Promoting Students' Reflective Thinking of Multiple Quantifications via the Mayan Activity

Contributed Research Report

Kyeong Hah Roh
Arizona State University
khroh@math.asu.edu

Yong Hah Lee
Ewha Womans University
yonghah@ewha.ac.kr

The aim of this presentation is to introduce the Mayan activity as an instructional intervention and to examine how the Mayan activity promotes students' reflective thinking of multiple quantifications in the context of the limit of a sequence. The students initially experienced a difficulty due to the lack of understanding of the meaning of the order of variables in the definition of convergence. However, such a difficulty experienced was resolved as they engaged in the Mayan activity. The students also came to understand that the independence of the variable ε from the variable N is determined by the order of these variables in the definition. The results indicate the Mayan activity promoted students' reflective thinking of the independence of ε from the variable N and helped them understand why the order of variables matters in proving limits of sequences.

Keywords: Quantification, Reflective Thinking, Proof Evaluation, Convergent Sequence, Cauchy Sequence

Introduction

The purpose of this paper is to introduce the Mayan activity as an instructional intervention and to give an account of its effect on students' understanding of multiple quantifications in the context of the limit of a sequence. The ε-N definition of the limit of a sequence is of fundamental importance and is very useful in studying advanced mathematics; however, many students encounter difficulties when learning the ε-N definition (e.g., Mamona-Downs, 2001; Roh, 2009, 2010). In particular, students' difficulty is caused by their lack of understanding of multiple quantifications in general (Dubnisky & Yiparaki, 2000) as well as the logical structure of the ε-N definition (Durand-Guerrier & Arsac, 2005). Many students cannot perceive the importance of the order between ε and N in the ε-N definition, and they cannot recognize the independence of ε from N (Roh, 2010, Roh & Lee, in press). Accordingly, in order to improve students' understanding of the ε-N definition of limit, it is important to enable the students to understand the role of multiple quantifiers in the definition. The Mayan activity is specially designed with the intention of helping students understand the *independence of ε from N* in the ε-N definition of the limit of a sequence. By comparing students' responses before and after the Mayan activity, this study addresses the following research question: How do students develop their understanding of the role of the order of variables in the ε-N definition via the Mayan activity?

Theoretical Perspective

The theoretical perspective is based on Dewey's theory of reflective thinking. According to Dewey (1933), when an individual is opposed to his or her knowledge or belief, he or she experiences perplexity, difficulty, or frustration; then in the process of resolving it, the reflective thinking is necessarily accompanied. Dewey divides reflective thinking into three situations as follows: The *pre-reflective* situation, a situation experiencing perplexity, confusion, or doubts; the *post-reflective* situation, a situation in which such perplexity, confusion, or doubts are dispelled; and the *reflective* situation, a transitive situation from the pre-reflective situation to the post-reflective situation. In addition, Dewey characterized the reflective situation in terms of suggestions, intellectualization, hypotheses, reasoning, and tests of hypotheses by actions, which are not always in the order but some phases can be omitted or include sub-phases. In line with this perspective, the Mayan activity introduced in this paper was designed to provide arguments described in a way that ε is selected depending on N and against student knowledge or belief about limit, and to present a tractable context later in which the students can properly activate their reasoning and perceive the independence of ε from N, hence resolve their perplexity, difficulty, or frustration about their problem.

Research Methodology

The research was conducted as part of a design experiment (Cobb, Confrey, diSessa, Lehrer, & Schauble, 2003) at a public university in the USA. The tasks designed were iterated 4 times from fall 2006 to spring 2010. Such an iterative nature of the design experiment allowed for frequent cycles of prediction of student learning, analysis of student actual learning, and revision of the tasks. This paper reports two studies from the design experiment: Study 1 in the fall semester of 2006 and Study 2 in the spring semester of 2010. The participants were mathematics students or preserive mathematics teachers, and had already completed calculus and a transition-to-proof course. The author of this paper served as the instructor in both studies. The classes in both studies mainly followed an inquiry approach, in which students often made conjectures, verified their argument, or evaluated whether given arguments were legitimate as mathematical proofs. In this manner, the students studied the limit of a sequence and its related properties, in particular, the ε-N definition, and its negation, and limit proofs using the ε-N definition. Also, the similar discussions related to Cauchy sequences followed prior to the days of this study.

