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**Abstract**: In this talk, we characterize the nature of students' thinking about real-world problem situations that mathematicians might choose to reason about using ideas from linear algebra such as eigen theory, matrix equations, and/or systems of linear equations. We documented students working in groups on the "Car Rental Problem," a task that our research team specifically designed to elicit students' thinking about problem contexts that might be modeled in the aforementioned ways. We will describe the models students create to reason in this problem context, illustrating the variety in the final solutions of four different groups of linear algebra students and discuss the trends that appeared across the four groups as they worked toward their solution. Our analysis follows Lesh & Kelly's (2000) multi-tiered approach, and will focus on the mathematical topic areas drawn upon, the inscriptions created, and the quantitative reasoning that the students engaged in as they worked toward a solution.

**Key Words**: Linear Algebra, Modeling, Student Thinking

**Students' Modeling of Linear Systems: The Car Rental Problem**
**RUME 2011: Contributed Research Report**

## Introduction:

Linear algebra has the potential to provide students with powerful tools for analyzing and understanding systemic problems in many areas of mathematics, engineering, and sciences. These tools include the use of systems of linear equations, matrices, and eigen theory for modeling real world phenomena. Research shows that students struggle to bridge their informal and intuitive ways of thinking with the formalization of concepts in linear algebra (Carlson, 1993; Dorier, Robert, Robinet & Rogalski, 2000).

## Research Objective:

The central objective of this work is to characterize the nature of students' thinking about real-world problem situations that mathematicians might choose to reason about using ideas from linear algebra such as eigen theory, matrix equations, and/or systems of linear equations. The reason for this is two-fold: first, it offers insight into what it means to understand these ideas at a very fundamental level. Second, it offers insight into the informal and intuitive ways students have for thinking about these ideas -- ways that might then be leveraged instructionally. To this end, we documented students working in groups on a task that our research team specifically designed to elicit students' thinking about problem contexts that might be modeled in the aforementioned ways. We will refer to this task as the Car Rental Problem. In our talk, we will describe the models students create to reason in this problem context. We will illustrate the variety in the final solutions of four different groups of students and discuss the trends that appeared across the four groups as they worked toward their solution. The three questions that will guide our analysis are: (1) What mathematical topic areas do students draw upon to reason about such situations? (2) What are the nature and role of the inscriptions students develop to structure their thinking about such situations? (3) What role does quantitative reasoning play in the development of students' solutions?

In the car rental problem, students are presented with a scenario where there is a car rental company that has three locations in a city. Patrons of the company are allowed to return cars at any of the three locations and the problem describes what percent of cars from each location are returned where (see Figure 1).

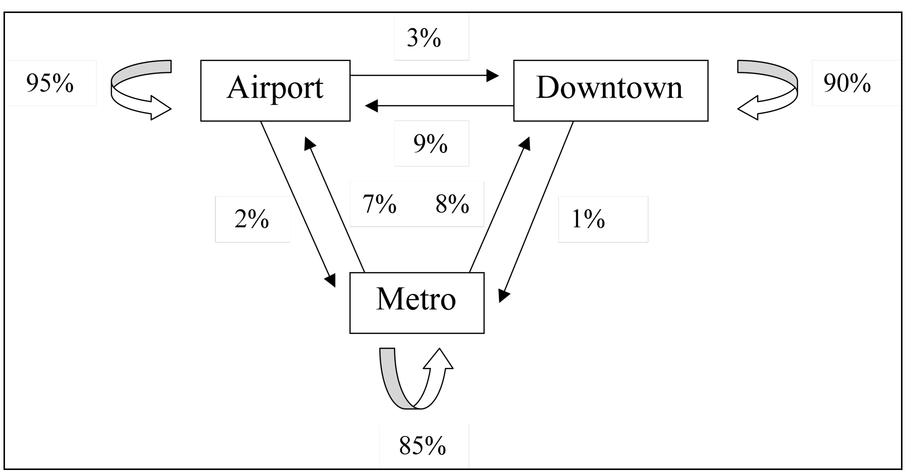


Figure 1: Diagram of Redistribution Rates in the Car Rental Problem

In particular, each week, about 95% of the vehicles rented from the Airport location are returned at the Airport location, about 3% rented at the Airport are returned Downtown, and about 2% of the cars rented from the Airport location are returned at the Metro location. Students are given an initial distribution of cars, and asked to describe the long-term distribution of the cars if the cars are returned at the described rates. They are also asked whether changing the initial distribution of cars would change the long-term distribution, and told to develop a business plan for the company so that they do not have to keep reshuffling the cars.

