**Relationships between Quantitative Reasoning and Students' Problem Solving Behaviors**
Contributed Research Report

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***Abstract:*** This presentation reports on the results of a study into precalculus students' reasoning when solving novel problems. The study intended to identify students' mental actions that support or hinder their ability to provide meaningful and correct solutions, while also characterizing the role of quantitative reasoning in the students' solutions. Analysis of clinical interviews with each student revealed that a student's propensity to reason about quantities and a problem's context significantly influenced his or her problem solving approach. Students who spent a significant amount of time orienting to a problem by identifying quantities and relationships between quantities leveraged the resulting mental images throughout their problem solving activity. Contrary to this, students who focused on recalling procedures and performing calculations spent little time reasoning about a problem's context and encountered difficulty providing meaningful and correct solutions. These findings offer insights into the relationship between students' reasoning and their problem solving behaviors.

Key Words: Precalculus, Problem Solving, Student Reasoning, Quantitative Reasoning

## Introduction

Problem solving has been a focus of mathematicians and mathematics educators for well over the past half-century. The focus on problem solving has ranged from suggesting that curricula be designed to promote learning through problem solving (NCTM, 2000) to characterizing the problem solving processes and performance of students and mathematicians (M. Carlson, 1999; M. P. Carlson & Bloom, 2005; Lester Jr., 1994; Pólya, 1957; Schoenfeld, 2007). Such investigations have labeled problem solving as a complex process of interrelated factors and phases, including planning, monitoring, affect, and orienting. Recent studies (M. Carlson, 1999; M. P. Carlson & Bloom, 2005; Schoenfeld, 2007) have begun revealing the intricate role these factors play in problem solving, while emphasizing the importance of exploring students' problem solving behaviors and the role of problem solving in learning mathematics.

Contributing to the body of research on problem solving, recent reports (Moore, Carlson, & Oehrtman, 2009; Smith III & Thompson, 2008) have illustrated the importance of quantitative reasoning in students solving novel problems. These reports describe that a student's mental image of a problem's context (e.g., a mental scene consisting of quantities and relationships between quantities) significantly influences his or her solution to the problem. This finding highlights the delicate and complex nature of problem solving, and advocates the need to further investigate the role of quantitative reasoning in problem solving and learning mathematics.

This study sought to build on the current body of problem solving research by investigating precalculus students' reasoning as they engaged in problem solving activity. The goal of the study was to identify relationships between quantitative reasoning and students' behaviors during the various problem solving phases identified by Carlson and Bloom (2005). In doing so, this study's findings add to the limited knowledge on students' problem solving behaviors at the secondary and undergraduate mathematics level. The results presented in this paper focus on various behaviors that occur during the problem solving phases, and how a student's propensity to reason quantitatively influences the mental actions driving these behaviors. These results offer insights into the types of reasoning that either hinder or support students' problem solving abilities, and how these reasoning patterns influence each problem solving phase.

## Background

In an attempt to provide a finer characterization of problem solvers' cognitive processes, Carlson and Bloom (2005) investigated the problem solving activity of 12 mathematicians. Drawing from analysis of interviews with the mathematicians, as well as previous research on problem solving (Lester Jr., 1994; Pólya, 1957; Schoenfeld, 2007), the authors created the *Multidimensional Problem-Solving Framework*. This framework identifies multiple problem solving cycles within four problem solving phases: orientation, planning, executing, and checking. Additionally, Carlson and Bloom's study revealed various problem solving attributes (e.g., monitoring and affect) that influence a problem solver's behaviors.

Carlson and Bloom (2005) noted that much is still to be learned relative to the problem solving processes of students, as their study focused on mathematicians. In response to this call, Moore, Carlson, and Oehrtman (2009) examined precalculus students' problem solving behaviors. Findings from this study identified the critical role of quantitative reasoning (Smith III & Thompson, 2008) when a student orients to a novel problem. The students involved in the study often constructed incorrect mental images of a problem's context when orienting to a problem. The students subsequently constructed incorrect solutions, where these solutions were consistent with their images of the problem's context. After the students reflected on their solutions and refined their images of a problem's context, they corrected their solutions to reflect their modified quantitative structures. These actions enabled the students to provide meaningful explanations of their corrected solutions. These findings illustrate the importance of the orientation phase, as well as a need to further explore the role of quantitative reasoning in problem solving.