In Study 1, the instructor asked the students to evaluate Statement 1: *If a sequence* $\{a_n\}_{n=1}^{\infty}$ *in* $\mathbb{R}$ *is a Cauchy sequence, then for any* $\varepsilon > 0$*, there exists* $N \in \mathbb{N}$ *such that for all* $n > N$*,* $|a_n - a_{n+1}| < \varepsilon$. After the group discussion about Statement 1, the instructor asked the students to evaluate Ben's argument: *Consider* $a_n = 1/n$ *for any* $n \in \mathbb{N}$. *Since the sequence* $\{a_n\}_{n=1}^{\infty}$ *is convergent to 0, it is a Cauchy sequence in* $\mathbb{R}$. *Let* $\varepsilon = 1/\{(N+1)(N+2)\}$ *for all* $N \in \mathbb{N}$. *Let* $n = N+1$. *Then* $n > N$. *But* $|a_n - a_{n+1}| = |a_{N+1} - a_{N+2}| = 1/\{(N+1)(N+2)\} = \varepsilon \geq \varepsilon$. *Therefore, Statement 1 is false.*

On the other hand, the Mayan activity (Roh & Lee, in press) implemented in Study 2 consists of three steps: The first is to evaluate Sam's argument and Bill's argument. Sam's argument is a proper argument showing the sequence $\{1/n\}_{n=1}^{\infty}$ converges to 0 whereas Bill's argument draws an erroneous conclusion that $\{1/n\}_{n=1}^{\infty}$ does not converge to 0 by selecting ε dependently on N; the second is to evaluate the Mayan stonecutter story (see Figure 1) in which the priest's argument is compatible to Bill's argument, but is relatively easier than Bill's or Ben's argument to track on the logical error made by reversing the order of two variables from the craftsman's argument in the story; and the third is to evaluate Statement 1and Ben's argument to Statement, which were used in Study 1. Comparing results from Study 1 with those from Study 2, this paper addresses the role of the Mayan activity as an instructional intervention in promoting students' reflective thinking of the independence of ε from N.

The Mayan stonecutter story

One of the famous Mayan architectural techniques is to build a structure with stones. These stones were ground so smoothly that there was almost no gap between two stones. It was even hard to put a razor blade between them. One day a priest came to a craftsman to request smooth stones.

Craftsman: *No matter how small of a gap you request, I can make stones as flat as you request if you give me some time.*

Priest: *I do not believe you can do it. If I ask you to flatten stones within 0.01 mm, you won't be able to do it.*

Craftsman: *Give me 10 days, and you will receive stones as flat as within 0.01 mm.*

Ten days later, the craftsman made two stones so flat that the gap between them was within 0.01 *mm*. On the 11th day, the priest came to see the stones and argued that,

Priest: *These stones are not flat within 0.001 mm. What I actually need are stones as flat as within 0.001 mm.*

Craftsman: *Okay, if you give me 5 more days, I can make the stones as flat as within 0.001 mm.*

Five days later, the craftsman made the two stones so flat that the gap between them was within 0.001 *mm*. On the 16th day, the priest came to see the stones and argued that,

Priest: *But these stones are not flat within 0.0001 mm and I meant 0.0001 mm. You don't have that kind of skill, do you?*

If the priest keeps arguing this way, is the priest really fair showing that the craftsman does not have the ability to flatten stones within any margin of error?

Figure 1. The Mayan stonecutter story.

Results and Discussions

It is expected that when two conflict arguments to each other are suggested, students can recognize that at least one of the arguments is false. However, it is not assured that they will select the true statement between the two conflict arguments. In Study 1, many students initially accepted Statement 1 as a true statement, but they reversed their determination of Statement 1 to accept Ben's argument. Although the students had considerable experiences with rigorous proofs about the convergence of sequences and their reasoning was proper in deriving the truth of Statement 1, they had deficiency of perception of the independence of ε from N, and could not give their refutation against invalid conclusions derived from allowing ε to be selected dependently on N. This result from Study 1 indicates that in order to properly promote students' reflective thinking of the independence of ε from N, it is needed to exclude the possibility that students can accept an argument, such as Ben's argument, that is described by choosing ε dependent on N, hence to be false.

