a solution method (based on prior procedural knowledge) as important to solving mathematics problems.

These two seminal studies highlighted the role of making sense of a problem and proposing a solution in solving mathematics problems. They also framed problem solving in a way that did not attend to students' mental actions as part of the problem solving process. This study sought to describe how making sense of a situation and proposing a solution method (described as something students *do* by Polya and Schoenfeld), might be explained by the students' mental actions and ways of thinking. We drew heavily on Carlson & Bloom's (2005) taxonomy that highlighted the importance of attending to student thinking in a problem solving context.

Carlson & Bloom (2005) proposed a general problem solving taxonomy highlighting the mental actions driving students' behavior in problem solving situations. Carlson & Bloom (2005) described *orientation* as an important problem-solving phase. Making sense of the problem, organizing information, and constructing possible modes of solution are the key behaviors associated with the orientation phase. We adopted the idea that making-sense of the problem, organizing and proposing modes of solution are mental actions that manifest themselves in behavior. Attending to students' mental actions allowed us to describe problem solving as an active, highly individualized process while Schoenfeld and Polya's helped us identify the behaviors students exhibit during problem solving in order to propose ways of thinking that led students to exhibit specific problem solving strategies.

Carlson and Bloom (2005) identified that the mathematicians in their study were paying attention to attributes of a situation and constructing solutions based on how they related those attributes. For example, when asked to describe the relationship between the volume of water in a bottle and the height of that water in the bottle, mathematicians coordinated measuring the volume of the water with measuring the height of the water. We suggest that the mathematicians were extremely proficient in working this problem because they had conceived of two attributes of the situation, height of water in the bottle and volume of water in the bottle, and were able to imagine measuring these attributes. Conceiving of a situation in that way we believe is an example of *quantification*.

The Bottle Problem

Imagine this bottle filling with water. Construct a rough sketch of the graph of the height as a function of the amount of water that is in the bottle.

1 **Figure 1: The Bottle Problem from Carlson & Bloom (2005)**

Quantitative and Covariational Reasoning

In Thompson's (1993, 1994) work, there is parallel work on quantity and quantitative reasoning that fleshes out the mental processes in identifying and constructing quantities. He identified identify and constructing quantities was an important tool for students' conceiving of a novel problem situation. Carlson and Bloom said nothing about quantity, but pointed out that what Thompson referred to as quantity was critical in understanding the way mathematicians made sense of a problem.

Thompson (1990, 1993) described a quantity as a conceived attribute of an object that one can imagine being measured. For instance, a student might conceive of the water in the bottle, the volume of the water, and measuring the volume in cubic inches. Thompson describes the water in the bottle as the object, the volume of the water the attribute of the object, and the cubic inches as a way of measuring an attribute of the object. Together, these conceptions are an example of conceiving of a situation quantitatively. Thompson (1990) emphasized the importance of quantitative reasoning for conceiving of novel word problems. He argued that when a student conceives quantitatively of a situation in terms of its measurable attributes, it provides a powerful problem-solving mechanism for that student. For example, if a student conceives of the volume of the water in a bottle, constructing a measuring process to measure, conceiving of that measuring process entails relating other measurable attributes of the water in the bottle.

Carlson et al. (2002) and Saldanha & Thompson (1998) describe covariational reasoning as holding in mind the sustained image of two simultaneously varying quantities. Carlson (2002), Oehrtman, Carlson & Thompson (2008) , and Saldanha & Thompson (1998) emphasize the importance of covariational reasoning for conceiving of a situation. For example, if a student thinks about the varying relationship between the height and volume of the water in a bottle, the student must first conceive of how both height and volume of water vary individually. Imagining how they vary individually entails conceiving of measuring process for height and volume. When a learner coordinates the variation of height and volume and holds an image in mind of these two varying simultaneously, he/she is able to conceive of how height and volume of the water vary together. Carlson et al. (2002) and Oehrtman et al. (2008) suggest that conceiving of a problem in terms of quantities varying in tandem is important for making sense of and constructing modes of solution to novel word problems.

In sum, we have suggested that orientation, including making sense of, planning, and constructing possible modes of solution for novel word problems is a key problem solving phase for both calculus students and advanced mathematics problem solvers. Thus, we argue that orientation to problem solving is an important phase for understanding how students solve complex word problems. We have also shown that quantitative and covariational reasoning are important in making sense of and constructing solutions for novel word problems. However, revealed in the review of literature, there have been no empirical investigations into describing the role that quantitative and covariational reasoning play in students' orientation to novel precalculus word problems.

