Undergraduate Students' Interpretations of the Equals Sign

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Abstract: While many instructors may assume that their students have a good conception of equality, recent investigations on students' algebraic reasoning suggest that this may not be the case. This report examines the ways undergraduate students interpret expressions involving the equals sign and use the equals sign to represent situations involving comparisons. The study describes two theoretical perspectives for interpreting the results (mental models and a semiotic perspective) and discusses the motivation for making a theoretical shift.

Keywords: Equality, Algebra, Semiotics

Although increased attention has been given to middle-school students' understanding of equality (e.g. Alibali, Knuth, Hattikudur, McNeil & Stephens, 2007), relatively little is known about how undergraduate students think about equality and use the equals sign. Many teachers (and researchers) at the undergraduate level may assume that their students indeed have a good conception of equality. However, recent investigations on students' solution strategies for specific types of algebra word problems suggest that this may not be the case.

For example, Weinberg (2007) interviewed a student who used the equals sign to represent a comparison between two groups. He asked students to write an algebraic expression to represent the statement: "There are twelve times as many students as professors at this university." One of the students in the study wrote "$12s=p$" and said: "I put students to professors, and then I originally put dots for a ratio, then I wasn't sure if a ratio would be right, so I put equal—I'm not sure if that's the same thing." Other students in the study used the equals sign in different, nonstandard ways.

The aim of this paper is twofold. First, the paper will report the results of a study that investigates students' conceptions of equality from a cognitive perspective. The analysis of these results motivated a shift in theoretical perspective, from focusing on students' mental models to using ideas from semiotics; the second aim of this paper is to describe this theoretical shift and re-interpret the results.

Background and Initial Theoretical Perspective

Equality is a concept that is introduced to students early in their mathematical careers but is given little formal instructional attention after elementary school (Alibali et al., 2007). Researchers have generally described students' conceptions of the equals sign as either *operational* or *relational* (Behr, Erlwanger & Nichols, 1980; Ginsburg, 1982; Kieran, 1981). Students with an operational conception view the equals sign as a signal to "do something" (e.g. say that the solution to "2+3=__+1" is 6) while a student with a relational conception recognizes that the equals sign indicates equivalence. Larson, Zandieh, Rasmussen and Henderson (2009) used this perspective to describe the conceptions of students in a linear algebra class. They found that the meaning students ascribe to the equals sign is context-dependent–a view that is supported by McNeil and Alibali (2005). In addition, McNeil (2007) found that students' conception of equality evolves based on their previous understanding.

Equality can be viewed as an important component of the mental models that students develop for solving problems. Although there are few descriptions of models that specifically describe conceptions of equality, numerous researchers have described the construction of problem models in general (e.g. English & Halford, 1995; Hegarty, Mayer & Monk, 1995; Nathan, Kintsch & Young, 1992). The study described here used multiplicative comparison problems (e.g. the "student-professor" problem described above) as a tool to evoke students' conceptions of equality. Other researchers have described students' mental models for these multiplicative comparisons (e.g. Clement, 1982; Lewis & Mayer, 1987; MacGregor & Stacey, 1993; Verschaffel, 1994). Although Rosnick and Clement (1980) noted that the equals sign may play a significant role in these models, no subsequent research has addressed this perspective.

Research Questions

Weinberg (2007, 2009) investigated the mental models students' constructed when solving multiplicative comparison word problems. He found that students used multiple solution strategies and that these strategies changed when the problem was presented in a new context or when students were asked to answer a different question in the same context. The students' strategies indicated that they were using several mental models, three of which had not been previously described. Weinberg (2009) noted, "equality and the equals sign play[ed] a different role in each of these models" (p. 715) and hypothesized that students' conceptions of equality and the equals sign may play a significant role in students' mental models.

The initial study described here set out to address two research questions:

1. How do students think about equality, as evidenced by their use of the equals sign?
2. Are students' conceptions of equality and the equals sign related to either their success on algebra word problems or the strategies they use to solve them?

