SIGMAA on RUME 2011 Proposal
Contributed Research Report

Designing and Implementing a Limit Diagnostic Tool

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Abstract:

The purpose of this study is to create and utilize a tool that evaluates students' comprehension of the logical structure and implications of the formal definition of limit. This study continues the trajectory of recent limit research involving classroom-based interventions that reveal student metaphors and conceptions (Boester, 2010; Oehrtman, 2009; Roh, 2008, 2010). The diagnostic tool, based on seven concepts embedded in the formal definition, uses a set of delta/epsilon diagrams that students must explain, either accepting them as correct, or augmenting them to make them correct. The assessment was used after giving students in a conceptually-based calculus class a problem meant to introduce the logical structure of the formal definition. While students did not spontaneously show many of the concepts based on the problem alone, an interview protocol following the assessment prompted the students to rethink the implications of the problem, thus promoting the missing concepts.

Keywords: calculus, limit, assessment, conceptual decomposition

Cornu (1991) first summarized research on students' spontaneous conceptions, mental models, and epistemological obstacles concerning the formal definition of limit. Since then, research has grown from cataloging misconceptions (Bezuidenhout, 2001; Davis & Vinner, 1986; Tall & Vinner, 1981) to describing possible frameworks of student conceptions (Cottrill et al., 1996; Lakoff & Núñez, 2001; Williams, 1991, 2001). Some of the most recent research (Boester, 2010; Oehrtman, 2009; Roh, 2008, 2010) has described classroom-based interventions that both further our understanding of students' conceptions of limit, while documenting how and why limits were taught to students using particular problems, activities, or manipulatives.

In order to assess the effectiveness of emerging pedagogical strategies for limit instruction, it would be nice to have a generic tool to gauge students' comprehension of the logic contained within the formal definition. The purpose of this study is to create such a tool, then use it to assess students' comprehension of the logical structure and implications of the formal definition of limit following a classroom-based intervention.

Limit Diagnostic Tool

Using the following statement of the formal definition of limit at a point

$$\lim_{x\to a} f(x) = L \text{ means that}$$
$$\forall\varepsilon > 0,\ \exists\delta > 0, \text{ such that } 0 < |x-a| < \delta \Rightarrow |f(x)-L| < \varepsilon$$

the concepts contained within this definition, which should be assessed by the diagnostic tool, need to be established. Through discussion with other limit researchers, the following list was created:

1) We control delta, not epsilon.
2) Delta interval must fit inside (cannot be outside) epsilon interval.
3) Delta interval can be strictly inside epsilon interval.
4) Delta and epsilon do not have to be equal.
5) In order for the limit as x approaches a to be L in the continuous case, $f(a) = L$.
6) The length of the interval on each side of a / L must be the same, because you can be the same distance (delta / epsilon) away in both directions.
7) For a non-linear graph, one side of the delta interval may be the same as the epsilon interval, but for the other side, the delta interval may be strictly inside the epsilon interval.

A written assessment, consisting of six standard delta/epsilon diagrams (an example is shown in Figure 1), was then created to test for these seven concepts. For each diagram, the student must either confirm that this pairing of delta and epsilon in the diagram is appropriate for the given graph of a function, or explain why it is not appropriate and correct the diagram to create an appropriate pairing. For instance, if a student states that the diagram in Figure 1 is not an appropriate pairing, and proposes redrawing the delta interval to "correct" the diagram in Figure 2, this would reveal that the student has failed to grasp concept #3, that the delta interval can be strictly inside the epsilon interval. A student could alternatively redraw the epsilon interval to line up with the delta interval, showing that they have also failed to grasp concept #1, that we can only change delta.

In addition to the written assessment, an interview protocol was also created. After a few preliminary questions eliciting feedback about limits in general, students are asked to explain their responses to all six diagrams. The interviewer questions their responses only after the student has completed explaining all the diagrams, in order to provide a baseline for their responses. For correct responses, the questions probe for recognition of the embedded concepts. Did the student actually answer the question following the intent of the concept? In other words, were the students' beliefs robust enough to withstand deeper questioning, or were they easily malleable by the researcher? For example, if a student correctly states that the diagram in Figure 1 follows the definition, the protocol indicates to show the student Figure 2 and ask "Another student might say that the delta is too small, and should be bigger to match the epsilon. Do you need to do this?" For partially correct or incorrect responses, the questions probed for the related concepts in their apparent absence. Would students recognize or maintain their misconceptions of the definition? If a student proposed Figure 2 as a correction to Figure 1, the student would be asked "Is it ok for the delta interval to be inside of the epsilon interval? Do delta and epsilon have to be equal?" (This addresses concept #4, as well as concept #3.)

Implementation of the Tool Following a Classroom Activity

This diagnostic tool was first used in Math 348, Concepts of Calculus for Middle School (Pre-Service) Teachers, a course taught by the researcher during the Spring 2010 quarter at a mid-size, Midwestern university. Math 348 focuses on the ideas, rather than the procedures, of calculus. Even though the goal of the course is to enable students to recognize how the fundamental idea of change relates to the functions commonly presented in middle school curricula, limits *are* introduced in the course, mainly to promote derivatives and integrals later on.

