The Role of Conjecturing in Developing Skepticism: Reinventing the Dirichlet Function.
Contributed Research Report

The study presented in this research report was born out of the desire to develop pathways for students from informal to formal modes of thinking. The data from this report stems from a series of small group interviews using a process of guided reinvention incorporating frequent student conjectures in order reinvent the definitions of limit and continuity. During these interviews, students used the practice of skepticism in order to suspend judgment on various mathematical statements. In the process of exploring a developed conjecture, the students' suspension of judgment allowed them to alter their initial beliefs about the nature of continuity and their interactions with functions.

Keywords: Conjecturing, Skepticism, Calculus, Continuity, Dirichlet Function

The study reported on in this presentation was born out of the need to develop pathways for students from intuitive to advanced mathematical thinking. As noted by Gravemeijer and Doorman (1999) "guided reinvention offers a way out of the generally perceived dilemma of how to bridge the gap between informal knowledge and formal mathematics" (p. 112). In spring 2010 four small groups of multivariable calculus students participated in a series of eight interviews aimed at reinventing the core concepts of limit and continuity. The groups each experienced different levels of uncertainty and skepticism throughout the project and frequently used the practice of conjecturing to reframe and clarify their skepticism.

According to Dewey (1933) uncertainty plays an integral role in cognitive development as it is the primary component in reflective thinking, which he describes as "a state of doubt, hesitation, perplexity, [and] mental difficulty" (p. 12) accompanied with actions seeking to resolve these feelings of doubt and uncertainty. Similarly Cornu (1991) set forth a model of learning marked by overcoming cognitive obstacles which may be characterized in part by the uncertainty that they create in those encountering them.The theoretical framework employed in this report was set forth by Zaslavsky (2005) and expanded by Brown (2010).

Zaslavsky (2005), in her work, describes the origins of uncertainty in different mathematical tasks. She details three types of mathematical tasks that lead to different types of uncertainty: competing claims, unknown paths or questionable conclusions, and non-readily verifiable outcomes. Brown (2010) expanded upon Zaslavsky's work by recognizing that in Zaslavsky's work uncertainty was coupled with a *lack of belief* about the truth of the premise at hand. Brown went on to define skepticism as doubt coupled *with a belief* regarding the truth of the premise at hand. She further describes skepticism "as a state of being; that is, a collective or individual can, at a particular point in time, both obtain evidence for a conjecture and view the conjecture as of unknown truth value" (p. 2). She goes on to point out that viewed from this lens, skepticism can be perceived as the classroom practice of "suspending judgment against a backdrop of empirical or experiential evidence" (p. 2).

This research report presents the results from three of the above mentioned interviews during which the students involved spontaneously reinvented the Dirichlet Function, confronting several of their preconceived, informal beliefs regarding the concept of function. Throughout the semester, and in particular, throughout the interviews, students were encouraged to develop and write conjectures which captured their beliefs about the task. The students were then encouraged to attempt to either resolve the conjectures or refine their conjectures into new statements to be resolved.

The interviews used in this study were developed from the perspective of guided reinvention with the use of both predetermined and spontaneous context problems (Gravemeijer and Doorman, 1999). The context problems provided by the researcher were designed to complement the student's conjecturing activities by providing environments of uncertainty for the students. The students were then asked to participate in the process of creating and resolving and refining conjectures in order to address their uncertainty about the problem.

Much like the students in Brown's (2010) study, the students observed in this study frequently encountered uncertain situations with a well-developed belief about the truth of the premise being discussed. However, through the disciplines of skepticism and conjecturing, they were able to suspend their judgment on the premise in order to fully resolve the conjecture at hand. In the three interviews considered for this report, this process of suspending judgment allowed the students to further explore and refine their conjectures and eventually demonstrate that their initial beliefs about the nature of continuity were false.

This practice of skepticism and conjecturing exposed several of the students' naïve beliefs about the nature of continuity. Nunez (1999) in his argument that all mathematics is the result of embodied experience proposes two metaphors for continuity based on embodied experiences: *natural continuity* which is based on the metaphor that a continuous graph is the result of motion along that line and *Cauchy-Weierstrauss continuity* which is based on the metaphors that a line is a collection of gapless points with continuity defined as the preservation of closeness between those points. Based on these descriptions, the group in question unanimously mirrored the metaphor of natural continuity described by Nunez, thus leading them to believe that continuity must only exist on open intervals. However, though the use skepticism this group was able to reinvent the Dirichlet function (a function which takes the value of 1 for all rational numbers and 0 for all irrational numbers) and as a result alter their metaphor of continuity.

Additionally, the change in metaphor corresponded to a change in the way the students interacted with functions. The interviews exposed a view of function as existing outside the students' control. However, the change in metaphor resulted in a shift in control towards the students, allowing them to have power over the individual values of the function, thus allowing them to embrace the formal definition of function.

This research report is a case study analysis that took place in the first iteration in a process of development research aimed at better understanding how students transition from informal to formal understanding of calculus. The author welcomes all suggestions for improvement in future iterations of the study.

Brown, S. (2010). The role of skepticism in the emergence of the practice of proving. *Proceedings of the 13th Annual Conference for Research on Undergraduate Mathematics Education.* Raleigh, NC: MAA SIGMAA on RUME

Cornu, B. (1991). Limits. In D. Tall (ed.) *Advanced Mathematical Thinking* (pp. 153-166). Dordrecht: Kluwer Academic Publishers.

Dewey, J. (1933). *How We Think: A Restatement of the Relation of Reflective Thinking to the Educative Process*, D.C. Heath and Co., Boston.

Gravemeijer, K, and Doorman, M (1999). Context problems in realistic mathematics education: A calculus course as an example. *Educational Studies in Mathematics, 39*, 111-129.

Nunez, R. E., Edwards, L. D., and Matos, J. F. (1999). Embodied cognition as grounding for situatedness and context in mathematics education. *Educational Studies in Mathematics, 39*, 45-65.

Zaslavsky, O. (2005). Seizing the opportunity to create uncertainty in learning mathematics. *Educational Studies in Mathematics*, *60*, 297-321.