In Step 1 of the Mayan activity implemented in Study 2, two conflict arguments were also given to students: One is Sam's argument that students can be convinced of the truth of its conclusion, and the other is Bill's argument that is contradictory to Sam's argument by choosing ε dependent on N. Unlike Study 1, students in Study 2 could perceive that Bill's argument induces an erroneous conclusion. Pointing out that a negation was attempted in Bill's argument, the students also intellectualized the problem of Bill's argument, and took note of that a negation was tried in Bill's argument. It indicates that they were beyond just suggesting the invalidity of Bill's argument, but further explored intellectually the problem of Bill's argument. Nonetheless, similar to the students in Study 1, the students in Study 2 were unable to find the logical fallacy in Bill's argument. When a student Matt asked "how did he [Bill] not correctly [conclude] it? I guess that's part of the question here," other students encountered a difficulty in explaining the reason why such an erroneous conclusion could be derived. These students did not develop any proper hypothesis and did not make any proper reasoning, to the problem of Bill's argument. Consequently, they failed to resolve their perplexity caused from Bill's argument.

It is worth noting that in Step 2 of the Mayan activity while evaluating the priest's argument in the Mayan stonecutter story, the students in Study 2 instantly suggested the priest unfair. In addition, they perceived that the priest attempted to disprove the craftsman's claim, and intellectualized that in order to disprove the craftsman's claim, the priest should prove the negation of the craftsman's claim. After comparing the negation of the craftsman's claim and the priest's argument in terms of quantified statements, the students recognized that the order between the margin of error and time in the priest's argument was reversed from that in the negation of the craftsman's claim. The students then hypothesized that the reversal of the quantifiers in the priest's argument entailed the illogical conclusion that the priest made. They also reasoned out that while attempting to disprove the craftsman's claim, the priest generated an irrelevant argument to the negation of the craftsman's claim. Eventually the students found why the priest's argument is invalid. As a consequence, they came to understand why the order of variables in these arguments is improperly determined. Furthermore, in Step 3 of the Mayan activity, the students were convinced of their reasoning by confirming that the reversal of the order of the variables in Ben's arguments is the same logical problem as that in priest's argument.

The results from this study indicate that the Mayan activity played a crucial role as an instructional intervention in promoting students' reflective thinking and helping them understand the role of the order of variables in the ε-N definition. The Mayan activity enables students to experience first-hand the meaning of the independence of ε from N. In fact, the activity introduces the Mayan stonecutter story from which students concretely realize the problem of describing ε dependently on N. In addition, the priest's argument is logically *compatible* with Bill's argument but is *tractable* so that students easily understand the logical structure and perceive the logical fallacy in the argument. Furthermore, the stonecutter story is a *transferrable* context in the sense that students can properly link the variables (gaps between stones and days) in the priest's argument to the variables (ε and N) in Bill's argument.

References

Cobb, P., Confrey, J., diSessa, A., Lehrer, R., & Schauble, L. (2003). Design experiments in educational research. *Educational Researcher*, *32*, 9-13.

Dewey, J. (1933). *How we think: A restatement of the relation of reflective thinking to the educational process*. NY: D.C. Heath and Company.

Dubinsky, E., & Yipataki, O. (2000). On student understanding of AE and EA quantification. *Research in Collegiate Mathematics Education*. Vol. 4, 239-289.

Durand-Guerrier, V., & Arsac, G. (2005). An epistemological and didactic study of a specific calculus reasoning rule. *Educational Studies in Mathematics*, *60*, 149-172.

Mamona-Downs, J. (2001). Letting the intuitive bear on the formal: A didactical approach for the understanding of the limit of a sequence. *Educational Studies in Mathematics*, *48*, 259-288.

Roh, K. (2009). Students' understanding and use of logic in evaluation of proofs about convergence. *Proceedings of ICMI Study 19 Conference: Proof and proving in mathematics education, vol. 2, 148-153.*

Roh, K. (2010). An empirical study of students' understanding of a logical structure in the definition of limit via the ε-strip activity. *Educational Studies in Mathematics*, *73,* 263-279.

Roh, K., & Lee, Y. (in press). The Mayan activity: A way of teaching multiple quantifications in logical contexts. To appear in *Problems, Resources, and Issues in Mathematics Undergraduate Studies*.