How Intuition and Language Use Relate to Students' Understanding of Span and Linear Independence

Contributed Research Report

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This report describes a case study in an undergraduate elementary linear algebra class about the relationship between students' understanding of span and linear independence and their intuition and language use. The study participants were seven students with a range of understanding levels. The purpose of the research was to explore the relationship between students' "natural" thinking and their conceptual development of formal mathematics and the role of language in this conceptual development. Findings indicate that students with low indicators of intuition and stronger language skills developed better understanding of span and linear independence. The report includes possible instructional implications.

Keywords: Intuition, Language use, Linear algebra, Linear independence, Span

In an essay about his experiences teaching linear algebra, David Carlson (1997) posed a question that has become emblematic of students' learning in linear algebra: Must the fog always roll in? This question, he writes,

> refers to something that seems to happen whenever I teach linear algebra. My students first learn how to solve systems of linear equations, and how to calculate products of matrices. These are easy for them. But when we get to subspaces, spanning, and linear independence, my students become confused and disoriented. It is as if a heavy fog rolled over them, and they cannot see where they are or where they are going. (p. 39)

Research into the teaching and learning of linear algebra has spanned several decades, but the issue of how to clear the fog for students is still outstanding. In this report, I describe a research study designed to contribute to the understanding of how students learn concepts in linear algebra.

The purpose of this study was to address two outstanding issues in the learning of advanced mathematics. The first issue is a theoretical difference between the ways in which students learn "naturally" and the formal structure of mathematics, and how this difference may or may not influence students' mathematical understanding. The second issue is the relationship between students' language use and their mathematical understanding and how this might relate to students' natural ways of learning. My research question was:

> How do students' intuition and language use relate to the nature of their understanding of span and linear independence in an elementary linear algebra class?

Existing research supports the existence of the issues this study was designed to address. In his epilogue of *Advanced Mathematical Thinking*, Tall (1991) noted that many of the book's contributors believed students' difficulties in learning advanced mathematics could be explained by the discrepancies between the way students viewed mathematics and classroom instruction, which is often based on the formal structure of mathematics. More recently, in their discussion of advanced mathematical thinking, Mamona-Downs and Downs (2002) suggested traditional teaching of mathematics does not "connect with the students' need to develop their own intuitions and ways of thinking" (p. 170). An impediment to developing instructional theory based on students' intuitions is an incomplete understanding of how people develop abstract mathematical knowledge. Pegg and Tall (2005) compared several theories of concept development and derived a fundamental cycle of concept construction underlying each of the theories. However, there is no consensus on the mechanism of how this concept development occurs. Some evidence exists to suggest language may play a role in this development (Dehaene, Spelke, Pinel, Stanescu, & Tsivkin, 1999; Devlin, 2000). Pugalee (2007) contends "language and competence in mathematics are not separable" (p. 1). MacGregor and Price (1999) and Boero, Douek, and Ferrari (2002) believe that metalinguistic awareness is necessary for students to coordinate the various notation systems in mathematics. Yet, little research exists that explores the relationship between students' language abilities and mathematics learning (Barwell, 2005; Huang & Normandia, 2007; MacGregor & Price). Interestingly, though, just as mathematics education researchers have found a contrast between intuitive thinking and formal mathematics, language researchers have found this same contrast between everyday language use and the demands of formal school language (Schleppegrell, 2001, 2007). It is possible, then, that language plays an important role in how students move from intuitive, everyday thinking to understanding formal mathematical concepts and theory.

The literature about learning linear algebra in general and learning about span and linear independence specifically reflects the issues reported in the literature about intuition in learning mathematics. This includes students' difficulty with understanding and using formal definitions (Medina, 2000) and students relying upon surface features, prototypical examples, and intuitive models rather than conceptual understanding (Harel, 1999; Hristovitch, 2001; Medina). Lacking in this literature, though, is a clear picture of the interaction between instruction, students' intuition, and the nature of students' understanding. In particular, it does not reveal the components of understanding of span and linear independence that are sufficient for an elementary linear algebra class nor the individual differences in intuition and language use that may account for variation in student learning.