**Theoretical Lens: Modeling & Quantitative Reasoning**

The Models and Modeling (M&M) perspective adopts the view that many mathematically significant ideas that need to be learned by students originate from real world contexts, and that meaningful learning occurs when students are given the chance to reason about such ideas in their rich contexts (Lesh & Doerr, 2003). This work draws heavily on the literature and research tools developed in association with the M&M perspective in two important ways. First, we appeal to Lesh and Doerr's characterization of a model: "*Models* are conceptual systems… that are expressed using external notation systems, and that are used to construct, describe, or explain the behaviors of other system(s) – perhaps so that the other system can be manipulated or predicted intelligently" (2003, p. 10). Second, we have relied on the design principles associated with this perspective to inform the design of the task posed to students in this study.

In order to simplify our analysis of students' problem-solving efforts, we decided to focus largely on the quantitative reasoning that was related to each group's final solution. According to Kaput (1998), "Quantitative reasoning… can be regarded as modeling – building, usually in several cycles of improvement and interpretation, mathematical systems that act to describe and help reasoning about phenomena arising in situations" (p. 16). In our analysis we chose to use Smith and Thompson's (2008) description of quantitative reasoning; namely, quantitative reasoning is reasoning with and about quantities, where quantities "are measureable attributes of objects or phenomena; it is our capacity to measure them – whether we have carried out those measurements or not – that makes them quantities" (p. 10).

**Methods**

The students in this study were enrolled in an introductory undergraduate linear algebra course at a public university in the southwestern United States during the Spring of 2007. Successful completion of two semesters of calculus was prerequisite to the course, so students had a strong mathematical background. There were 32 students in the class, 8 of whom participated in the problem-solving interviews used as data for this study. Information is not available on the breakdown of the majors of the students in the class as a whole. However, of the 8 participants, 4 were electrical or computer engineering majors (one of whom had a double major in mathematics), 3 were computer science majors, and 1 was a graduate student in the business school. In the problem solving interviews, students worked in groups of 2-3 (although one student ended up working individually because the others in his group did not show up for their interview) for approximately 90 minutes. Data were collected on four groups of students working on the car rental problem. The first two groups completed the task about halfway through the semester, and the other two groups completed the task at the end of the semester. The interviews were videotaped and student work was collected.

Data were analyzed using Lesh and Kelly's (2000) multi-tiered approach to create accounts of the models created by each group of students. While our accounts of each group individually focus primarily on their final solution to the problem, we looked for themes across the problem solving sessions in order to identify points of commonality among the groups with regards topic areas, inscriptions, and quantitative reasoning.

**Results**

In the interest of space, we will focus our discussion here on just the first of our three research questions. Our first question is "What mathematical topic areas do students draw upon to reason about such situations?" Viewing each group's final solution individually illustrates the variety of topic areas that students drew upon. The students in Group 1 drew heavily on ideas from calculus to reason about the patterns they saw, considering them as sequences whose rates of change were decreasing. The students in Group 2 drew on their knowledge of computer programming and created general algebraic expressions for the computations to be performed. The students in Group 3 focused on a (perceived as constant) weekly rate of cars gained or lost, and drew on ideas from psychology and business (considering customer's needs and desires and working to meet them while maintaining a profit). The student in Group 4 drew primarily upon ideas from linear algebra, using a matrix to model the system of linear equations and reason about the system's behavior.

However, looking across the problem-solving activity of all groups showed that Group 1's idea from calculus (namely the general idea that 'if the change in the number of cars from one week to the next is decreasing, then the number of cars must converge') was echoed by nearly every group, even though none of the other groups appealed to the argument in their final solution. For example, early in his interview Matthew (Group 4) made a passing comment "Metro looks like it's decreasing the amount it's going down. Downtown is decreasing the amount it's going up, and so is airport. It seems like they're going to converge somewhere maybe." Group 3 offered a similar explanation very late in their interview (after they had written their final solution and been pushed by the interviewer to try to extend and generalize their argument). The only group that did not offer an argument of this nature was Group 2, who had focused their efforts on their computer simulation and never considered the difference in cars from one week to the next at a given location as a quantity of interest.

In our talk, we will delve into the second and third research questions as well, using examples of student work to illustrate themes that emerged across groups. We will illustrate the ways in which students' inscriptions (especially their symbolic expressions) served to support their quantitative reasoning. We will also argue that in students' solutions, quantitative reasoning served as a basis for the aforementioned symbolic expressions students developed to aid computation and further symbolization. Across groups, we show how the inter-relatedness of the quantities created a need for computational efficiency as well as symbolism that (1) supports computational needs, (2) represents interrelatedness of quantities, and (3) aids in conceptualization of the system as a whole.

### Final Remarks

In a way, this work serves to illustrate the ways in which students draw on their experiences and coordinate multiple topic areas as they engage in new mathematical problems. Here, this highlights the need for students to expand their mathematical horizons with the additional computational and notational tools linear algebra has to offer. Having conducted this analysis, we are now exploring ways in which this task can be leveraged to help students develop ideas about matrix multiplication as a tool to aid in computation and modeling of systemic level change.

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