Methodology

Students in nine sections of first- and second-semester Calculus classes (n=210) at a northeastern comprehensive college completed a written assessment with four word problems and ten equality problems. All students were invited to participate in an open-ended interview; 27 students were randomly selected from the volunteers. In the interviews, the students were asked to explain their reasoning on each of the problems; the interviews were videotaped and transcribed.

On the written assessment, students worked with algebraic aspects of four word problems that involved multiplicative comparisons (e.g. "four pigs for every three cows"). Students were asked to perform one of three tasks: write an equation to represent the statement (*e.g.* $3p=4c$), write an equation that allowed them to predict the value of one quantity if they knew the value of the other (e.g. $p=4/3c$), or compute the value of one quantity if they knew the value of the other. Students were also presented with ten equations and were asked to decide whether they were correct. Notably, four of the equations were "run-on" expressions (e.g. 2+3=5+2=7; $g(x) = x^2 = g'(x) = 2x$) and one equated an expression with an integer multiple of itself ($2x+12=x+6$).

Results

On the written assessments, students routinely misidentified the "run-on" expressions as correct, and over a third of students identified as correct the equation in which the quantity on one side was a multiple of the quantity on the other side (see **Table 1**).

Expression	Percent identifying as correct
$f(x)=2x^3=f'(x)=6x^2$	88%
$g(x)=x^2=g'(x)=2x$	83%
$2+3=5+2=7$	66%
$5\times 4=20+3=23$	36%
$2x+12=6+x$	37%

Table 1. Student Responses to Equations

A student who said that one of the run-on expressions was correct was not more likely to say that any of the other four were correct Q(4, n=210)=141.67, p<.05. There was also no association between students' responses on the fifth expression and their answers on the first $\chi^2(1, n=210)=3.286$, p>.05, second $\chi^2(1, n=210)=1.498$, p>.05, third $\chi^2(1, n=210)=.436$, p>.05, and fourth $\chi^2(1, n=210)=.765$, p>.05 run-on expressions. There were no significant relationships between getting the equality questions correct and getting any of the word problems correct or using particular strategies on the word problems.

In the interviews, some students who had initially identified the "run-on" expressions as incorrect questioned their written response. Conversely, most students who had initially identified these expressions as correct changed their answer. For example, **Table 2** contains excerpts from an interview with a student who demonstrated this varying interpretation of the run-on expressions:

Expression	Response
$f(x)=2x^3=f'(x)=6x^2$	This is just the derivative. Bring down the three, multiply it by two, six, and subtract the power, so it's six *x* squared.
$2+3=5+2=7$	Because two plus three does not equal five plus two, but five plus two equals seven.

Table 2. A student's responses to run-on equations.

Other students expressed the idea that the run-on equations could be both true and untrue:

Student: If you just take it five doesn't equal seven and seven, but if you want to say two plus three equals five, plus two equals seven, take that separately, that's true, but I was just looking at it as five doesn't equal seven...

Interviewer: So in some readings it might be true, and in others...

S: Yeah.

Over a third of students indicated that the expression $2x+12=6+x$ was correct because they were able to find a value of *x* that made it true (i.e. they could "solve for *x*"). For example, one student responded: "That's just a problem you would solve. It's not really right or wrong, it depends what *x* is." Others reported that the expression was true because they could transform it into a reflexive identity. For example, the student described his solution as follows:

I guess twelve *x* plus, or two *x* plus twelve, set that equal to zero... just get *x* alone.... I guess that's wrong actually, could it be negative, two *x* plus twelve equals negative twelve, divided by two over the *x* equals negative six... or, no that is right, it's just...times two. Just the same amount. If I take the two out, then *x* plus six equals six plus *x*.

The language this student used to describe his solutions to several of the word problems suggested that he viewed the equals sign as *actively* equating two quantities. Table 3 shows how this student's language presents the equals sign as having an operation aspect, much in the same way elementary students describe "3+5=8" as "3 and 5 make 8" (Ginsburg, 1982).

Algebraic Situation	Response
There are twelve times as many students as professors.	I guess I was right the first time... *s* equals twelve *p* [writes], for three times... three times the amount of professors *equals to* the thirty-six students.
There are three trucks for every five sedans.	I set *t* equal to trucks then *s equals to* sedans

Table 3. A student's operational language (emphasis added).

