Reaching out to the Horizon: Teachers' use of Advanced Mathematical Knowledge

Abstract. This paper explores teachers' use of advanced mathematical knowledge (AMK) – that is, the knowledge acquired during undergraduate university or college mathematics courses. In particular, our interest is in the use of advanced mathematical knowledge as an instantiation of knowledge at the mathematical horizon (KMH). With this tie to undergraduate mathematics education, we re-conceptualize the notion of knowledge at the mathematical horizon and illustrate its value with excerpts from instructional situations.

Key words: advanced mathematical knowledge; horizon knowledge; group theory; calculus

Contributed Research Report

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Knowledge at the mathematical horizon is a category of teacher's knowledge included by Hill, Ball and Schilling (2008) in their refinement of Shulman's classic categorization of teacher's Subject Matter Knowledge (SMK) and Pedagogical Content Knowledge (PCK). While Hill et al. (2008) develop several subcategories of SMK and PCK, they omit elaboration on KMH – Knowledge at the Mathematical Horizon. Our interest, in this paper, is to explore the idea of teachers' knowledge at the mathematical horizon, what is there, how it may be used, and to what benefit. We develop a notion of KMH, which is influenced by educational and philosophical perspectives, and explore examples that illustrate how KMH in conjunction with knowledge acquired in undergraduate studies at university or college – that is, Advanced Mathematical Knowledge, AMK (Zazkis and Leikin, in press) – may be used to advantage in teaching and in teacher education.

At the horizon of teachers' knowledge

Ball, Thames, and Phelps (2008) describe horizon knowledge as "an awareness of how mathematical topics are related over the span of mathematics included in the curriculum (p. 403), while Ball and Bass (2009) explain in more detail the notion of mathematical horizon:

> We define horizon knowledge as an awareness – more as an experienced and appreciative tourist than as a tour guide – of the large mathematical landscape in which the present experience and instruction is situated. It engages those aspects of the mathematics that, while perhaps not contained in the curriculum, are nonetheless useful to pupils' present learning, that illuminate and confer a comprehensible sense of the larger significance of what may be only partially revealed in the mathematics of the moment (p. 6).

They further describe their idea of horizon knowledge as consisting of four elements (ibid):

1) A sense of the mathematical environment surrounding the current "location" in instruction.
2) Major disciplinary ideas and structures
3) Key mathematical practices
4) Core mathematical values and sensibilities

Their attention, however, seems to be focused on teachers' knowledge of *students'* mathematical horizon. For instance, they remark, "that teaching can be more skillful when teachers have mathematical perspective on what lies in all directions, behind as well as ahead, for their pupils, that can serve to orient their navigation of the territory" (p.11).

Horizon knowledge also has philosophical roots in Husserl's notions of 'inner' and 'outer' horizon. Briefly, Husserl's notion of inner horizon corresponds to aspects of an object that are not at the focus of attention but that are also intended, while the outer horizon of an object includes features which are not in themselves aspects of the object, but which are connected to the world in which the object exists (Follesdal, 1998, 2003). Connecting this notion to mathematics, and to Ball and Bass's (2009) description, we interpret the inner horizon of a mathematical object as the features of the object which are not at the focus of attention, but which surround its current "location", and this includes major disciplinary ideas and structures. The outer horizon, that which is not part of the object but is connected to the 'world', includes key mathematical practices, values, and sensibilities.

In this paper we extend the idea of knowledge at the mathematical horizon by focusing on teachers' 'inner' horizon knowledge and exemplifying the value of knowledge acquired during teachers' undergraduate studies in mathematics (their AMK) as it informs their understanding of the mathematical environment and major disciplinary ideas and structures.

Horizon and AMK

Teachers' horizon knowledge is, for us, deeply connected to their knowledge of advanced (university/college level) mathematics – that is, to their AMK (Zazkis and Leikin, in press). We consider application of AMK in a teaching situation as an instantiation of KMH. Our view is influenced by the metaphor of horizon as a place "where the land meets the sky" and we interpret this as the place where advanced mathematical knowledge of a teacher (the sky) appears to meet mathematical knowledge reflected in school mathematical content (the land). In what follows, we offer two examples of teachers' KMH.

Example 1

Miss Scarlett's Grade 12 students had just finished a unit on inverse functions. The unit test had been poorly done; Miss Scarlett observed several instances of confusion in notation and this was leading to miscalculations among other errors. The majority of her students were writing $1/f(x)$ where they meant $f^{-1}(x)$, and she suspected that students were unclear as to when the reciprocal of a function was, or was not, also its inverse.