Objectives and Research Question

In response to the lack of research describing the role of quantitative and covariational reasoning in orientation to novel word problems, we designed a study to investigate how a curriculum that focused on developing quantitative and covariational reasoning impacts students' development of these abilities. Our major research question was to understand what role quantitative and covariational reasoning play in a student's orientation to a novel problem-solving context. In order to answer this question we identified three research goals.

1) Describe the ways of thinking about solving novel mathematics problems students had prior to receiving instruction in a redesigned precalculus course.
2) Describe and explain how students' ways of thinking and problem solving abilities changed or remained the same over the course of one semester.
3) Describe, explain and theorize about the role construction of quantity and covariation of quantity played in students' orienting to novel applied precalculus problems.

We attempted to identify and describe the mental processes that students used to construct and relate quantities when responding to a novel problem. We were particularly interested in describing the relationship of these mental processes to a student's ability to conceive of functional relationships and reason covariationally with regards to these relationships. In the next section, we describe our assumptions about learning and understanding, and describe how these assumptions framed the design of our investigation drove the analysis of study data.

Theoretical Framework

The investigation drew from Glasersfeld's (1995) view of radical constructivism that learning begins and ends with the learner. He argues that each learner's mathematical reality is independent and unknowable to others. Thus, we as researchers cannot be completely confident in our understanding of a student's mathematical reality, but we can describe ways of thinking so that if a student had them, their actions and descriptions in the context of solving a problem make sense to the observer. The classroom is a place for students to develop ways of thinking about mathematics, although each student will use a different way of thinking, since the reality he/she experiences is fundamentally tied to his/her prior experiences. We also assume that what the learner does and says in a given situation is coherent for that learner. By this, we mean that though an observer might conceive of a students' actions as nonsensical, the learner has ways of thinking that make those actions reasonable. As a result of these assumptions, an aim of this study was to suggest ways of thinking so that if the students had them, their ways of making sense of, and proposing possible solutions to a problem might be coherent.

In order to support this aim, our interview questions and classroom interactions with the students focused on helping students reveal attitudes, beliefs, behaviors, and verbal descriptions while working to solve novel word problems in precalculus. Because we had previously conjectured that quantitative and covariational reasoning were important to how students thought about novel word problems, we focused on asking questions and viewing the data through the lens of students' quantitative reasoning in the context of the orientation phase. Making sense of the problem, organizing information, and constructing possible modes of solution are the key behaviors associated with the orientation phase. According to Thompson (1994), quantitative reasoning is the analysis of a problem-solving situation in terms of the measurable attributes of a conceived object. A quantitative structure is a network of quantities and relationships between those quantities. Quantitative reasoning focuses on the importance of developing students' ways of thinking to conceive of problem situations in terms of measurable attributes of that problem. Thus, our particular lens for the study was considering students' classroom interactions and responses in order to describe the role of quantitative reasoning to the orientation phase of approaching a novel word problem. Next, we describe the subjects, data collection, and analytical method.

Methods

Subjects and Setting

The authors tracked six students over the course of their enrollment in a one-semester, redesigned, precalculus course. All subjects voluntarily participated, and received monetary compensation for their time. Two students were female, and four were male. All were enrolled in a redesigned precalculus course, which focused on developing students' ability to reason conceptually about functions and quantity in the context of developing an ability to solve conceptually based mathematical problems. These students were identified as high-performing, middle-performing, and low-performing based on a pre-calculus assessment instrument designed to predict success in a conceptually based precalculus course(Carlson, Oehrtman, & Engelke, 2009). Individual interviews revealed that all six students completed high school with a GPA of 3.5 or above on a 4.0 scale, and each had completed at least one course that aimed to teach precalculus. Two of the students enrolled, and passed an AP calculus course in high school, but neither scored above 2 on the Advanced Placement exam. At the conclusion of the semester in which we conducted the study, all six students passed the precalculus course. Two students earned a C, three earned a B, and one student earned an A. Three of the six students planned to enroll in calculus I the following semester, while the remaining three students were unsure about what mathematics courses they would take in the future.