In his solution to one of the word problems, this student explicitly described interchanging the equals sign and a symbol for a ratio. In this example, the student had originally written $3t$=$5s$ as his solution:

Interviewer: Okay. So then this—was this an equals sign or a ratio sign?

Student: This should be a ratio [writes colon over the equals sign] - three trucks for every five sedans, fifteen trucks for every 25 sedans.

After replacing the equals sign with a ratio, the student went on to describe the way he used this equality/ratio relationship to think about the problem. He reported thinking of the two quantities as connected and that the appropriate way to interact with the quantities was to perform the same action on both of them:

So you multiply that times five [points to $3t$] and that times five [points to $5s$] because you're doing the same... it's by the same amount you're multiplying this by.

Even though this student used the equals sign to represent something other than true equality, he was still able to construct a problem model that enabled him to solve the problem correctly.

Other students also used multiple conceptions of the equality and the equals sign when working on a single problem. However, not all students were able to successfully solve the problems. In the following example, a student used proportional reasoning in the same way as the previous student to solve the trucks-and-sedans problem:

Student: Okay. I did a ratio, there are 3 trucks, 5 sedans, and there's a total of 165. So this side is my trucks, so 3 trucks, and eventually I want to determine how many trucks there are, so, this is sedans, 5 sedans, 165, and then I cross multiplied? Yes, then I cross-multiplied, and there are 99 trucks.

Interviewer: Okay. Are you confident with that answer?

Student: Yes.

When the interviewer asked the student to create an algebraic representation of the situation, she gave the same (in correct) answer as the previous student:

Interviewer: What if in this one I asked you to do something similar to the cows and pigs one? So there I wanted you to write an expression to represent the relationship. So here, what if you didn't know there are 165 sedans—what if you just knew there are 3 trucks for every 5 sedans? Could you write me an equation to represent this relationship, maybe using *t* for number of trucks and *s* for number of sedans?

Student: Does it have to be solvable? Can it be more than one variable?

Interviewer: there's *t* and there's *s*.

Student: Alright... *t* times [write 3*t*] yeah, equals [writes 5*s*] 5*s*... because... the number of, wait, no, hang on a sec... yeah, no. This is a weak spot in my math education... No, I don't know what to do.
Interviewer: So, you don't - this one [points to $3t=5s$] doesn't seem right?
Student: I'm not confident about it being right. But, it makes sense to me.
Interviewer: And how did you get the $3t$ equals $5s$?
Student: Cause the number of trucks times 3 should equal the number of sedans times 5.

However, unlike the previous student, this student was subsequently unable to work successfully in the problem situation:
Interviewer: And would that give you—I assume it would give you the same result as the 99 here.
Student: Definitely. I don't know. Yes—no it wouldn't, because... maybe it would... if I plugged in 165 sedans, times 5 divided by 3, would that equal 99? No, because it didn't work for the cows and pigs...
Interviewer: This is trucks and sedans, not cows and pigs...
Student: More fuel-efficient, I don't know... I know it's wrong, I just don't know what to do to fix it.

Although this student was unable to construct a general algebraic model for this particular problem, she *was* able to do so for some of the other multiplicative comparison problems.

Discussion

Students used the equals sign flexibly to represent multiple ideas. In some of the run-on equations, they recognized that it indicated equality between multiple expressions, while in others they viewed it as indicating a directed relationship between the expressions. In the fifth equation, some students took the equals sign to indicate that they should investigate whether two algebraic expressions *could* be equal for specific values of *x*, or *would* be equal for all values of *x*, while other students attempted to find a value of *x* that would make the two expressions equal. In addition, the students viewed the equals sign as playing different roles in the different word problems. For example, a single student in this study used the equals sign in all of the following ways:

1. Describe an equality between two quantities as simultaneously having relational and operational characteristics
2. Denote a relation between identical objects
3. Indicate that one should perform an operation on an algebraic expression
4. Indicate the presence of a non-equality relationship or comparison
5. Link and compare expressions in run-on equations

Students' solution strategies for the word problems suggested that they constructed multiple types of mental models, each of which incorporated different conceptions of equality and the equals sign. The models they used changed depending on the problem, with some students appearing to use multiple models within the same problem. What is striking about these different conceptions of equality—and the resulting uses of the equals sign—is that some students were able to successfully solve problems even if they did not use the equals sign in a "mathematically correct" way. At the same time, other students appeared to construct similar mental models, yet they were unable to solve the problems.

