From Intuition to Rigor:
Calculus Students' Reinvention of the Definition of Sequence Convergence

Contributed Research Report

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Abstract

Little research exists on the ways in which students may develop an understanding of formal limit definitions. We conducted a study to i) generate insights into how students might leverage their intuitive understandings of sequence convergence to construct a formal definition and ii) assess the extent to which a previously established approximation scheme may support students in constructing their definition. Our research is rooted in the theory of Realistic Mathematics Education and employed the methodology of guided reinvention in a teaching experiment. In three 90-minute sessions, two students, neither of whom had previously seen a formal definition of sequence convergence, constructed a rigorous definition using formal mathematical notation and quantification nearly identical to the conventional definition. The students' use of an approximation scheme and concrete examples were both central to their progress, and each portion of their definition emerged in response to overcoming specific cognitive challenges.

Keywords: Limits, Definition, Guided Reinvention, Approximation, Examples

Introduction and Research Questions

A robust understanding of formal limit definitions is foundational for undergraduate mathematics students proceeding to upper-division analysis-based courses. Definitions of limits often serve as a starting point for developing facility with formal proof techniques, making sense of rigorous, formally-quantified mathematical statements, and transitioning to abstract thinking. The majority of the literature on students' understanding of limits (Bezuidenhout, 2001; Cornu, 1991; Davis & Vinner, 1986; Monaghan, 1991; Tall, 1992; Williams, 1991) describes informal student reasoning about limits, with particular attention given to the myriad of student misconceptions. However, there is a paucity of research on student reasoning about formal definitions of limits. The general consensus among the few studies in this area seems clear – calculus students have great difficulty reasoning coherently about the formal definition (Artigue, 2000; Bezuidenhout, 2001; Cornu, 1991; Tall, 1992; Williams, 1991). What is less clear, however, is how students come to understand the formal definition. Indeed, this is an open question with few empirical insights from research to inform it (Cottrill et al., 1996; Roh, 2008; Swinyard, in press). Oehrtman (2008) proposed a coherent approach to developing the concepts in calculus through a conceptually accessible framework for limits in terms of approximation and error analysis. Students were recruited to participate in our study from a course that relied heavily on Oehrtman's approach. This study addressed the following research questions:

1. What are the cognitive challenges that students encounter during a process of guided reinvention of the formal definition for sequence convergence?
2. What aspects of their concept images do students evoke during this reinvention?
3. How do students' evoked concept images and their solutions to cognitive challenges encountered support more advanced mathematical thinking about limits of sequences?

Theoretical Perspective and Methods

We adopted a *developmental research* design, described by Gravemeijer (1998) "to design instructional activities that (a) link up with the informal situated knowledge of the students, and (b) enable them to develop more sophisticated, abstract, formal knowledge, while (c) complying with the basic principle of intellectual autonomy" (p.279). Task design was supported by the *guided reinvention* heuristic, rooted in the theory of Realistic Mathematics Education (Freudenthal, 1973). Guided reinvention is described by Gravemeijer, K., Cobb, P., Bowers, J., and Whitenack, J. (2000) as "a process by which students formalize their informal understandings and intuitions" (p.237).

The authors conducted a six-day teaching experiment with two students at a large, southwest, urban university. The full teaching experiment was comprised of six 90-120 minute sessions with a pair of students currently taking a Calculus course whose topics included sequences, series, and Taylor series. The central objective of the teaching experiment was for the students to generate rigorous definitions of *sequence*, *series*, and *pointwise convergence*. The research reported here focuses on the evolution of the two students' definition of sequence convergence over the course of the first three sessions of the teaching experiment. The design of the instructional activities was inspired by the *proofs and refutations* design heuristic adapted by Larsen and Zandieh (2007) based on Lakatos' (1976) framework for historical mathematical discovery. Activities commenced with students generating prototypical examples of sequences that converge to 5 and sequences that do not converge to 5. The majority of each session then consisted of the students' iterative refinement of a definition to fully characterize sequence convergence. The students were to evaluate their own progress by determining whether their definition included all of the examples of convergent sequences and excluded all of the non-examples.

Results

Three broad areas of findings emerged from our data analysis: the role of students' use of examples, the effect of a scheme for limits based on approximation language, and the students' adoption and appreciation of quantifiers and efficient mathematical expressions.