Data Collection & Analysis

Students attended every class session over the course of the semester. The research team interviewed each student in an individual clinical interview at six different times during the semester. The sequence of six interviews focused on revealing insights into students' understanding of function, exponential growth, and trigonometric relationships from a covariation perspective. The students attempted to solve three novel mathematics problems in each interview session. Students were videotaped, with permission, during classroom sessions and clinical interviews. Students' written work was scanned as a backup if the camera was not able to capture what the student wrote during the interview or classroom session.

The video data was synthesized and coded using a combination of open and axial coding as described by (Strauss & Corbin, 1998). We openly coded the video data to identify episodes we believed fell into the orienting phase of problem solving. The next level of coding broke these episodes into making sense of the problem, organizing information, and constructing possible modes of solution. The video data was then coded for episodes the authors believed exhibited a student's quantitative and/or covariational reasoning abilities. The authors then used axial coding to characterize how the categories were related. The authors were able to take over thirty emergent categories from open coding and abstract them into three major categories. The major explanatory categories that emerged from the data were identified as covariation of number, covariation of quantity, quantification, and mental images. We describe these emergent categories below.

Results

Making Sense of the Box by Attending to Quantities

This section presents results from our analysis of interview data from students as they responded to the box problem (Figure 1). The student responses to this task suggests that thinking about attributes of the box, such as length, width and height of the box, along with their possible measures were key for their construction of an image of the box's volume. All proper names are fictitious and do not represent the actual name of the subject.

> A box designer has been charged with the task of determining the volume of various boxes that can be constructed from cutting four equal-sized square corners of a 14-inch by 17-inch sheet of cardboard and turning up the sides. Construct a formula that relates the volume *V* of the box, to the length of the side of the cutout *x*.

Figure 2: The Box Problem

Adam's Understanding of the Box

STU: I'm voicing out the variables. What I would do is I would write out the basic formula of a volume for our box, which is a formula V equals length times width times height. X is our length of the cutout, and I would draw a diagram of our sheet, label it 14 by 17 and draw the squares I would cut out to make the box.
INT: What do you mean by V?
STU: Volume, so you get this when you multiply length by the height by the width of the box, it's a number when you plug in for the values of length, width and height.
INT: Okay.
STU: Well, as long as I make square cutouts, then it's pretty clear that the box will have the same height all the way around because the however big my cutouts are, that's the height of my box, so if all the cutouts are umm…, the same then the height has to be the same.
INT: I noticed that you wrote 14-2x here, could you tell me what that represents?
STU: Yeah. Well, I know we have a piece of paper that has a width of 14. But, its not always going to be 14.
STU: Well, at first, I thought I would just multiply 14, 17, and the height. Then I thought about what would happen if I made the box. The width would actually change depending on how big I made the cutout thingy. It would actually depend on twice the size of the cutout. Umm, that's not right. What I mean is you have to take away two cutouts from 14 to find the width, and those same two cutouts from 17 to find the length. That is where I get 14-2x and 17-2x.

Adam stated he was voicing out the variables (line 1), and we propose at that instance that he conceptualized the fixed quantities of length and width of the paper, and the varying quantity *x* which is the length of the side of the cutout. Adam also reveals a strong conceptualization of the meaning of volume as composed of length, width and height of the box (lines 6-7). We suggest he was making sense of the problem by constructing an image of a box (lines 9-11). Adam revealed he initially attempted to use the numbers in the problem (line 16), but then was able to imagine the width of the box as a varying value (lines 16-17). We suggest that Adam's attention to the measurable attributes, or quantification, of the situation (lines 19-21) was key in his thinking about the length of the box as 17-2x instead of 17 and the width as 14-2x instead of 14. Lines 18-22 indicate that Adam not only imagined the width as related to the cutout length, but he was imagining the length of the base of the box as 14 reduced by two cutout lengths. We also suggest that Adam viewed the width of the base of the box as something that could be measured indicated by him seeing width as a quantity that takes on different values (lines 21-22). This passage supports the idea that quantification of a complex word problem allows a student to place the numbers given in a problem in context. Adam's quantification of the situation supported his constructing a viable image of the problem, which in turn allowed him to think about and organize the complexities of the situation. We now turn to an example of quantification's role in orienting to the box problem.