The two students described above highlight these issues. The first student initially appeared to be directly translating the description of the problem into a string of algebraic symbols (Hegarty, Mayer & Monk, 1995). However, he was able to use his representation of "equality" as a tool to help him work manipulate two quantities (the numbers of trucks and sedans), suggesting that he had constructed a model of *systematic comparison* (Weinberg, 2009). In contrast, the second student initially seemed to be using a systematic comparison model in the same situation. However, after she attempted to represent her model algebraically, she was unable to use this representation to perform computations in a way she found satisfactory.

From the perspective of mental models, there seem to be few patterns and little consistency in the ways students conceived of equality and the equals sign. We may hypothesize that students constructed a wide variety of mental models to work with these multiplicative comparison problems and that a wide range of problem types, contexts, and tasks may have activated each type of model. Equality seems to play a distinct role in each model, although it was not always easy—or possible—to determine which conception of equality a student is using based solely on their external representations.

An Alternate Perspective

Although it is possible to describe students' conceptions of equality using mental models, this theoretical perspective made it difficult to create descriptions of students' thinking that had clear relationships to their mathematical activity. In particular, these models did not easily describe students' representational acts, seemed to play a central role in their activity. For example, consider the first student's solution for the trucks-and-sedans problem: he created an external representation to *use* while he solved the problem; even though the equals sign did not match his "internal" conception (i.e. a ratio), it still enabled him to work within the problem context and produce a solution.

This suggests that shifting the theoretical focus from students' internal cognition to their representations might yield more satisfying descriptions of their mathematical activity. The previous discussion of students' conceptions of equality was based on a cognitive perspective in which internal and external representations are seen as distinct and separate. As an example, consider the original research questions:

1. How do students think about equality, as evidenced by their use of the equals sign?
2. Are students' conceptions of equality and the equals sign related to either their success on algebra word problems or the strategies they use to solve them?

The first research question implicitly frames the "use" of the equals sign as an external act that is distinct from the internal act of thinking. Similarly, the second research question presents a "conception" and "strategy" as something that is "inside" a student's mind and is distinct from the external representations they use when solving the word problems.

This focus on representation suggests that a semiotic perspective might offer a viable theoretical alternative. In short, semiotics is about the creation and interpretation of (mathematical) signs. There are many types of semiotic perspectives that are being used in mathematics education research (see, e.g., Berger, 2004; Morgan, 2006; Radford, 2000, 2003, 2009; Sáenz-Ludlow, 2006) and a full presentation of the theory is beyond the scope of this paper. However, Rotman (2006) summarizes the main idea:

> Those things which are 'described'—thoughts, signifieds, notions—and the means by which they are described—scribbles—are mutually constitutive: each causes the presence

of the other; so that mathematicians at the same time think their scribbles and scribble their thoughts. (p. 121-122)

That is, semiotics focuses on the *interaction* between thinking and representing. As Radford (2000) notes, adopting this perspective entails a "theoretical shift from what signs *represent* to what they *enable* us to do" (p. 241). The semiotic perspective is in contrast to theoretical foci that separate internal and external representations; these foci might view external representations as indicators of mental constructs or view internal thinking as a response to external representations.

Since semiotics focuses on mathematical *activity* and the *use* of representations, using a semiotic perspective entails using specific methods for generating data. For example, it no longer makes sense to ask students how they interpret the equals sign in run-on expressions; asking a student how they think about a pre-made symbol will not produce results that we can analyze. In contrast, we can ask students to solve problems in which they represent mathematical situations and then analyze the way their representations shape their activity.

A full analysis of students' semiotic activity would be beyond the scope of this paper. However, there are two ideas from semiotics that offer insight into students' mathematical activity: the distinction between icons, indexes, and symbols; and the distinction between personal and cultural semiotic systems.

