Translating Definitions Between Registers as a Classroom Mathematical Practice

Contributed research report

Abstract: Many have noted that mathematical definitions constitute a duality between a category of objects and the definition that delineates that category (Alcock & Simpson, 2002; Edwards & Ward, 2008; Mariotti & Fischbein, 1997; Tall & Vinner, 1981). Prior research has readily identified conflict between these two elements of students' conceptions, but reliable mechanisms for explaining and resolving such conflicts are still forthcoming. The present study observed a real analysis classroom in which the duality was embodied and addressed directly in class dialogue and activities. Particularly, three linguistic registers (metaphorical, common, and symbolic) arose to express different aspects of the definitions themselves (conceptual and formal). Translation across these registers provided a mechanism by which some students were able to segue their concept image and concept definitions successfully. Some students corrected errors in their concept image as a result of this practice.

Keywords: mathematical defining, real analysis, translating definitions, harmonisation, classroom communication

Mathematical definitions constitute an integral part of proof-based mathematics, but often receive less pedagogical attention than do the theorems and proofs built thereupon. This may result from the fact that, in the logical realm, definitions require no proof and do not add to a body of deductive theory (nothing is true with a definition that was not true without it) (Mariotti & Fischbein, 1997). In the cognitive realm however, nothing can be proven about a category or property that has not been defined, thus definitions are of premium pedagogical importance.

This emphasizes definition's role, which is to delineate a particular category of objects or aspect of a class of objects. Definitions thus constitute a duality between the class or concept being identified and the formal definition used to isolate that class (Alcock & Simpson, 2002; Edwards & Ward, 2008; Tall & Vinner, 1981). Many previous studies reveal the pertinence of this duality within definitions in light of the conflict and divergence of the two aspects.

Mariotti and Fischbein (1997) propose that, at least in the realm of plane geometry, concepts have two cognitive aspects: the *figural* aspect that relates to the concrete and visual nature of the concept (members of a category) and the *conceptual* aspect that expresses the abstract and theoretical nature of the concept (the property establishing category membership). Though they note the common discord between these two aspects, the authors propose that the process of defining involves "harmonisation" between the two aspects by ferrying back and forth mentally between the two viewpoints until they can be brought into sufficient agreement. In one reported interchange between students, Mariotti and Fischbein (1997) note how one students was focusing on the figural aspect while the other focused on the conceptual, such that their dialogue embodied direct interaction between these two aspects within the class discussion.

The present study identifies a classroom mathematical practice that developed in an undergraduate real analysis classroom and provided a mechanism for harmonisation between student's concept image and concept definition. The classroom discourse and activities openly addressed the duality that existed within definitions and the classroom mathematical practice of translating definitions through three linguistic registers helped them mediate between the aspects.

Methods

This study represents part of a larger study of classroom communication in undergraduate real analysis. This study's data comes from two 15-week semester courses of real analysis taught by the same instructor at a medium sized university in the southwest. The course included a proof-based development of real numbers, sequences, limits of functions, continuity, and uniform continuity. All class meetings were observed and written records were kept of the overall classroom dialogue and activities. A set of student volunteers (about 5 per semester) were interviewed weekly throughout the semester regarding their understanding of course content and classroom dialogues and activities. The interviews particularly focused on the appearance of classroom diagrams, lines of reasoning, and language as students articulated their own understanding and reasoning or as they worked on presented tasks.

Results

The three registers of definition articulation appeared early in and throughout each semester of study. For example, the class formulated the definition of one-to-one as:

1. If the function is viewed as arrows being shot from the domain to the target, then no one in the target gets hit with two arrows (metaphorical register).
2. No two inputs have the same output (common register).
3. For every x_1, x_2 in the domain such that $f(x_1)=f(x_2)$, $x_1=x_2$ (symbolic register).

The metaphorical register played different roles in the discussion of different definitions. Dawkins (2009) presents a detailed account of these classes' metaphor use and comprehension.