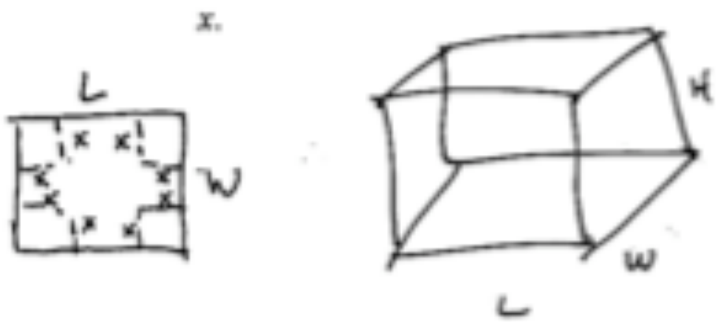

Figure 3: Adam's Construction of the Box

Chris's Understanding of the Box

STU: [Reads problem aloud]. Well, umm, let me think. [Long pause] I think the volume would be length minus two x times width minus two times x times x.
INT: Could you say more about what you mean?
STU: The height is as tall as the cutout is long, I, umm, that's the easiest way to say it. Then the formula comes from multiplying the length and width and height of the box, which aren't the same measurements as the diagram.
INT: I see that you have written L = 2x-14 and W = 2x-17, could you explain what these mean?
STU: Sure. What I thought about was if I want to know the length, I have to subtract the cutout. At first, I thought it would just be taking away one cutout, but I drew the diagram and realized you would need to take away two cutouts from the length and width you start with. So the L [points to the formula] is the length of the box and the W is the width of the box. All you do is plug in your cutout for x, and you have the length and width.
INT: Could you show me what you mean?
STU: So let's say that I pick a cutout like 2, then I just plug into my formula for volume. So I multiply -15 times -12 times 2, and I get a volume of 360, in whatever units I measure. So I know that my formula is probably accurate. Wait, wait… the length and width

cannot be negative, oops, I said it right, but I did not write it right. I should reverse the equations that I have (see star in figure 3). There, that makes more sense with what I said.

Chris revealed a strong conceptualization of box volume as formed by length, width and height of the box (line 27) and revealed he was thinking of length and width of the box as varying quantities (lines 27-28). Chris understood the height of the box as coupled with the length of the cutout (lines 30-31), and revealed he made a distinction between the paper he used to construct the box and the box itself (lines 30-31). When asked to describe the formulas for length and width of the box, Chris revealed a strong image of the situation, in which he imagined the length and width of the box varying (lines 34-3). We suggest that Chris conceived of the box quantitatively, or in a way that allowed him to relate the numerous attributes in the problem. Lastly, we argue that Chris's strong imagery of the dynamic box allowed him to make sense of why he calculated a negative length and width (lines 41-44). Chris made sense of the problem by constructing a mental image of the measurable attributes of the box. We suggest that the image of the box was the mechanism by which he made sense of the problem situation, and further, that the image he constructed was a result of him quantifying the situation. Thus, his quantification of the situation produced an image of what was going on, and that image allowed Chris to make sense of the problem and propose a method for solution (multiplying the varying quantities length, width and height of the box).

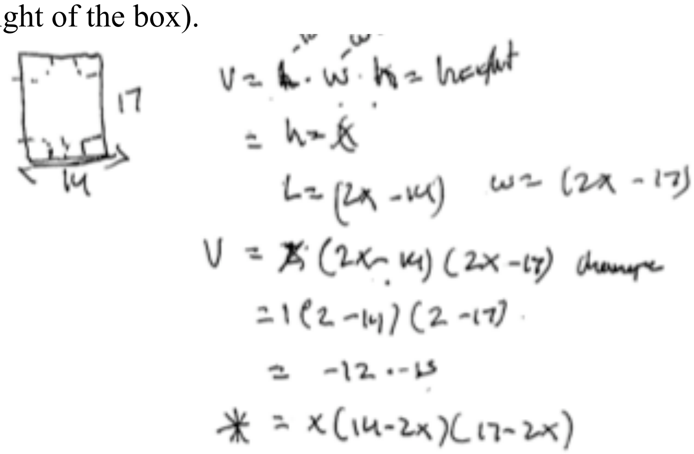

Figure 4: Chris's Understanding of the Box

Discussion

We have proposed that Adam and Chris made sense of the box problem by constructing a diagram of the situation, which then guided their construction of a formula for the volume of the box. In both cases, their imagery of the situation represented the varying quantities they conceived of, and after constructing an image of each quantity varying individually, they tied the three together into their formula for volume. We propose that the construction of the diagram was a representation of the students' mental quantification of the problem. Adam and Chris attended to the measurable attributes of the box, and represented the range of values the attribute could take on using the variable x. Because both students conceived of interrelated quantities in the context of a strong image of the situation, they were able to justify their solutions and in Chris's case, resolve competing calculations results of the problem. These results suggest that students' quantification of the situation in terms of fixed and varying quantities allows them to create a dynamic mental image of the situation. This mental image then guides and checks their understanding as they make sense of, and then suggest a solution to the problem.