Icons, Indexes, and Symbols

An *icon* is a representation that resembles "what it stands for" (e.g. four tally marks are four things that may be counted). An *index* is a representation that "points" directly to what it stands for (e.g. four tally marks might represent a collection of four objects other than the tally marks). A *symbol* is a representation that is associated with other representations (e.g. the symbol "4" might represent the four tally marks, a collection of properties of the number, etc.).

For example, a student who wrote $3t$ might interpret t as an index that points to an imagined truck and $3t$ as an index that points to an imagined collection of three trucks. In contrast, when mathematicians write $3t$, it is typically interpreted as a symbol that represents all of the possible values that result from multiplying 3 by an unknown number of (imagined) trucks.

Personal and Cultural Systems

Students create *personal* semiotic systems when they begin to solve problems; these systems may not match the *cultural* systems that mathematicians have agreed upon (Berger, 2004). As students work on a mathematical problem, they may use "=" as part of a sign to which they ascribe meaning. The meaning that students construct depends on the activity in which the sign is used, and—reflexively—the evolving meaning of the sign mediates the way the student engages in the mathematical activity. This negotiation of meaning is situated in the context of the activity. That is, the way students understand and use the equals sign is shaped by their previous understanding and experiences.

For example, the first student began working on the trucks-and-sedans problem by generating a semiotic system in which he used $3t=5s$ as an initial representation. In his personal system, the symbols $5s$ and $3t$ seemed to be indexes, pointing to mental or discursive objects. The symbol "=" enabled him to compare and manipulate the $3t$ and $5s$ according to a specific set of rules.

When the interviewer later asked the first student to compute the number of sedans when there were 15 trucks, the student was able to successfully work within his system; if he had tried to use the rules of the cultural semiotic system (i.e. substituting 15 for t and multiplying by 3) he would have had difficulty producing an answer that made sense to him. This is precisely what happened with the second student: her personal system—using indexes—was incompatible with the cultural system, in which *t* and *s* are symbols. Even though she had previously produced a different semiotic system that enabled her to compute values for the numbers of trucks and sedans, when she tried to represent the algebraic relationship, the clash between the personal and cultural semiotic systems left her unable to solve the problem.

Conclusion

The results described here suggest that students may think about the equals sign in multiple ways that depend on the mathematical activity in which they engage. While mathematics teachers would like their students to use the equals sign in a prescribed, mathematically correct way, students may be able to use the equals sign in other ways yet still work productively on mathematical problems.

From a cognitive perspective, the equals sign can represent a wide variety of ideas depending on the student's mental model and the problem situation. However, this perspective can make it difficult to explain all of the ways students think of equality and all of the ways they use the equals sign when they solve problems.

From a semiotic perspective, students create personal semiotic systems to solve mathematical problems. They use signs—among them the equals sign—to represent a wide range of ideas; these signs both enable the students to engage in mathematical activity and, at the same time, shape and constrain the way they engage in the activity. Even though the students' personal semiotic systems may not match the cultural systems of mathematicians, the students may still be able to successfully solve problems. However, if students try to operate in multiple semiotic systems simultaneously, they may not be able to work productively.

For mathematics teachers, these results underscore the need to understand the ways students symbolize their own mathematical ideas and ascribe meaning to formal mathematical symbols. By viewing this symbolization process as a negotiation—between the student, the teacher, formal mathematics, and the context in which the activity is situated—teachers can help their students use these symbols meaningfully in ways that are compatible with standard, formal mathematical notation.

For mathematics education researchers, these results highlight the importance of understanding the ways undergraduate students conceptualize equality and use the equals sign. This study shows how a semiotic perspective may be used to make sense of students' mathematical activity. In addition, it shows the potential benefits of adopting new theoretical perspectives when old perspectives make it difficult to interpret results.

References

Alibali, M., W., Knuth, E. J., Hattikudur, S., McNeil, N. M., & Stephens, A. C. (2007). A longitudinal examination of middle school students' understanding of the equal sign and equivalent equations. *Mathematical Thinking and Learning, 9(3)*, 221-247.