## Methods and Subjects

The subjects of this study were nine undergraduate precalculus and college algebra students at a large public university in the southwest United States. The students were chosen on a voluntary basis and they received monetary compensation for their participation. Clinical interviews (Clement, 2000; Goldin, 2000) were conducted with each student, during which they were asked to solve a set of novel problems. During the interviews, the interviewer prompted the students to explain their thinking in order to gain insights into the reasoning processes driving their problem solving behaviors. Due to the cognitive nature of problem solving, the clinical interview setting was critical in identifying reasoning that would not have been revealed in a classroom setting or collected student work. Also, this study rested on the stance that each student engages in unique reasoning, and hence the clinical interview methodology offered data that enabled characterizing each student's reasoning processes.

The data was analyzed following an open coding approach (Strauss & Corbin, 1998). The students' behaviors were analyzed in an attempt to determine the mental actions that contributed to their solutions. The mental actions inferred from the students' behaviors were then characterized in terms of the problem solving phases identified by Carlson and Bloom (2005). This phase of the data analysis involved identifying how the students' mental actions influenced their behaviors during the four problem solving phases. This approach to analyzing the data enabled classifying how various reasoning patterns related to the students' problem solving behaviors. Lastly, the students' mental actions and problem solving behaviors were compared and contrasted. This stage of analysis led to the finding that the students held varying problem solving dispositions that paralleled their propensity to engage in quantitative reasoning.

## Results

Analysis of the students' solutions revealed that their propensity to engage in quantitative reasoning significantly influenced the nature of their problem solving behaviors and their ability to provide meaningful solutions. Students who extensively focused on a problem's context developed a mental image of the context that they leveraged during the problem solving phases. Contrary to this, when students focused on performing procedures and calculations, they did not build an image of a problem's context that supported their solution process.

When orienting to a problem, students with a propensity to focus on a problem's context frequently drew and labeled a diagram of the situation. This act included identifying known and unknown measurements and discussing various relationships between quantities. As these students continued to focus on a problem's context, they were observed revisiting the problem statement to identify the goal (and sub-goals) of a problem in terms of the quantities of the situation. During the planning phase of problem solving, they continued to spend a significant amount of time reasoning about a problem's context. They planned their solutions by identifying relationships between quantities and reasoning about these relationships in ways that enabled them to *anticipate* performing calculations. By reasoning about relationships between quantities without performing numerical operations, the students were able to engage in the *conjecture-imagine-evaluate* cycle identified by Carlson and Bloom (2005) to mentally play out their solutions. Similarly, these students recalled formulas during the planning phase and described these formulas in terms of the quantities of the situation. This enabled the students to use formulas to represent relationships between values without having to evaluate the formulas.

When executing their planned solutions, the students described calculations in terms of the quantities of the situation and consistently illustrated the quantity referenced by a newly obtained value. Also, due to their calculations being grounded in quantitative relationships, the students constructed a quantitative meaning for a value before obtaining a numerical result. That is, the students did not need to determine the meaning of a result of calculating, as they had developed a meaning previous to the calculation. The students' images of the problems' contexts also supported their monitoring the appropriateness of the calculated values. When they obtained values that were not consistent with their image of a problem's context, the students returned to the context to further orient to the problem, check their solution, and modify their solution if needed. These actions enabled the students to identify incorrect solutions and use the context of the problem to justify alterations to their solution.

