Students' Reinvention of Formal Definitions of Series and Pointwise Convergence

Jason Martin
Arizona State University
Jason.H.Martin@asu.edu

Michael Oehrtman
University of Northern Colorado
Michael.Oehrtman@unco.edu

Kyeong Hah Roh
Arizona State University
khroh@math.asu.edu

Craig Swinyard
University of Portland
Swinyard@up.edu

Catherine Hart-Weber
Arizona State University
Catherine.Hart@asu.edu

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The purpose of this research was to gain insights into how calculus students might come to understand the formal definitions of sequence, series, and pointwise convergence. In this paper we discuss how one pair of students constructed a formal ε-N definition of series convergence following their prior reinvention of the formal definition of convergence for sequences. Their prior reinvention experience with sequences supported them to construct a series convergence definition and unpack its meaning. We then detail how their reinvention of a formal definition of series convergence aided them in the reinvention of pointwise convergence in the context of Taylor series. Focusing on particular x-values and describing the details of series convergence on vertical number lines helped students to transition to a definition of pointwise convergence. We claim that the instructional guidance provided to the students during the teaching experiment successfully supported them in meaningful reinvention of these definitions.

Keywords: Reinvention of Definitions, Series Convergence, Pointwise Convergence, Taylor Series

Introduction and Research Questions

How students come to reason coherently about the formal definition of series and pointwise convergence is a topic that has not be investigated in great detail. Research into how students develop an understanding of formal limit definitions has been largely restricted to either the limit of a function (Cottrill et al., 1996; Swinyard, in press) or the limit of a sequence (Roh, 2010). The general consensus among the few studies in this area is that calculus students have great difficulty reasoning coherently about formal definitions of limit (Bezuidenhout, 2001; Cornu, 1991; Tall, 1992; Williams, 1991). The majority of existing research literature on students' understanding of sequences and series concentrates on informal notions of convergence (Przenioslo, 2004) or the influence of visual reasoning or beliefs (Alcock & Simpson, 2004, 2005). Literature on pointwise convergence is typically in the context of Taylor series addressing student understanding of various convergence tests (Kung & Speer, 2010), the categorization of various conceptual images of convergence (Martin, 2009), the influence of visual images on student learning (Kidron & Zehavi, 2002), and the effects of metaphorical reasoning (Martin & Oehrtman, 2010). We recruited a pair of students from a second-semester calculus course incorporating approximation and error analysis as a coherent approach to developing the concepts in calculus defined in terms of limits (Oehrtman, 2008). The goal that they reinvent the formal definitions of sequence, series, and pointwise convergence. For this paper we posed:

1. What are the challenges that students encountered during guided reinvention of the definitions for series and pointwise convergence?

2. What aspects of the students' definition of sequence convergence supported their reinvention of series convergence? What aspects of the students' definition of series convergence supported their reinvention of the definition of pointwise convergence?

Theoretical Perspective and Methods

To investigate our research questions, we adopted a *developmental research* design, described by Gravemeijer (1998) "to design instructional activities that (a) link up with the informal situated knowledge of the students, and (b) enable them to develop more sophisticated, abstract, formal knowledge, while (c) complying with the basic principle of intellectual autonomy" (p.279). Task design was supported by the *guided reinvention* heuristic, rooted in the theory of Realistic Mathematics Education (Freudenthal, 1973). Guided reinvention is described by Gravemeijer, K., Cobb, P., Bowers, J., and Whitenack, J. (2000) as "a process by which students formalize their informal understandings and intuitions" (p.237).

The authors conducted a six-day teaching experiment with two students at a large, southwest, urban university. The full teaching experiment was comprised of six, 90-120 minute sessions with a pair of students who were currently taking a Calculus course whose topics included sequences, series, and Taylor series. The central objective of the teaching experiment was for the students to generate rigorous definitions of *sequence convergence*, *series convergence*, and *pointwise convergence*. The research reported here focuses on the evolution of the two students' definitions of series and pointwise convergence over the course of the last three sessions of the teaching experiment following the students' reinvention of a formal definition of sequence convergence. The design of the instructional activities was inspired by the *proofs and refutations* design heuristic adapted by Larsen and Zandieh (2007) based on Lakatos' (1976) framework for historical mathematical discovery.

The teaching experiment activities on series began with students producing and subsequently unpacking details of convergent series graphically. We then asked the students to generate a definition by completing the statement, "A series converges when…" To address pointwise convergence, we asked the students to produce a graph of e^x with several approximating Taylor polynomials and discuss several details of convergence on the graph. The students where then prompted to talk about what Taylor series were, and finally instructed to produce a definition for Taylor series convergence. The majority of each session consisted of students' iterative refinement of a definition and the unpacking of their intended meanings for individual elements within each definition.