The Role of Examples. The students' reinvention efforts were aided considerably by the presence of the examples they constructed at the start of the experiment. These examples served as sources of cognitive conflict when their definition failed to fully capture the necessary and sufficient conditions under which sequences converge. For example, the students' initial definitions were predictably couched in language that was vague, intuitive, and dynamic. Their first written definition was "A sequence converges to 5 as $n \to \infty$ provided that the number approaches or is 5 and no other number." The students immediately identified weaknesses in this definition as they applied it to their examples that increase monotonically to 4, alternate around 5 or behave erratically before eventually looking like a standard example of a convergent sequence. Having identified these weaknesses, they also looked to their examples to provide direction for their revisions. This pattern of evaluating and refining their definitions against the examples repeated over 18 cycles during the first three days of the teaching experiment.

The Effect of an Approximation Scheme for Limits. The students' familiarity with a previously established approximation scheme mirroring the structure of the formal definition but framed in more accessible terms (Oehrtman, 2008) provided students significant leverage for i) focusing on relevant quantities in the formal definition, ii) fluently working with the relationships between these quantities, and iii) making the necessary but difficult cognitive shift to focus on N as a function of ε (Roh, 2008; Swinyard, in press). For example, during the first 12 minutes of the teaching experiment, the students did not invoke language about approximations to describe aspects of a sequence $\{a_n\}$. During this time they did not discuss or represent the quantity $|a_n - 5|$ in any form and all descriptions of convergence involved informal dynamic language. But once they invoked an approximation scheme, they described the limit as the value being approximated, the terms a_n as the approximations and the distance between them as the error which they immediately represented as $|a_n - 5|$. These ideas became an integral part of their arguments and the students shifted to discussing how close the terms needed to get to 5 to consider the sequence convergent. After another 14 minutes, the students invoked the idea of an error bound (corresponding to ε in the formal definition) to address this question and focused on how to make the error smaller than this bound. Nine minutes later, they introduced the idea of there being "some point n" (corresponding to N in the formal definition) at which this must happen. Afterwards, they consistently reasoned that this "point n depends [on] what the acceptable error is." For the remainder of Day 1 and throughout Days 2 and 3 of the teaching experiment, the students continued to rely on this approximation scheme to describe the relevant quantities and to keep track of the relationships among them.

Adoption of Quantifiers and Mathematical Expressions. Powerful use of logical quantifiers and mathematical expressions emerged only after the students had i) fully developed the underlying conceptual structure of convergence in informal terms, ii) wrestled with the problem of how to rigorously express those ideas, and iii) seen the quantifiers and expressions as viable solutions to these problems. Early in the first day of the teaching experiment, one student recalled the use of universal and existential quantifiers. While she used them correctly neither student applied them to resolve any problem they were wrestling with and they soon dropped the quantifiers. On Day 3 of the teaching experiment, the students were consistently verbalizing all elements and appropriate logic of the ε-N definition, but lacked the terminology or notation to construct what they considered an acceptable written definition. As they struggled with these issues, brief reminders of the quantifiers they had used earlier but discarded were seized upon as perfect solutions to their difficulties. Ultimately the students settled on the definition

"A sequence converges to U when $\forall\ \varepsilon$, there exists some N, $\forall\ n \geq N$, $|U - a_n|<\varepsilon$."

The students expressed strong appreciation for the power of the quantifiers and mathematical notation in their definition, citing multiple problems that each part efficiently resolved.

Limitations, Implications and Conclusions

The two students in this teaching experiment had only experienced instruction aimed at developing a systematic approximation scheme for reasoning about limits for a portion of one calculus course. Consequently, it is not surprising that they did not immediately invoke this scheme as they began to wrestle with generating a definition of sequence convergence and that the scheme emerged in pieces. Nevertheless, it did not take them long to turn to approximation ideas, and each portion of their evoked scheme emerged in response to particular problems for which it was well-suited to address. We note that these students progressed much more quickly towards a formal definition and through resolving several cognitive challenges than students not introduced to the approximation framework (Swinyard, in press). Once evoked, the students' ideas about approximation remained consistent, and their images and application of their scheme was sufficiently strong to provide them considerable guidance and conceptual support for reasoning about the formal definition.

This study drew from data collected in a teaching experiment with only two students and we acknowledge that each individual will follow unique paths. Further, orchestrating this type of discussion for an entire class will certainly involve significant differences from what was possible with focused attention on two students. Nevertheless, these students' reinvention of the definition serves not only as an existence proof that students can construct a coherent definition of sequence convergence, but also as an illustration of *how* students might reason as they do so. Our findings shed light on several relevant cognitive challenges engaged by the students, how they resolved these difficulties, and the resulting conceptual power derived from their solutions. These results are guiding our future work to develop, evaluate and refine classroom activities for introductory analysis courses.

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