Students with a propensity to reason about calculations and procedures engaged in problem solving behaviors significantly different to the behaviors previously outlined. When orienting to a problem, students with a tendency to focus on calculations and procedures often drew a diagram, but they infrequently labeled known and unknown values on the diagram and spent limited time verbally discussing a problem's context. Instead, they regularly referred to previously completed problems deemed similar to the current problem. Subsequently, these students attempted to recall the steps or calculations made when solving a similar problem. In the cases that they recalled previous solutions, the students progressed to the executing phase without further explaining or analyzing the recalled solution. In the cases that they could not recall a previous solution, they experienced difficulty progressing and suggested calculations to the interviewer (who did not provide feedback). When asked to explain a meaning of their suggested calculations, these students expressed a need to first calculate a numerical value, as opposed to attempting to explain the calculation previous to performing the calculation.

After executing a suggested calculation, the students experienced difficulty determining how the obtained value related to a problem's context or the goal of the problem. The students frequently gave multiple meanings to the determined numbers (e.g., using a number to refer to multiple lengths), and the students relied on the aesthetic quality of their answers (e.g., values not "too big" or "too small") to check their solutions. In the cases that the students believed their solution was incorrect, they looked to the interviewer for assistance or attempted to recall another procedure.

## Conclusions and Implications

The varying problem solving approaches exhibited by the students of this study reveal how a student's problem solving disposition can influence his or her ability to solve novel problems. These insights should inform curriculum designers and teachers about the reasoning abilities and problem solving behaviors they should strive to engender in students. Also, students were sometimes observed alternating problem solving dispositions from problem to problem, as well as within a single problem. Further research should explore reasons for such transitions, and the instruction necessary to promote students developing a disposition that supports their constructing meaningful and correct solutions to novel problems. Students with a quantitative disposition also appeared to be more reflective during their problem solving activity. This may have been a result of their reasoning creating a foundation for reflective actions. Future research should investigate this phenomenon, and its implications for using problem solving to promote students learning mathematics.

## References

Carlson, M. (1999). The mathematical behavior of six successful mathematics graduate students: Influencs leading to mathematical success. *Educational Studies in Mathematics, 40*(3), 237-258.

Carlson, M. P., & Bloom, I. (2005). The cyclic nature of problem solving: An emergent multidimensional problem-solving framework. *Educational Studies in Mathematics, 58*(1), 45-75.

Clement, J. (2000). Analysis of clinical interviews: Foundations and model viability. In A. E. Kelly & R. A. Lesh (Eds.), *Handbook of Research Design in Mathematics and Science Education* (pp. 547-589). Mahwah, NJ: Lawrence Erlbaum Associates, Inc.

Goldin, G. A. (2000). A scientific perspective on structured, task-based interviews in mathematics education research. In A. E. Kelly & R. A. Lesh (Eds.), *Handbook of Research Design in Mathematics and Science Education* (pp. 517-545). Mahwah, NJ: Lawrence Erlbaum Associates, Inc.

Lester Jr., F. K. (1994). Musings about mathematical problem-solving research. *Journal for Research in Mathematics Education, 25*(6), 660-675.

Moore, K. C., Carlson, M. P., & Oehrtman, M. (2009). *The role of quantitative reasoning in solving applied precalculus problems.* Paper presented at the Twelfth Annual Special Interest Group of the Mathematical Association of America on Research in Undergraduate Mathematics Education (SIGMAA on RUME) Conference, Raleigh, NC: North Carolina State University.

NCTM. (2000). *Principles and standards for school mathematics*. Reston, VA: NCTM.

Pólya, G. (1957). *How to Solve It, A New Aspect of Mathematical Method* (2 ed.). Princeton, NJ: Princeton University Press.

Schoenfeld, A. H. (2007). Problem solving in the United States, 1970-2008: research and theory, practice and politics. *ZDM Mathematics Education, 2007*(39), 537-551.

Smith III, J., & Thompson, P. W. (2008). Quantitative reasoning and the development of algebraic reasoning. In J. J. Kaput, D. W. Carraher & M. L. Blanton (Eds.), *Algebra in the Early Grades* (pp. 95-132). New York, NY: Lawrence Erlbaum Associates.

Strauss, A. L., & Corbin, J. M. (1998). *Basics of qualitative research: Techniques and procedures for developing grounded theory* (2nd ed.). Thousand Oaks: Sage Publications.