THE LIMIT NOTATION: WHAT IS IT A REPRESENTATION OF?

Contributed research report

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Abstract

Student difficulties with the notion of limit are well-documented by research. These studies suggest that students mainly realize limits through dynamic motion, which can hinder further realizations of the concept. Some studies mention the overemphasis on the dynamic aspects of limits in classrooms but research on the teaching of limits is quite scarce. This work investigates the development of discourse on limits in a beginning-level undergraduate calculus classroom with a focus on the limit notation and uses a communicational approach to learning, a framework developed by Sfard (2008). The study explores how the limit notation is utilized by an instructor and his students and compares the realizations of limit in their discourse. The findings indicate that the shifts in the instructor's word use when talking about the notation supported students' realizations of limit as a process despite the frequency with which the instructor talked about limit as a number in his discourse.

Keywords: teaching of calculus, limits, the limit notation, discourse analysis

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Introduction

Being the building block of many fundamental calculus concepts, the notion of limit has drawn significant attention from researchers and student difficulties about the notion are well-documented by research. These studies suggest that dynamic motion dominates students' realizations of limit, which can interfere with other aspects of limits such as the formal realization of the concept (Bezuidenhout, 2001; Tall, 1980; Tall & Schwarzenberger, 1978; Tall & Vinner, 1981; Williams, 1991). In particular, the representational tools (e.g., verbal, visual, and symbolic) used by students while thinking about limits may lead to additional difficulties (Bagni, 2005; Cottrill et al., 1996; Williams, 1991). Further, some of the problems students encounter as they work on limits result from difficulties related to the underlying concepts such as functions and the notions of infinitely large and small (Parameswaran, 2007; Sierpińska, 1987). Therefore, the concept of limit presents students with two challenges: the need to make the transition from its intuitive to formal realization, and the need to cope with the compatibility of the conceptual and representational tools within the intuitively realized aspects of limits.

Some researchers highlight that the intuitive aspects of limits are perpetuated in teaching and curriculum. Parameswaran (2007) considered the reliance of calculus textbooks on graphing as problematic since it can lead to the incorrect idea that limit is a process of approximation. Cornu (1991) mentioned that "in teaching mathematics, certain aspects of the limit concept are given greater emphases, which are revealed by a review of the curriculum, the textbooks and examinations" (p. 153). Bezuidenhout (2001) argued the learning and teaching approaches stressing the instrumental rather than conceptual aspects of limits can result in students' realization of the notion as isolated procedures.

Although existing studies imply possible links between instruction and students' realizations of the limit concept, there is not extensive research on the teaching of limits to justify these claims. This work is part of a case study that investigates the development of the discourse on limits in a beginning-level undergraduate calculus classroom. The study uses a communicational approach to learning, a framework developed by Sfard (2008), to focus on the elements of one instructor's and his students' discourse on limits. In this paper, the main focus is on the limit notation as a symbolic representational tool in the discourse of limits since, besides graphing, it is the main visual mediator with which ideas about limit are communicated. The study addresses the following questions: (a) How is the limit notation utilized by the instructor in a beginning-level undergraduate calculus course and what kinds of realizations of limit does the notation support?, and (b) How do the elements of the instructor's discourse on the limit notation compare and contrast with the students' discourse?

Theoretical framework

One of the highlights of the *commognitive* framework (Sfard, 2008) is the interrelationship between communication and thinking. By defining thinking as the individualized form of communication, Sfard (2008) argues that the "cognitive processes and interpersonal communication processes are thus but different manifestations of basically the same phenomenon" (p. 83). Given this, the term *commognitive* entails the combination of the terms *cognitive* and *communicational*. This framework considers discourse as its central unit of analysis in which the main focus is on activities, patterns of interaction and communicational failures. Sfard (2008) defines the term discourse as " the different types of communication set apart by their objects, the kinds of mediators used, and the rules followed by participants and thus defining different communities of communicating actors" (p. 93).

The commognitive framework views mathematics as a particular type of discourse, which is distinguishable by its *word use*, *visual mediators*, *routines,* and *narratives.* Although number or quantity related words can be found frequently in daily life, "mathematical discourses as practiced in schools or in academia dictate their own, more disciplined uses of these words" (Sfard, 2008, p. 133). Given the abstract nature of mathematical objects, *word use* is a critical element of a mathematical discourse because possible differences in participants' use of those words can hinder mathematical communication. An important feature of mathematical *word use* is objectification. Objectification results in replacing the talk about processes and actions with states and objects (Sfard, 2008). For a mathematical discourse, objectification is a means for formalization and enhances the effectiveness of our communication. However, the objectified mathematical discourse is abstract and hides the discursive layers and metaphors it is composed of. Therefore, being explicit about the underling discourses and metaphors of an objectified mathematical concept can be quite important for students at the beginning stages of their learning.