Covariation of Quantity: Coordinating Changes in Aspects of the Box

In orienting to the box problem, LeAnn (another student in the study) and Chris revealed two prevalent means of operating covariationally. The first of these, covariation of values, explains the behavior of the formula for volume by plugging in numbers for the length, width and height of the cutout. The second, covariation of quantity, explains the behavior of the volume of the box by attending to variations in length, width, and height.

LeAnn: Covariation of Values

STU: The problem asks me to now explain how the volume of the box changes as the length of the cutout increases. Alright, so I have this formula 2x(17-2x)(14-2x), but it's harder than just plugging in a value, because you have to plug in a lot of values to test it out. I think I am going to start by just working with the length. (reveals that in order to test a formula, one has to check calculations, an inductive way of thinking rather than deductive)

INT: Could you explain what you mean?

STU: Okay. I am just going to look at the length of the box and the length of the cutout to simplify things first [Student plugs in values of 1, 2, 3, and 4] for the length of the side of the cutout and writes down the resulting values]. (Focuses on using the formula for only length, rather than thinking about volume. Sees it as part of thinking about the volume.)

INT: Okay, so what did your calculations tell you?

STU: They told me the corresponding values of volume for when the cutout is 1, 2, 3 and 4 inches.

INT: Okay, does that help you describe how length of the box changes as the cutout increases in size?

STU: [Reaches for calculator]. Well the formula for the length of the box is 17-2x. So what I Would do is plug this into a table. I used the table function on my calculator, and it says the length gets smaller when x gets bigger.

INT: Does that make sense with what you know about the box?

STU: It if gets smaller like it does, well, hmm. I guess it says that when the cutout gets bigger, the length has to get smaller. Umm, do you have a sheet of paper?

INT: Yeah. Here you go [hands sheet of paper to student]

STU: Okay, yeah [holds and turns around sheet of paper]. I can see if I made the cutout bigger then the length would have to be smaller. So I guess what I found in the formula, it makes sense with how I drew the box before. So each time I increase the length of the cutout by 1, the length of the box decreases by the same amount each time.

LeAnn produced a viable formula for the volume of the box, but revealed she did not conceive of the situation as dynamic. By this, we mean she may have written a viable formula (line 46), but she had not attended to how each part of the formula, length, width, and height, varied as the cutout length changed. She had conceived of the problem in terms of multiplying the length, width and height of the box, but she did not initially attend to the quantities varying (lines 60-62) as evidenced by her reliance on the calculator to construct numerical patterns. LeAnn focused on working through the problem by tracking the values of the length of the side of the box and the length of the cutout (lines 53-56). We say she tracked the values because she focused on plugging in values for the cutout length to produce a length of the box, but did not evidence she saw the cutout length and box length varying in tandem. She made sense of the problem by coordinating changes in the value of the cutout length with the length of the side of the box (lines 51-53), but was not able to anticipate how the length of the box and the cutout length would change in tandem. LeAnn noticed that the length of the box decreased as she inputted bigger values of the cutout length. She explained this pattern (lines 61-64) by referring to the sheet of paper and thinking about why the length of the box would decrease as the cutout length increased (lines 67-69). Her response while holding the sheet of paper also suggests that she could imagine forming the box from a rectangular piece of paper. This pattern of constructing solutions and explanations to problems by testing out value patterns between two related things (we use things to contrast the situation with quantity) persisted over the course of the entire semester.