During the second semester of study, the professor discussed with the students how the limit of a sequence should be defined without presenting the definition itself. One student suggested that if the limit was like a party, then you could find the party by seeing where all of the people (the elements) are. The professor adopted this language and began to verbally explain the sequence definition in terms of people going to a party. She wrote on the board a common register definition that stated, "A sequence converges to the real number L if we can make the terms of the sequence stay as close to L as we wish by going far enough out in the sequence." She translated this statement verbally into the metaphorical domain saying it is only a party if for any size party you pick, after some point everyone shows up at the party. In another formulation, she said "only finitely many guys can be outside the room for you to have a party." She then proceeded to translate the common language definition on the board into symbolic language replacing "far enough out in the sequence" with index terminology and "as close to L as we wish" with epsilon neighborhood terminology. This pattern of translation between the metaphorical (party, time), common language (close enough, far in the sequence), and symbolic (epsilon intervals, indices) appeared repeatedly in the classroom discussion across the various topics of the course and throughout the discussion of sequence limits.

When asked three days after the introduction of the party metaphor what sequence convergence means, Tidus explained the definition primarily in the common language register. He did not directly reference the party language, but rather said "at some point" adopting a time-based metaphor for sequences and said "numbers will be in 4's neighborhood" treating the neighborhood as a place rather than a set.

When I asked Tidus later in that same interview to explain to me the roles epsilon, K, and n played in the definition, he said, "K represents how far n has to go on the number line to get into the epsilon neighborhood." Tidus expressed a correct correspondence between the mixed metaphorical and common register explanation he had previously provided of sequence convergence and its symbolic register translation in terms of indices.

After the class took the test over sequence convergence (about two weeks later), I asked Tidus to explain the definition of sequence convergence and he said, "you pick an epsilon. For any epsilon that you pick, an infinite amount of terms will be in that epsilon neighborhood and a finite amount of terms will be outside." However, when I asked him about the role of epsilon, K, and n, he explained by giving me the formal definition. When asked how he understood the definition, he accurately elaborated his understanding in terms of the party metaphor.

During the first semester, the professor sought to help students identify which functions are uniformly continuous by describing that they contained a steepest point. She indicated that this distinction showed why $\sqrt{x}$ is uniformly continuous though $\ln(x)$ is not. During interviews, three students reasoned from this criterion that if the point (0,0) is deleted from $\sqrt{x}$, then the function no longer uniformly continuous because it has no steepest point. Only one of these three, Aerith, then looked at the formal definition and concluded that:

> Well, cause by definition it says x_1-x_2 will be less than delta and $f(x_1)$-$f(x_2)$ will be less than epsilon and there is like you can find two points from here and that holds the definition. I just think when, it's like, this thing basically is saying when x_1 and x_2 getting closer, the image of, I mean the value of these two points will getting close, too.

Aerith translated the formal definition back into the common register and concluded that $\sqrt{x}$ would not cease to be uniformly continuous by the deletion of a point.

Discussion

Both Tidus and Aerith mirrored the classroom mathematical practice of translation across registers as they discussed their understanding of analysis definitions. Tidus began expressing himself in a mixture of the metaphorical and common registers, and initially was unable to articulate the symbolic register definition. He could explain the correspondences between his less formal definition and the symbolic definition's elements. Over time, he constructed his concept definition, but did so with strong ties to his concept image expressed in the metaphorical and common registers. It appears that the translation process provided a means of harmonisation between his concept image and concept definition of sequence convergence. Also, he gave more prominence over time to the formal, symbolic definition rather than the metaphorical.

Aerith, along with several classmates, developed a misconception in their concept image of uniform continuity based upon the professor's non-standard criterion of a "steepest point." Though not a metaphor per se, this alternate notion does not directly represent the concept or definition of uniform continuity and requires some translation. Once Aerith translated the formal definition into the common register, she was able to develop an alternative to the "steepest point" within her concept image ("when x_1 and x_2 getting closer, the image of, I mean the value of these two points will getting close, too") that helped her correct her misconception.