Visual mediators refer to the visible objects created and operated upon for the sake of communication. Daily life discourses are generally mediated by the images of concrete objects whereas mathematical and scientific discourses are often mediated by symbolic artifacts. *Routines* refer to the set of metarules that characterize the patterns in the activity of participants of a discourse. *Narrative* is “any sequence of utterances framed as a description of objects, of relations between objects, or of processes with or by objects, that is subject to *endorsement* or rejection with the help of discourse-specific substantiation procedures” (Sfard, 2008, p. 134, italics in original). Narratives of a given discourse that are endorsed by the majority of the discourse community, in particular by “experts”, are considered as “true”.

The focus of this paper is on one instructor's and his students' use of the limit notation as a *visual mediator* as well as their *word use* and *endorsed narratives* associated with the notation to explore the similarities and differences between the instructor's and students' discourse.

Research methodology

The participants of this study were one calculus instructor and his section of undergraduate students taking a beginning-level calculus course in a large Midwestern university. For the instructor's discourse, the data consisted of video-taped classroom observations and field notes. The observation data consisted of eight 50-minute sessions in which the instructor discussed limits and continuity. For the students' discourse, part of the data included 23 students' responses to a diagnostic survey, which was taken from Williams (2001). The survey informed the selection of four students for an individual task-based interview session. The data for the analysis of the interviews came from students' written work and field notes taken during the interviews. The interviews were audio-taped and lasted between 53-76 minutes. Participation in the survey and the task-based interviews was voluntary.

The transcripts for the video and audio-taped sessions included what the participants said and what they did. Therefore, the transcripts also coded participants' actions as they were referring to the limit notation. For this study, the units of analyses were the instructor's and students' word use, and the limit notation as a visual mediator. Both the instructor's and the students' word use was analyzed with respect to the degree of objectification in their discourse on limits. The word use on limits was considered objectified if the participants talked about limit as an end-state or a number; it was considered operational if participants talked about limit as a process. Particular attention was also given to the use of metaphors and endorsed narratives underlying participants' word use. The analysis then focused on the similarities and differences between the instructor's and the students' discourse on the limit notation.

Results

The analysis of the instructor's overall discourse on limits revealed that he mostly talked about limit as an end-state of the limiting process: a specific number. In the context of the limit notation, however, he shifted his word use and referred to limit as a process based on dynamic motion. The analysis also showed that the instructor's word use on the limit notation depended on the following three mathematical contexts: computing limit at a point; limit at infinity; and infinite limits. In each of these contexts, the ways he talked about limits and infinity as end-states or processes differed. However, the shifts in his word use remained implicit for the students.

The analysis of the diagnostic survey and the individual interview sessions showed that, unlike the instructor, students rarely referred to the limit L as a number when talking about the limit notation $\lim_{x \to a} f(x) = L$. Instead, they adopted the elements of the instructor's discourse that referred to limit and infinity as processes. Therefore, although the instructor could flexibly talk about limit and infinity as processes or as end-states depending on the context, the notions remained as processes in students' discourse.

In summary, although the instructor's discourse on limits was mainly objectified, the shifts in his word use when talking about the limit notation supported students' operational word use. As a result, the students heavily relied on the metaphor of continuous motion whereas the instructor alternated between the metaphors of motion and discreteness. Moreover, the students only endorsed the narrative *limit is a process*, whereas the instructor mainly endorsed *limit is a number*.

Conclusions and implications for mathematics education

The students in the study developed the realization of limit as a process despite the instructor's general word use on limits, which was objectified. Talking about the limit notation was one mathematical context in which the instructor's word use alternated between the objectified and operational aspects of limits. Note also that the operational and objectified word use on limits utilize distinct metaphors: the former is based on the metaphor of continuous motion whereas the latter is based on the metaphor of discreteness. The tacit nature of these metaphors and students' adoption of the instructor's operational word use as the dominant means with which to talk about limits signal the importance of explicitness during instruction. As the insiders and the experts of the mathematical discourse, instructors can "lose the ability to see as different what children cannot see as the same" (Sfard, 2008, p. 59). This study provides evidence of the ways in which connected aspects of a mathematical concept can remain distinct and implicit for learners.

In addition, the study problematizes the utilization of the limit notation. The instructor primarily used the limit notation $\lim_{x \to a} f(x) = L$ to represent the end result of a limiting process (the limit is *equal to L*). Yet, students used it to represent the limiting process "the function *f(x) approaches the limit L as x approaches a"* (Hughes-Hallett et al., 2008; Thomas et al., 2008). So, although a symbolic and abstract visual representation, the limit notation might inevitably support dynamic motion and assumptions about continuity that underlie students' intuitive realization of limits.

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