Results

The reinvention of series convergence began at the end of day 3 and continued on into the 4th day. The students initially drew a graph of an alternating series, and after considering the harmonic series, they remembered that the series was divergent. When considering other formulas they were unable to produce another graph besides alternating series graphs. However, when prompted to not focus on finding a formula, the students compared these graphs to sequences and expressed that series graphs "are harder to throw out there." After they stopped focusing a formula they were able to produce a series graph increasing toward 7 and partial sums eventually became constant. Afterwards, these graphs were available for the students to refer to when defining series convergence. The students' initial definition of series convergence to 7 was simply that a series converges when "the a_n's are going to 0 and s_n's are going to 7." On day 4, after briefly looking at their graphs of series from the previous day, the students almost immediately started to perceive "graphically" series as "very similar to sequences because you could still set the error bound within certain- whatever range you want- any error bound, and then determine the point N where all the partial sums are within the error bound." Likewise, the students stated the meaning of series convergence in terms of terminologies and notations from the approximation frame. They also reinterpreted each element (N, error bounds, quantifiers) from their prior definition of sequence convergence as elements in a definition for series convergence. Furthermore, they recognized the need to replace a_n with the partial sum s_n. However, the students did not *just* change a_n to s_n, they considered each dot in the series graph as representative of partial sums, "You're adding a_1 to a_2 to a_3 to get each one of these dots on the graph of a series." After a few revisions, they constructed a definition for series convergence as follows: "A series converges to U when $\forall \varepsilon > 0$, there exists some N s.t. $\forall n \geq N\ \left|U - S_n\right| \leq \varepsilon$."

In initially discussing Taylor series, the students employed informal reasoning as they described various graphical attributes of Taylor polynomials approaching e^x. While giving these informal descriptions the students were not attending to the convergence of Taylor series for particular values of the independent variable. When they were prompted to discus error, however, one of the students suggested considering a specific point, and they subsequently highlighted errors as vertical distances between the values of e^x and a Taylor polynomial at a particular x-value. Even though their focus had moved to a particular x-value, they continued to employ informal reasoning that entailed Taylor polynomials and generating functions as being exactly the same once the graphs were "on" each other. They reasoned that this "on-ness" would occur for a Taylor polynomial of relatively low degree for x-values close to the center while a larger degree was needed for x-values away from the center. Only once they had used a Taylor series equation for e^x to find an explicit series for e did they realize that a finite number of terms merely approximated e^x because the remaining terms not used in the approximation "had value."

Once the students recognized that focusing on a single x-value produced a series, they attempted to leverage their definition of series convergence to define Taylor series convergence. During this process they demonstrated considerable confusion between the independent variable, x, and the index, n. One of the students questioned the existence of an N for which all subsequent Taylor polynomial approximations would be within a given error bound of e^x, but in her explanation N appeared to correspond to some lower bound of x-values rather than n-values. After being instructed to explain their definition of series convergence using only a vertical number line, the students recognized the role of N on a vertical number line. This shift to viewing series in a vertical orientation eventually freed them to see the graphs of Taylor series as comprised of convergent series at each x-value where N is dependent upon x as well as ε. Subsequently they expressed a need to "expand" their series definition to capture all x-values. In their first attempt they simply added $\forall x$ at the end of their definition, but they later expressed discomfort with finding one N such that all subsequent Taylor polynomials would be within ε of e^x for all x. The students then quickly latched onto a suggestion to move $\forall x$ to the beginning of their definition, acknowledging how this movement expressed the dependence of N upon x. Their final definition of pointwise convergence in the context of a Taylor series with an infinite interval of convergence was as follows: "A Taylor series converges to $f(x)$ when $\forall x$, $\forall \varepsilon > 0$ there exists some N such that $\forall n \geq N\ \left|f(x) - S_n(x)\right| \leq \varepsilon$."

Even though this definition for Taylor series convergence captures much the formal meaning of pointwise convergence, one student commented that the consecutive universal quantifiers felt "goofy." Even so, they continued to view it as best capturing Taylor series convergence.

Conclusion and Discussion

It is remarkable that the students reinvented and unpacked the formal definition of series and pointwise convergence within such a short time. The students faced challenges that ranged from seeing graphical attributes of series and Taylor series to the ordering of quantifiers. We claim that the instructional guidance provided to the students during the teaching experiment successfully supported them engaging these challenges and their subsequent reinvention of these definitions. First of all, the instructors' asking students to produce graphs of series and Taylor series convergence gave students a reference point for which they could refer to during the construction of their definitions. Second, the prior activity of defining sequence convergence became a means for supporting the students' definition of series convergence as they recognized similarities between sequence and series convergence in the context of their graphs and their definitions. Similarly, their prior activity of defining sequence and series convergence supported students' definition of pointwise convergence. Finally, their emerging approximation scheme helped the students to meaningfully recognize similarities between definitions and interpret each component within a definition. The approximation terminology that they had learned from class allowed them to meaningfully interpret the role of approximations, error, and error bounds in and across definitions.

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