Title: A Multi-Strand Model for Student Comprehension of the Limit Concept

Category: Contributed Research Report

Abstract: In analyzing interview transcripts to assess student understanding of limits for first year calculus students, the application of the 7 Step Genetic Decomposition created by Cottrill, et. al. (1996) indicated that the interviewed students possessed no higher than a 3rd step understanding. Despite an inability to clearly articulate their understanding in terms of the expected lexicon, several students were able to create valid examples and counterexamples while justifying their answers. This suggests that these students possessed a better understanding of the limit concept than they were able to articulate. Thus, this study concludes that there exists additional criterion that should be taken into account in order to accurately diagnose student understanding of the limit concept. In particular a model for student understanding of limits should contain strands reflecting the student's method for solving a problem involving limits, the student's justification for the solution, and the applicability of the student's method and justification within the context of the problem.

Keywords: limits, student understanding, calculus, interview methodology

Author information:
Gillian Galle
Mathematics and Statistics, University of New Hampshire
get7@unh.edu

Introduction and research questions

The failure rate in undergraduate calculus courses is traditionally quite high (Ferrini-Mundy & Graham, 1991) and many reasons have been offered as explanations for this phenomenon. Insufficient or unsatisfactory background courses and student difficulty with the important concepts that form the underpinnings of calculus are just a few of the explanations that have been put forth (Burton, 1989; Ferrini-Mundy & Graham, 1991). In particular, Davis and Vinner (1986) point out that students' troubles with the concept of limits may play a strong role in the rate at which they fail out of calculus. Research suggests that a strong conceptual understanding of this abstract concept is needed in order for students to understand the topics which follow, i.e. derivatives and integrals (Bezuidenhout, 2001; Hardy, 2009; Orton, 1983). Since typical undergraduate courses in calculus often rely on the limit concept to explain continuity of functions, define derivatives, and define integrals, a student holding misconceptions of the limit concept runs the risk of developing flawed conceptions of these later topics that will negatively impact the rest of his or her mathematical understanding. Must this understanding include symbolic proficiency with the formal definition of limit or can a student develop a strong conceptual understanding of the limit concept that does not rely on the symbolic interpretation?

Up to now, the most comprehensive model of the stages of understanding students must pass through before achieving mastery of the limit concept has been the genetic decomposition created by Cottrill, et. al. (1996). However, this decomposition seems to account for only one strand of the many that make up the web of student understanding of the limit concept. Inspired by the model for comprehension of mathematical proof designed by Mejia-Ramos, et. al. (2009), this study seeks to identify additional strands of knowledge needed by students to fully master the limit concept.

1 Literature

Several researchers suggest that most of the research conducted on student understanding of the limit concept seems to fall into one of two or three categories. These classifications can be summarized as research on the informal notions of limit held by students, research on how students reason about limit in the context of the formal definition, and research into the obstacles students face as they try to make sense of the limit concept (Swinyard, 2009; Williams, 2001).

Much of the research that has been conducted has focused on developing classification schemas or identification rubrics to identify commonly held conceptions and describe the levels of understanding students might hold regarding the limit concept. Throughout the available research on student understanding of limit the most influential categorization seems to be the 7 Step Genetic Decomposition devised by Cottrill, et. al. (1996). Through their work to identify student conceptions and misconceptions about the limit concept they developed a 7 step classification for the phases students undergo as they make sense of the formal limit definition for themselves (pp. 177 - 178). As evidenced by their decomposition, their emphasis is on students starting with an x-value understanding which then culminates in a formal definition.

Swinyard (2009) attempted to enhance their 7 step genetic decomposition by filling in more detail in the last few stages of the genetic decomposition, steps 5 – 7, which focus on "the transition from informal to formal reasoning" (p. 20). According to the work of Swinyard these stages can be enhanced by the consideration of how students "define limit" and "define closeness in a concrete and increasingly restrictive manner" (pp. 22, 24). This study addresses the following questions: Is the genetic decomposition sufficient for determining a student's understanding of the limit concept? If not, what other strands should be taken into account when assessing what a student understands about the limit concept?