Miss Scarlett decided to take time clarifying this confusion when taking up the test. She illustrated with a handful of examples instances when the reciprocal and the inverse are the same function and when they are not. She also recalled students' work with reciprocal and inverse of numbers, noting that the reciprocal of a number depends on the operation of multiplication, but that the inverse of a number can refer to its additive inverse or its multiplicative inverse (the latter being equivalent to the reciprocal).

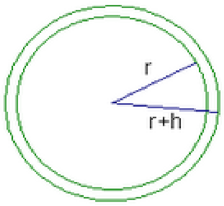
The concept of inverse is one that is prevalent in many mathematics courses, however it was during a university course in group theory that Miss Scarlett acquired an understanding of the inverse of a group element with respect to the particular operation of that group. For example, the set of integers with a corresponding operation of addition has a group structure – it includes an identity element, is closed with respect to addition, and necessarily contains inverses but *not* reciprocals. Miss Scarlett drew on this understanding to help her address her students' confusion. A similar instance of confusion and resolution was reported in (Zazkis and Zazkis, 2011) where a teacher used her understanding of group theory to help her student interpret the meaning of an exponent of negative one.

Example 2

During a lesson on applications of derivatives, Mrs. Peacock's pre-calculus students were given a set of 'real-world' problem in which they were to take derivatives of various formulae. The lesson was designed to reinforce calculation techniques through application to standard word problems. The students were unfamiliar with limits, as it was not part of the course curriculum.

As the class worked on their exercises, one student noticed when working with the sphere and circle, that the derivative of the volume formula yielded the formula for surface area, and the derivative of the area formula yielded the formula for circumference, respectively. That is, $\frac{dV}{dr} = \frac{d}{dr}\left(\frac{4}{3}\pi r^3\right) = 4\pi r^2$ and $\frac{dA}{dr} = \frac{d}{dr}(\pi r^2) = 2\pi r$. The student asked why this relationship held for the sphere and the circle, and not in other cases such as with the cube and square.

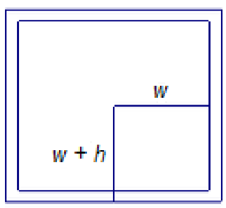
The connection between surface area and volume is one that Mrs. Peacock made during a university calculus course. She recalled a geometric representation for the derivative of a circle's area, and was aware of an analogous argument for the derivative of a sphere's volume. Mrs. Peacock understood the significance of the diagram:



and knew that the derivative of the volume (for e.g.) could then be defined as:

$$\lim_{h\to 0}\frac{\frac{4}{3}\pi(r+h)^3 - \frac{4}{3}\pi r^3}{h} = \lim_{h\to 0} 4\pi r^2 + 4\pi rh + \frac{4}{3}\pi h^2 = 4\pi r^2.$$

She further knew that it was possible to similarly represent and define the derivative of the volume of a cube. She recalled the figure:



where *w* is equal to half the length of one side. From this diagram, she knew that the derivative of the volume of the cube could be written as:

$$\lim_{h\to 0}\frac{8(x+h)^3 - 8x^3}{h} = \lim_{h\to 0} 24x^2 + 24xh + 8h^2 = 24x^2.$$

While it was beyond the scope of the lesson to introduce the definition and calculation of limits in this class, Mrs. Peacock gave an intuitive and geometric explanation for why this relationship holds. It was her knowledge of mathematics acquired in her university studies that heightened her awareness of the important observations her student had made and of the potential connections that might result.

Discussion

The two examples presented above illustrate how teachers' knowledge of major disciplinary ideas and structures beyond what was addressed in the secondary school curriculum (e.g. group structures and inverses; derivatives and geometric interpretations of limits) were useful in a teaching situation. The knowledge they acquired in university – their AMK – was not at the focus of their attention, however they were able to recognize its applicability and connection to the mathematics in question, and to access it easily and flexibly in order to address students' questions. In particular, Miss Scarlet and Mrs. Peacock were able to apply their AMK in a way that resonated with their students, and that showed their "sense of the mathematical environment surrounding the current 'location' in instruction" (Ball and Bass, 2009, p.6).

Although related to the work of Ball and Bass (2009), our notion of knowledge at the mathematical horizon differs from what they describe as "a kind of elementary perspective on advanced knowledge" (2009, p. 10). Rather, we see it as an advanced perspective on elementary knowledge. That is, as advanced mathematical knowledge (AMK) applied to ideas in the elementary or secondary (or undergraduate) curriculum. The two examples focus on what we interpret as the "inner" horizon of function or derivative – aspects of these mathematical objects that are beyond the scope of the student, but that are fundamental to the object and within the grasp of the teacher. We seek to further explore teachers' and prospective teachers' KMH with a particular focus on what could be included in undergraduate mathematics education and teacher preparation in order to encourage a flexible use of KMH.

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