Chris: Covariation of Quantities

STU: Alright, so I have this formula x(17-2x)(14-2x). I can look at the problem like this, so if I increase or decrease the cutout, which is x, then each part of the box, the length, width and height change.
INT: Let's follow that a little further. So how do the volume of the box and the length of the side of the cutout relate to each other?
STU: Well, I can say they are related by this formula. So I can plug in a number for x which gives a volume for that cutout length. I could do that for any x I wanted, except for when the volume went below zero, that isn't possible – we talked about volume not being negative in class.
INT: Now, what happens to the volume of the box as you increase the cutout?
STU: Well, it's not hard to figure out what's happening to each piece of the formula. The length and width get smaller as you increase x, which is the cutout size. But the height, which is also x, gets bigger. So you kind of have to figure out how the decrease in the length and width balance the increase in height.
INT: Could you say more about what you mean by balance?
STU: Well, you have to think about the volume, and the volume depends on umm… the length, width, and height of the box. So if the increase in the height outweighs the decreases in the length and width, the volume should increase. Umm… [long pause]. I'm just not sure how to determine the outweighing, I'd probably plug some values in.
INT: Okay, so let's follow what you are thinking.
STU: Well, I know there are only certain values I can plug in, what we talked about before. I could plug in tons of values and tell you how the volume is changing but it's easier to think about the box, then you can be smart about the values you might want to think about.

Chris understood that at each instant, if the length of the side of the cutout changes, so do the length, width and height of the box (lines 65-67). As he revealed in the initial part of the interview, his image of the situation was of a the quantities in the situation covarying as the cutout length varied. Whereas LeAnn used value patterns to explain how the length of the side and the length of the cutout changed, Chris used his image of the box to explain how the length and width of the box changed in tandem with the cutout length (lines 75-78). We propose that he had a sustained image of two quantities simultaneously varying. Chris's encountered difficulty when attempting to covary the length, width, and height of the box in tandem. We suggest he was doing this in order to understand how the volume of the box changed. His idea of determing the, "outweighing" (line 83), reveals that he understood that volume was a quantitative operation formed by the length, width, and height of the box. In constructing a possible solution to the box problem, Chris used his quantification of the situation to determine how length, width, and height of the box's values varied in tandem to produce the volume of the box.

Discussion

LeAnn and Chris made sense of the task asking them to describe how the volume of the box changed in relation to the cutout length by attempting to discern patterns in the length (LeAnn), and length, width and height (Chris) of the box. LeAnn attempted to *find* patterns by plugging in values for the length of the cutout and by focusing only on the length of the side of the box. He then explained the patterns based on his diagram of the box. Chris attempted to *predict* patterns of the volume of the box in relation to the length of the cutout based on his understanding of the box. We suggest Chris did not focus on plugging in values because he did not need to use values to explain the covariation of cutout length and box volume. Instead, his understanding of the quantities, as evidenced by his quantification of the box, provided him a powerful tool with which he conjectured how the volume and cutout length varied in tandem. We note that explaining the covariation of the length of the cutout and the volume of the box is a complex task without graphing the function. However, Chris and LeAnn's responses indicate that an initial quantification, and then covariation of the attributes of the situation is key to orienting to complex mathematical tasks.

Conclusions

The preceding analyses reveal that students' make sense of complex, novel problems by creating diagrams of the measurable attributes of the situation. We believe that the students' creation of these diagrams depicting relationships between the measurable attributes of a situation is reliant upon quantification. Students also attempt to find patterns to describe how independent and dependent variables change in a given situation. The previous passages indicate that describing these patterns is largely imagining two quantities or values changing in tandem. Adam and Chris revealed a distinction between finding a pattern (covarying values), and predicting a pattern (covarying quantities). Thus, we propose that students' quantification of a task in terms of its measurable attributes, and covariation of those measurable attributes, allows a student to make sense of a complex mathematical task by creating a diagram of the situation and allows a student to predict and explain patterns in the formulas that relate those measurable attributes.

In orienting to a problem solving task, if a student makes sense of the problem by thinking purely about plugging numbers into a formula he or she might be able to answer a question that asks about the volume of the box for a specific cutout value. However, as LeAnn's responses displayed, without quantifying the situation in terms of its measurable attributes and imagining those attributes changing in tandem, the student may not be able to explain why the numbers are changing in a specific pattern or decide what numbers are appropriate to plug into the formula. Thus, a student who does not attend to quantities or think about coordinating changes between quantities may appear to have difficulty imagining and relating quantities in a word problem. A quantitative structure allows the student to think about a dynamic problem situation in which there are numerous quantitative relationships and create an image of that situation which drives how they propose solutions to the novel task.

In future research, we believe it will be useful to identify and explain what factors play into students' formation of quantitative structure in order to better understand how a teacher might facilitate students' formation of these structures in a classroom setting. We also believe it will be important to investigate not only whether students construct a diagram of a situation, but how they use diagrams in precalculus, calculus, and beyond.

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