2 Data collection and methodology for analysis

A multiple choice written assessment was administered to 9 recitation sections of a first semester undergraduate calculus course. Individuals that represented low, medium, and high levels of understanding based on their assessment responses were then contacted to participate in follow up interviews. During the interviews the students revisited on their answers to the assessment items before being presented with two novel tasks.

During the first analysis of the transcribed interviews, a combination of both Cottrill, et. al.'s (1996) genetic decomposition and Swinyard's (2009) enhancements was used to interpret student answers and to categorize what understandings of the limit concept were held by the interviewees. Though it was possible to identify which stage of understanding the interviewed students reached, the decomposition did not seem to take into account several features of the students' explanations. Thus a second analysis was conducted which focused on identifying and defining other strands of knowledge held by the students that could reflect their understanding of the limit concept as depicted in their explanations.

3 Results

The first analysis of the transcribed interviews suggested that the interviewees had only reached the 3rd step of the 7 Step Genetic Decomposition. This finding was primarily based on the student responses to the second task, an item adapted from Bezuidenhout (2001). In this task the students were confronted with the limit of a difference quotient involving a function f and asked to determine the value of the limit, if it existed, based on a table of values for the function and its derivative. All interviewed students chose to find the limit of the whole expression rather than acknowledging that the problem could be interpreted as the limit of a combination of functions. Thus their answers failed to meet the criteria for step 4 of the genetic decomposition, "perform actions on the limit concept by talking about, for example, limits of combinations of functions" (Cottrill, et. al., 1996, p. 178).

However, several aspects of understanding arose during the first analysis of the interviews that did not seem to be adequately addressed by the combined genetic decomposition. The genetic decomposition did not account for the types of examples the students used during their explanations, the high level justifications offered for the existence of a limit, or their choice of methods for approaching the task of finding the limit. The second analysis then focused on these issues to identify and define the following strands of knowledge: the student's method for solving the presented task, the student's justification for the final answer, and the applicability of the student's method and justification in the context of the task.

There are many valid methods students are taught in order to determine the limit of an expression. These methods include the utilization of graphs and tables, but also encompass applications of stronger results such as L'Hopital's Rule. Thus one strand of student understanding addresses sophistication of the method the student selects for evaluating the limit. For instance, in one interview a student, Jim, utilized his knowledge that the existence of the derivative of the function in the table implied that the function was continuous, and subsequently that the limit of the function was equal to the value of the function at the given x value. This method for solution surpassed the demonstrated logic of his interviewed peers who focused on values provided in the table when they faced this task. Hence, there is an argument to be made for students' understanding being reflected by the method of solution they choose.

Another strand accounts for student justification, that is, the student's ability to correctly use the chosen method based on the information he or she perceived as applicable. The graphical examples and counterexamples offered by the interviewees as they responded to first task would be assessed along this dimension. The interviewees' decision to utilize L'Hopital's rule in the second task, after a direct evaluation of the limit of the difference quotient failed, was a selection of an appropriate method on the applicable data. However, as their computations were incorrect, they failed to reach the desired result and this would count as a failure of justification. Another example of justification that occurred in the second task was when Jim explained his use of the knowledge that the existence of the function's derivative at $x = 2$ implied both the continuity of the function and the existence of its limit.

The final strand is applicability which addresses the student's ability to see the relevance of given data and his own knowledge in the context of the problem. It is this factor that separated several of the interviewees' solutions and allowed the second analysis to distinguish between their respective understandings of limits. In one instance two students had provided very similar graphical counterexamples to a statement they were trying to prove true. Only one of them recognized the applicability of the counterexample to the task and used it to show the statement in question was false. The other interviewee failed to see the applicability of her example to the task at hand and persisted in trying to show the statement was true. This issue was also evident in both Jim's ability to see the relevance of his knowledge to the solution of the second task and Jim's selection of an inappropriate example for reasoning about the veracity of statements in the first task.