The theoretical perspective for this research was the *emergent perspective* described by Cobb and Yackel (Cobb, 1995; Cobb & Yackel, 1996; Yackel & Cobb, 1996). The emergent perspective is a type of social constructivism that coordinates the social and psychological (individual) views (Cobb & Yackel). The *interactionist* view of classroom processes (Bauersfeld, Krummheuer, & Voigt, 1988) represents the social perspective, while a constructivist view of individuals' (both students and teacher) activity (von Glasersfeld, 1984, 1987) represents the psychological perspective. I used the case study methodology for this research and delimited the setting of the study to one elementary linear algebra class. Broadly, the unit of analysis for this study was individual students. However, in alignment with my research question, I focused my analysis on students' understanding of span and linear independence and on their intuition and language use related to these understandings. I analyzed the overall level of students' understanding for the first four weeks of the course and then selected a set of seven students to represent as much as possible maximum variation in understanding levels.

This research depended on being able to operationalize the constructs of understanding, intuition, and language use. Based on the literature and the nature of my data, I found each of these constructs to be multi-dimensional. I defined understanding as the composition of definitional understanding and problem solving skills. Each of these elements had multiple components. Intuitions fell into two categories: self-evident intuitions and surface intuitions, with each category consisting of three different sub-types of intuitions. The salient characteristics of language use were understandability, completeness, and vocabulary use.

The overall findings of this research indicated an association between the quality of students' language use and the quality of their understanding. That is, the students with stronger language skills generally exhibited better understanding of span and linear independence. There was also an association between the degree to which a student's cognition had intuitive indicators and the quality of his/her understanding. The more a student's thinking had intuitive characteristics, the less likely he/she was to develop good understanding of span and linear independence.

A more detailed picture of the findings is as follows. Students' understanding was either functional or problematic. Students with fair or weak problem solving skills were classified as having problematic understanding, while those with good or strong problem solving skills were classified as having functional understanding. The quality of students' definitional understanding determined the level of understanding within each category. Within the functional category, students had strong, good, or fair definitional understanding. Within the problematic category, students had weak or poor definitional understanding. Students with functional understanding had low self-evident intuition indicators, while students with problematic understanding had medium or high self-evident intuition indicators. Students with fair, weak, or poor definitional understanding had more surface intuition indicators than students with strong or good definitional understanding. The quality of students' written explanations was associated with the students' level of understanding. However, language use quality more closely aligned with students' definitional understanding than with their problem solving skills.

There were several finding about the nature of students' learning of span and linear independence. While many students could learn the procedures related span and linear independence, some students struggled to develop conceptual understanding. In addition, many students eschewed knowing and understanding formal definitions in favor of using their own intuitive pseudo-definitions. Students who failed to develop conceptual understanding of foundational concepts, such as linear combination and solution, failed to develop conceptual understanding of span and linear independence. Students who were unclear about the objects associated with span and linear independence (e.g., did not associate linear independence with a set of vectors) did not reify these concepts, but instead viewed these concepts primarily as procedures.

The findings suggest possible classroom implications. While none of the instructional methods are new, this research may underscore their validity in supporting students' learning of mathematics by reducing the role of interfering intuitions. Instructional recommendations include helping students develop metacognitive awareness (Fischbein, 1987) and implementing compare and contrast activities (Marzano, Pickering, & Pollock, 2001). Several researchers have outlined more elaborate instructional tools. These include the instructional practices developed by researchers studying the role of beliefs in mathematics (Muis, 2004), conceptual change in science and mathematics (Vosniadou & Vamvakoussi, 2006), and in reducing misconceptions in mathematics (Stavy & Tirosh, 2000). In order to help students develop their language skills, which in turn may support their mathematical learning, it may be helpful to provide opportunities for students to engage oral and written language practice.

The study has several limitations. Because it was conducted in a single class, the findings may have limited transferability. Also, the nature of the data sources (student work and student interviews) may have limited the validity of the findings. Future research may refine or extend this study’s findings in other linear algebra classes. It may also be fruitful to explore this research question in other advanced mathematics classes, such as abstract algebra and analysis.

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