Behr, M., Erlwanger, S., & Nichols, E. (1980). How children view the equals sign. *Mathematics Teaching*, *92*, 13-15.

Berger, M. (2004). The functional use of a mathematical sign. *Educational Studies in Mathematics*, *55*(1-3), 81-102.

Clement, J. (1982). Algebra word problem solutions: Thought processes underlying a common misconception. *Journal for Research in Mathematics Education, 13(1)*, 16-30.

English, L. D., & Halford, G. S. (1995). *Mathematics education: Models and processes*. Hillsdale, NJ: Lawrence Erlbaum Associates.
Ginsburg, H. (1982). *Children's arithmetic: The learning process.* New York: D. Van Nostrand.
Hegarty, M., Mayer, R., & Monk, C. A. (1995). Comprehension of arithmetic word problems: A comparison of successful and unsuccessful problem solvers. *Journal of Educational Psychology, 87*, 18-32.
Kieran, C. (1981). Concepts associated with the equality symbol. *Educational Studies in Mathematics 12*, 317-326.
Larson, C., Zandieh, M., Rasmussen, C., & Henderson, F. (2009). Student interpretations of the equals sign in matrix equations: The case of Ax=2x. In *Proceedings of the 12th Conference on Research in Undergraduate Mathematics Education*. Raleigh, NC: North Carolina State University.
Lewis, A., & Mayer, R. (1987). Students' miscomprehension of relational statements in arithmetic word problems. *Journal of Educational Psychology*, *79*, 363-371.
MacGregor, M. and Stacey, K. (1993). Cognitive models underlying students' formation of simple linear equations. *Journal for Research in Mathematics Education*, *24(3)*, 217-232.
McNeil, N. M. (2007). U-shaped development in math: 7-year-olds outperform 9-year-olds on equivalence problems. *Developmental Psychology 43*(3), 687-695.
McNeil, N. M., & Alibali, M. W. (2005). Knowledge change as a function of mathematics experience: All contexts are not created equal. *Journal of cognition and development, 6*(2), 285-308.
Morgan, C. (2006). What does social semiotics have to offer mathematics education research? *Educational Studies in Mathematics,* 61(1), 219-245.
Nathan, M. J., Kintsch, W., & Young, E. (1992). A theory of algebra-word-problem comprehension and its implication for the design of learning environments. *Cognition and Instruction, 9*, 329-389.
Radford, L. (2000). Signs and meaning in students' emergent algebraic thinking: A semiotic analysis. *Educational studies in mathematics*, *42*(3), 237–268.
Radford, L. (2003). Gestures, speech, and the sprouting of signs: A semiotic-cultural approach to students' types of generalization. *Mathematical Thinking and Learning, 5*(1). 37-70.
Radford, L. (2009). Signs, gestures, meanings: Algebraic thinking from a cultural semiotic perspective. In *Proceedings of the Sixth Conference of European Research in Mathematics Education*, 1–19.
Rosnick, P. and Clement, J. (1980). Learning without understanding: The effect of tutoring strategies on algebra misconceptions. *Journal of Mathematical Behavior, 3(1)*, 3-27.
Rotman, B. (2006). Toward a semiotics of mathematics. In R. Hersh (Ed.), *Unconventional essays on the nature of mathematics* (pp. 97-127). New York: Springer.
Sáenz-Ludlow, A. (2006). Classroom interpreting games with an illustration. *Educational Studies in Mathematics,* 61(1), 183-218.
Verschaffel, L. (1994). Using retelling data to study elementary school children's representations and solutions of compare problems. *Journal for Research in Mathematics Education*, *25*, 141-165.
Weinberg, A. (2007). New Perspectives On the Student-Professor Problem. In T. Lamberg & L. Wiest (Eds.), *Proceedings of the 29th Annual Meeting of the North American Chapter of the International Group for the Psychology of Mathematics Education*, 164-170.
Weinberg, A. (2009). Students' mental models for comparison word problems. In S. L. Sward, D. W. Stinson, S. Lemons-Smith (Eds.), *Proceedings of the 31st Annual Meeting of the North American Chapter of the International Group for the Psychology of Mathematics Education*, 5, 709-717.