4 Significance and directions for further research

This study has opened several avenues for future research. Of particular importance is the consideration that the current genetic decomposition (Cottrill, et. al., 1996) alone is not sufficient for determining a student's understanding of the limit concept. Student understanding is comprised of many different strands and students may have developed methods for solving limit problems that do not rely on their appropriation of the formal limit definition. Are there limitations to the types of problems these methods can solve? Of particular interest would be problems that seem to require use of the formal definition but are solvable by students with a weak understanding of the formal definition.

This study suggested at least three strands that are evident when students interact with problems involving limits. Further research is necessary to verify the existence of these strands in a larger population and determine whether there are additional strands that need to be added to this tentative model. Such research should also refine the currently identified strands, especially the applicability strand. Clearly the function concept is strong for some of the interviewees as they constructed clear counterexamples that showed how a function's limit might not equal the value of the function at a particular *x*-value. However, the students' apparent inabilities to perceive that these were counterexamples and then use that knowledge to reach the correct conclusions in the first interview task is troubling. Additional study of this strand may also help educators understand why students select inappropriate examples when trying to make sense of a problem they are trying to solve.

Ultimately, this multi-strand perspective of student understanding could be a valuable aid to teachers. Knowledge of these strands could be incorporated into assessments which would enable instructors to determine their students' levels of understanding. Not only would this allow instructors to tailor their lectures to address any areas of weak understanding demonstrated by the students, but it would also serve as a guide toward the desired level of student understanding of the limit concept.

References

Bezuidenhout, J. (2001). Limits and continuity: Some conceptions of first-year students. *International Journal of Education, Science, and Technology*, *32*(4), 487 – 500.

Burton, M. (1989). The effect of prior calculus experience on "introductory" college calculus. *The American Mathematical Monthly*, *96*(4), 350 – 354.

Cottrill, J., Dubinsky, E., Nichols, D., Schwingedorf, K., Thomas, K., & Vidakovic, D. (1996). Understanding the limit concept: Beginning with a coordinated process scheme. *Journal of Mathematical Behavior*, *15*, 167 – 192.

Davis, R. B., & Vinner, S. (1986). The notion of limit: Some seemingly unavoidable misconception stages. *Journal of Mathematical Behavior*, *5*(3), 281 – 303.

Ferrini-Mundy, J. & Graham, K. G. (1991). An overview of the calculus reform effort: Issues for learning, teaching, and curriculum development. *The American Mathematical Monthly*, *98*(7), 627-635.

Hardy, N. (2009). Students' perceptions of institutional practices: The case of limits of functions in college level calculus courses. *Education Studies in Mathematics*. DOI: 10.1007/s10649-009-9199-8

Mejia-Ramos, P., Weber, K., Fuller, E., Samkoff, A., Search, R., & Rhoads, K. (2010, February). *Modeling the comprehension of proof in undergraduate mathematics*. Paper presented at the Thirteenth Conference on Research in Undergraduate Mathematics Education, Raleigh, NC. Retrieved from: http://sigmaa.maa.org/rume/crume2010/Archive/Mejia-Ramos%20et%20al..pdf

Orton, A. (1983). Students' understanding of differentiation. *Educational Studies in Mathematics*, *14*(3), 235 – 250.

Swinyard, C. (2009, February). Finding a suitable alternative to a potential infinity perspective: A watershed moment in the reinvention of the formal definition of limit. Retrieved on 12/15/2009. Conference on Research in Undergraduate Mathematics Education, Raleigh, NC.

Williams, S. (2001). Predications of the limit concept: An application of repertory grids. *Journal for Research in Mathematics Education*, *32*(4), 341 – 367.