Limits were first covered informally, based on a dynamic, approaching conception. Students were asked to solve routine limit problems for continuous, discontinuous, and piecewise-defined functions. Then the formal definition was introduced through a classroom activity centered around a story problem originally created for a teaching experiment (Boester, 2010). The bolt manufacturing problem allows students to explore the logical structure of the formal definition by thinking about the functional relationship between the input and output of a factory that makes bolts. After allowing students to discuss the bolt problem in groups, a whole class discussion (led by the researcher) was held to come up with the following statement:

> For every bolt length tolerance, there exists a raw materials tolerance, so that if an amount of raw material that falls within the raw material tolerance is put into the machine, the length of the bolt produced falls within the bolt length tolerance.

Written on the board next was a delta/epsilon diagram, whose graph had a slope roughly equal to one so that the delta and epsilon intervals would match and be equal. (The researcher tried to match the diagrams drawn by the students during their small group discussions of the problem.) This diagram was initially labeled with the terms from the bolt problem. The formal definition of limit was then written on the board. Finally, the researcher walked the students though a mapping of each representation onto the other, showing in particular how the symbols of the definition matched the pieces of the statement and the diagram.

The concepts on the above list were intentionally not highlighted during instruction. While all of these concepts are embedded in the bolt problem, could students actually unpack them without further explicit instruction? The students were reminded, however, to think of the bolt problem while constructing their answers and explanations for each diagram.

Students were then given the assessment tool as a written, optional homework assignment for extra credit. (Students did not have to agree to participate in the study to complete the assignment and receive the extra credit.) Those students who completed the assessment were also given the opportunity to be interviewed. Out of 24 students enrolled in the course, 13 returned the written survey, and 8 of those sat for a videotaped interview. For the students who were interviewed, their remarks were transcribed and compared to their written solutions in two passes, once for their initial explanations, and a second time when their responses were probed by the researcher. For those students that were not interviewed but turned in the assessment, their written remarks were coded as if they had sat for the first pass of the interview.

Results

A preliminary analysis of the results has revealed that the students struggled with the assessment. Most students showed that they grasped concept #2, that the delta interval cannot be outside the epsilon interval. Interviewed students attributed their explanations of concept #2 to the bolt problem, that an amount of raw materials too far away from the target amount would produce a bolt with a length outside the acceptable length range. Students also showed that they grasped concept #4, that the delta and epsilon intervals do not have to be equal. One student explained that delta and epsilon are describing different qualities, so why should they have to be equal?

However, few students expressed knowledge of concepts #1, #3 and #5. Evidence of this was shown through changing the epsilon interval to correct a diagram, expanding a delta interval to match an epsilon interval (as in Figure 2), and not recognizing that a misalignment of a and L in a diagram was a fundamental error, even if the delta interval was still within the epsilon interval. The lack of these concepts seemed to hinder their grasp of concepts later on the list (and tested in later diagrams in the assessment). While the last two errors could be attributed to metaphorical "baggage" of the bolt problem (in the context of the problem, these are acceptable or even desired actions), students should have recognized that the bolt manufacturer cannot control the customers' demands on the accuracy of the bolt length (epsilon), only their own measurement error of the raw material (delta).

Even with this disappointing performance, the second pass of the interview showed entirely different results. When probed, although a few students backed away from correct concepts, every student gained several concepts. The students' responses to the probes were consistently framed in the context of the bolt problem, which clearly helped the students come to grasp the concepts they heretofore lacked. Even with the researcher being careful to closely adhere to the interview protocol, six of the eight students left the interviews having expressed every concept on the list. Thus, while the bolt problem itself may not be sufficient to instill the logical structure and implications of the formal definition of limit, the problem clearly primed students for a discussion of the diagrams used in the diagnostic tool. This suggests that an instructional sequence involving the bolt problem should not be considered complete without an assignment like that of the limit diagnostic tool, and a discussion of the assignment that questions the reasoning of correct and incorrect responses.

References

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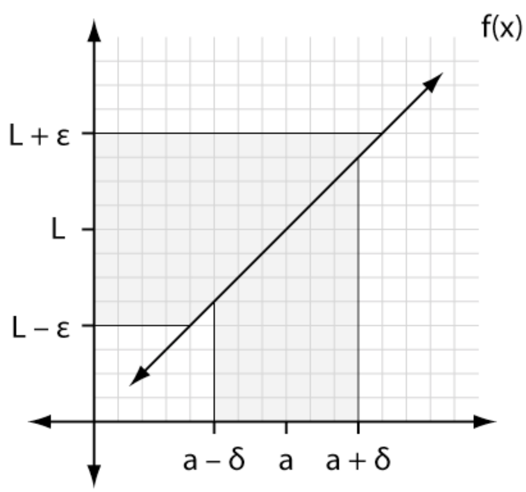

Figure 1. The delta/epsilon diagram from Question #2.

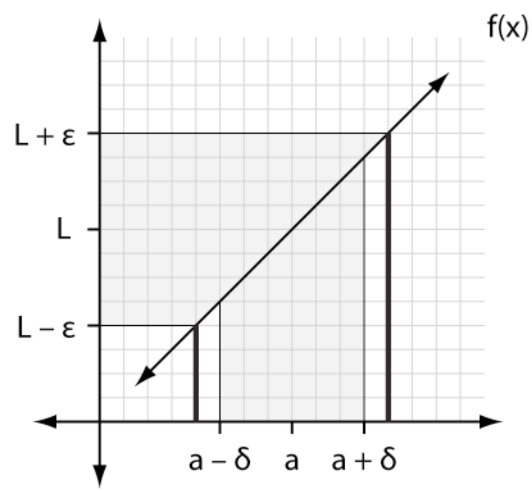

Figure 2. A proposed student correction to the delta/epsilon diagram in Question #2.