The professor engaged the class in the process of developing definitions to describe the behavior they observed in sets of examples. The dialogue separately referenced the "idea" and the "definition." In this way, she introduced the duality of definitions into the consensual domain. The "idea" was usually expressed using the metaphorical or common register, while the "definition" arrived by the end of the discussion in the symbolic register. The mathematical metaphors (Dawkins, 2009) employed in the classroom such as the party metaphor or the steepest point criterion had the effect of helping students develop their concept image of the particular concept. Similarly, the common register articulated the "idea" or the concept image. Thus, as Figure 1 presents, the dual aspects of definitions were acknowledged in the classroom discourse as the "idea" and the "definition." The linguistic registers embodied the dual aspects in the classroom discussion. The need to coordinate the two aspects (harmonisation) motivated the classroom mathematical practice of translation.

Thus, the classroom mathematical practice of translation between linguistic registers appears a viable tool for guiding students toward harmonising their concept image and concept definitions. This appears reasonable in light of the fact that this process encourages students to develop their concept definition out of their concept image, or at least with many connections between, rather than the two being introduced into the mind separately as when the formal definition is introduced in final form at the beginning of the discussion (Pinto & Tall, 2002).

This point also may explain the difficulty students experienced in working with uniform continuity. Most of the course definitions prior to this one had been introduced to students at the calculus level such that students already had concept images of function and sequence limits. Most students had no exposure to uniform continuity, and so they simultaneously constructed their concept image and concept definition from the professor's explanations and explorations.

Mariotti and Fischbein (1997) observed that classroom dialogue in which students embodied the different aspects of a definition seemed to promote harmonisation between the aspects. The ability of distinct linguistic registers to embody the different aspects of a definition appeared to promote similar negotiation both in the classroom setting and for individual students as they reasoned about classroom definitions in isolation. Further research is needed to identify the reliability of and conditions upon this classroom mathematical practice as a tool for helping students construct definition understanding and harmonise the dual aspects thereof.

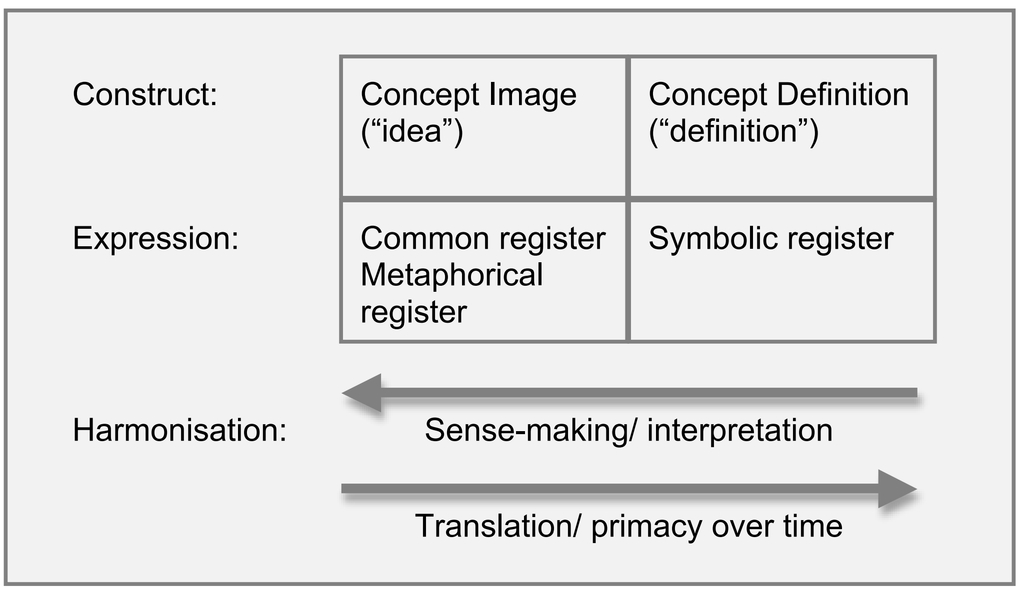

Figure 1: Dual Aspects of Defining

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